



## Bayesian analysis: Critical issues related to its scope and boundaries in a risk context



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### ABSTRACT

Bayesian analysis constitutes an important pillar for assessing and managing risk, but it also has some weaknesses and limitations. The main aims of the present paper are to summarize the scope and boundaries of Bayesian analysis in a risk setting, point to critical issues and suggest ways of meeting the problems faced. The paper specifically addresses the Bayesian perspective on probability and risk, probability models, the link between probability and knowledge, and Bayesian decision analysis. A main overall conclusion of the paper is that risk analysis has a broader scope and framing than Bayesian analysis, and that it is important for risk assessment and management to acknowledge this and build approaches and methods that extend beyond the Bayesian paradigm. To adequately assess and handle risk it is necessary to see beyond risk as commonly defined in Bayesian analysis.

### 1. Introduction

This paper discusses Bayesian analysis in a risk context. This context captures concepts, theories, principles, frameworks, approaches, methods and models for understanding, assessing, characterizing, communicating, managing and governing risk, simply referred to for short as ‘concepts for risk analysis’. It is defined by what can be seen as the scope of the risk field and science [2,21,44,45]. The issue raised is the role of Bayesian analysis in supporting risk analysis when ‘risk analysis’ is interpreted in this broad sense. In order to do this, it is also necessary to clarify the aims and scope of Bayesian analysis. Following text books on the topic and well-established nomenclature, Bayesian analysis can be viewed as a method of statistical inference that allows one to combine prior information with new information, using Bayes’ formula to guide the statistical inference process (e.g. [10,12,26,27,38]). Bayesian analysis is a cornerstone in decision analysis, and it is common to refer to Bayesian decision theory and Bayesian decision analysis. Bayesian decision analysis is built on two main components: the use of a prior subjective probability distribution  $P$ , and a utility function  $u$ , reflecting the decision maker's preferences. An optimal decision rule then follows by identifying the act that optimizes the overall expected utility,  $E[u] = \int u \, dP$ . Using Bayesian inference as explained above, the prior probabilities are updated when new information becomes available, using Bayes’ formula. Thus, there is a link between the Bayesian statistical inference and the Bayesian decision analysis, but, as the key pillars of the framework are  $P$  and  $u$ , many scholars just refer to this theory and analysis as decision theory and analysis and avoid the term ‘Bayesian’.

It is also common to refer to Bayesian probability, when probability

is understood as a subjective probability. Following such terminology, any use of subjective probability becomes a Bayesian analysis, whether Bayes’ formula is applied or not.

The present paper will discuss all these interpretations of Bayesian analysis. It will question the degree to which this type of analysis is suitable for supporting risk analysis, particularly on how to characterize risk and make decisions when facing risk problems. Considerable work has been conducted to understand both the strengths and weaknesses of the Bayesian thinking and approach, also in relation to risk; see Section 2 for an overview and discussions in Bernardo and Smith [12], Lindley [27], Bedford and Cooke [9], Singpurwalla [41], Paté-Cornell [32], Fenton and Neil [16] and Mayo [29]. The present discussion aims at reviewing current knowledge on the topic and also gaining new insights, by addressing some current fundamental issues in risk analysis and risk science, including the acknowledgment of the need to see beyond probability-based perspectives to adequately conduct risk analysis. To illustrate, the Society for Risk Analysis (SRA) has recently published a glossary [43] in which the concept of risk comprises two main components: i) the consequences of the activity considered and ii) related uncertainties. In line with this conceptualization, a main feature of risk analysis and science is the incorporation of measures and arrangements to strengthen robustness and resilience, to meet potential surprises and the unforeseen. We question how and to what degree Bayesian analysis can support this type of conceptualization and thinking, in relation to both risk characterization and risk handling.

Section 3 discusses challenges for the Bayesian analysis to deal with this type of fundamental issues in risk analysis. Finally, Section 4 provides some recommendations and conclusions.

The paper focuses on *challenges* of Bayesian analysis in a risk

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context. The strong scientific basis and the broad use of this type of analysis, including those related to risk, are well-documented in the literature but are only to a limited degree highlighted in the present paper. For some practical guidance for conducting Bayesian analysis, see for example Cowles [14] and Gelman et al. [19]. The discussion in this paper is relevant to all types of applications of risk analysis, including engineering, health, business and security [8,16,42,49].

The paper addresses generic and fundamental conceptual and methodological issues, including the meaning of the probability term, the use of probability models and Bayesian decision analysis. However, the aim is not only to make a theoretical contribution. The discussion is also very much important for the *practice* of risk analysis. For example, to effectively communicate risk in real life applications, the interpretation of probability could be critical. Furthermore, to assess, characterize and understand risk in practice it is essential to introduce and use probability models in a prudent way. And to apply the Bayesian decision analysis approach in practice it is crucial to understand the strengths of this approach, as well as its limitations. The paper clarifies what these issues are really about and how they should be dealt with in real life risk contexts.

## 2. The Bayesian perspective

The classical Bayesian set-up is illustrated by the following simple example. Let  $\theta$  be an unknown parameter, representing the health condition of a patient. If the patient is ill,  $\theta = 1$ , whereas  $\theta = 0$  otherwise. A test of the patient is conducted, indicating whether the patient is ill or not. Let  $X$  be the test result: 1 indicating that the patient is ill, and 0 indicating that the patient is not ill.

Bayesian analysis uses Bayes' formula to compute the posterior probability that the patient is ill, given the result of the test:

$$P(\theta = 1|X = x) = P(X = x|\theta = 1)P(\theta = 1)/P(X = x), \text{ where } x = 0 \text{ or } 1 \quad (2.1)$$

Here,  $P(\theta = 1)$  is the prior probability that the patient is ill, whereas  $P(X = 1 | \theta = 1) = 1 - P(X = 0 | \theta = 1)$  is the probability that the test is correct, i.e. shows a positive response when in fact the patient is ill. The prior probability can be based on historical data for a relevant population of patients, or it can simply represent the assessors' subjective probability judgement, given their knowledge about the illness and the patient before the test is conducted. The probability  $P(X = x | \theta)$  expresses the quality of the test.

In more general form, we can write (2.1) as

$$f(\theta|x) = f(x|\theta)f(\theta)/f(x) \quad (2.2)$$

where  $f$  is used as a generic symbol for a probability density function. Thus the posterior distribution  $f(\theta|x)$  is proportional to  $f(x|\theta)f(\theta)$ . Here,  $f(x|\theta)$  can be seen as a model of the 'world': how the observations  $x$  are generated for different world states  $\theta$ . The function  $f(x|\theta)$  is commonly referred to as a probability model for  $X$  with parameter  $\theta$ .

A probability model in a Bayesian setting can be viewed as a model formed by chances. A model is a simplified representation of the world, whereas a chance is interpreted as a limiting relative frequency or fraction of successes when performing a set of thought-constructed trials. Hence, the chance is a measure of variation – the trials either showing success or not. Consider, as an illustration, throwing a special die with not a normal shape. The focus is on the fraction of times  $q$  the die will show 6 (success) in the long run. Let  $Y_1, Y_2, \dots$  be the Bernoulli series for the outcome of the trials, where  $Y_i$  is equal to 1 if the die shows 6 and zero otherwise. Are the  $Y_i$ 's independent? No, clearly not, as, if we obtain the results of some trials, we learn about the die and can improve the predictions of future trials. However, if we were to know  $q$ , the  $Y_i$ 's would be independent, as the observations would not add anything to our knowledge. Unconditionally, the  $Y_i$ 's are outcomes from 'similar' trials but not independent ones. Formally, the idea of 'similar' is reflected by the concept of exchangeability defined as follows: If the

joint probability distribution of a set of  $Y_i$ 's is judged to remain unchanged (invariant) when switching (permuting) the indices, the series is said to be exchangeable [12,27].

Thus, we are led to a framework – the Bayesian one – which is based on two types of probabilities: subjective probabilities  $P$  and chances  $p$ . Different types of interpretations for a subjective probability  $P$  exist. Historically, the interpretation has been linked to betting situations (e.g. [15,37,39]). As an illustrative example, consider the event  $A$  that a specific hypothesis is true, and suppose a probability equal to 0.90 is assigned. Then, following de Finetti, this probability judgement can be understood as stating that 0.90 is "the price at which the person assigning the probability is neutral between buying and selling a ticket that is worth one unit of payment if the event occurs, and worthless if not" (see e.g. [7,40]). An alternative type of interpretation is provided by Lindley [25,27]. If a probability of 0.90 for an event  $A$  is specified, it means that the assessor's uncertainty and degree of belief in the event occurring (being true) is comparable to randomly drawing a red ball out of an urn comprising 100 balls, of which 90 are red.

In formulas (2.1) and (2.2), all probabilities (densities) are subjective. However, commonly, the subjective probability  $P(X = x|\theta)$  is replaced by a probability model reflecting variation. In the above illness case, the subjective probability, that the test instrument is showing positive results when in fact the patient is ill, is replaced by a frequency of the test instrument showing such results in general. Also, the prior probability  $P(\theta = 1)$  can be founded on such frequencies, as mentioned for the illness case, but in general it expresses the assessor's judgement, which could reflect other knowledge aspects than those produced by an observed relative frequency.

The subjective probability  $P$  is conditional on the assessor's knowledge  $K$ , as reflected when writing  $P(A) = P(A|K)$  for the probability of an event  $A$ . This knowledge is based on data, information, tests, argumentation, assumptions, etc.

This Bayesian statistical framework allows one to coherently combine prior information with new information, the tools being probability, probability models and Bayes' formula.

It is, however, also common to extend the above framework by incorporating the decision-making process; see e.g. Lindley [26] and Aven ([2], p. 133). As highlighted by Lindley [27], two more elements should be covered: the consequences should have their merits described by utilities, and the optimum decision needs to combine the probabilities and utilities by calculating subjective expected utility and then maximizing that. The rationale for the expected utility is strong, as thoroughly discussed in the literature (see e.g. [26]). Applying the theory is, however, challenging, as will be further discussed in the coming Section 3.

## 3. Challenges in a risk setting

From the review of Section 2, several issues need further reflections when applying the Bayesian perspective to a risk context. The first relates to the importance of distinguishing between i) 'Bayesian probability', ii) Bayesian inference, and iii) Bayesian decision analysis, defined as:

*Bayesian probability:* A subjective probability

*Bayesian inference:* A method of statistical inference that allows one to combine prior information with new information, using Bayes' formula

*Bayesian decision analysis:* The use of subjective expected utility as a basis for making decisions.

These three concepts and approaches support each other and are often integrated, but they can also be seen as three 'independent' instruments. For example, a risk analyst may support the use of subjective probabilities to express uncertainties about unknown quantities but not necessarily back Bayesian inference and decision analysis. Moreover, scholars supporting Bayesian inference may not favour Bayesian decision analysis (see discussion in Section 3.3). On the other hand,

Bayesian decision analysts would endorse all three components i), ii) and iii), forming a complete theory for risk and uncertainty analysis and decision-making. Some authors, like Dennis Lindley [26], go one step further, arguing that such a full Bayesian perspective is the only rational way of making inference and decision-making when faced with uncertainty.

Below we will discuss in more detail the following issues linked to the understanding and use of the instruments i), ii) and iii):

- 1) Several interpretations of the Bayesian (subjective) probability exist, but many cannot be justified. To describe uncertainties, Bayesian probabilities are in general not sufficient
- 2) Bayesian inference is based on probability models, but often these models cannot be justified
- 3) There is a need to see beyond Bayesian decision analysis when making decisions involving risk.

### 3.1. Bayesian (subjective) probabilities

The use of subjective probabilities represents a cornerstone of Bayesian analysis, and this may explain why these are referred to as Bayesian, despite the fact that Bayes' formula does not presume the use of subjective probabilities – it applies to any type of probabilities. As mentioned in Section 2, the literature on subjective probabilities is founded on early work by scholars like Ramsey [37], de Finetti [15] and Savage [39], and these authors are also commonly referred to today, particularly in economic-related risk analysis. These scholars have made important contributions to probability theory and uncertainty analysis, and their works represent an historical pillar in the development of subjective probabilities to express the assessor's uncertainty and degree of beliefs.

However, in a risk setting, all these interpretations are inappropriate. The problem is that they mix uncertainty assessment and value judgements, as the example in Section 2 demonstrates. When conducting an uncertainty or risk analysis, a basic requirement in practical risk analysis contexts is that the analysts provide 'pure' judgements of the uncertainties, not affected by their attitude to money and the betting situation [7]. The historical definitions of subjective probabilities are closely linked to expected utility theory and personal decision-making. However, risk assessment is mainly about producing a 'neutral' characterization of the uncertainties and risks, not being influenced by how the analyst likes or dislikes money. Being influenced by this type of value judgements would violate this neutrality. When subjective probabilities are adopted, subjectivity is acknowledged, but not lack of neutrality, as here described.

Fortunately, there is a well-established and solid theory for subjective probabilities which provides pure uncertainty judgements [25], as also indicated in Section 2. This interpretation of a subjective probability was mentioned by Kaplan and Garrick [23] in their renowned paper about risk quantification, but, surprisingly, it is not commonly referred to or used by risk analysts and researchers today [7]. This is unfortunate, as the interpretation is the only one, as far as the present author can see, that is theoretically sound and at the same time simple and easy to understand. The interpretation also allows for adjustments to treat imprecise probabilities, for example a statement that the probability is at least 0.90 (the assessor is not willing to be more precise): The assessor's uncertainty or degree of belief is comparable with randomly drawing a red ball out of an urn comprising 100 balls, of which 90 or more are red.

In much work on risk where subjective probabilities are used, interpretations are lacking. This is problematic, as the interpretation is often very important in the practical setting, for example on how to understand and communicate the results of the risk study and follow up the findings. It matters, for example, if the results presented are 'objective' representations of the world or 'just some' analysts' judgements about how the world performs. The subjectivity of the subjective

probabilities is commonly seen as a problem, conflicting basic scientific criteria of deriving objective knowledge not depending on the analysts' judgements. Two main types of argumentation are used to meet this challenge. The first is based on the use of imprecise (interval) probability, as exemplified by the above interval [0.9, 1.0]. This type of interval makes the transformation from the knowledge available to the probability statement less subjective than, say, a probability assignment of 0.94. The price is, however, a less informative statement, and still the statement could be highly subjective, for example expressing the view of one particular expert. The point made is that we also need to consider the knowledge on which the probability statements are based. It could, to varying degrees, be strong and objective.

Common Bayesian analyses do not, however, allow for or encourage such judgements of the knowledge supporting the probabilities, although they are increasingly often reflected in risk analysis and risk science (e.g. [43]). It still seems that many Bayesian analysts believe that probability in principle is a perfect tool for expressing uncertainties [28,30]. The thesis is that coherent judgements about uncertainty lead to the use of probability. As stated by Lindley ([28], p. 239), "If you want to handle uncertainty, then you must use probability to do it." This conclusion is based on different types of arguments [28], the use of scoring rules and logic. It is, however, based on one key assumption, that we restrict attention to a quantitative way of measuring uncertainty. Let us take one step back and consider all types of approaches for describing the uncertainties. To be concrete, let us consider the classical example with a coin showing head or tail. Suppose that you are to perform one and only one throw, and suppose in one case a) you have no knowledge about the coin (it could be non-symmetric) and the other b) where you have made a careful study of the coin, showing that it is perfectly symmetric. We are to assign a probability  $P(A|K)$  for both cases, where A is a coin showing head and K is the background knowledge supporting the assignment. In both cases, we are led to a probability equal to 0.5. However, the two situations are clearly fundamentally different, and building the uncertainty characterization on this probability alone would obviously be insufficient. As the notation  $P(A|K)$  shows, the uncertainty description is also a function of K, and hence we need to also pay attention to K. In the former case a), this knowledge is more or less empty, whereas, in the latter b), it is very strong. Clearly, this insight needs to accompany the probability assignment 0.5. Otherwise, the uncertainty characterization would be incomplete. For a discussion on how to conduct such knowledge strength judgments, see discussion in Aven [3]. See also discussion in Paté-Cornell [34] and Gilboa and Marinacci [20].

The argumentation leads to an adjusted approach: if you want to handle uncertainty, then you must use probability (precise or imprecise), together with judgements of the knowledge supporting the probability assignments.

A sensitivity analysis may show how a probability is affected by changing an aspect of the knowledge K, for example an assumption. Such an analysis would not, however, discuss the reasonability of the assumption made, as a knowledge strength judgement would, just the effect of changing the assumption.

Historically, many scholars have argued for so-called logical probabilities, reflecting the idea that the probability represents the objective degree of logical support that the evidence gives to the event or a statement to be true. The rationale for this type of probabilities can, however, be questioned. Dennis Lindley writes:

Some people have put forward the argument that the only reason two persons differ in their beliefs about an event is that they have different knowledge bases, and that if these bases were shared, the two people would have the same beliefs, and therefore the same probability. This would remove the personal element from probability and it would logically follow that with knowledge base K for an uncertain event E, all would have the same uncertainty, and therefore the same probability  $P(E|K)$ , called a logical probability. We do not share this view, partly because it is very difficult to say what is meant by two knowledge bases

being the same. In particular it has proved impossible to say what is meant by being ignorant of an event, or having an empty knowledge base, and although special cases can be covered, the general concept of ignorance has not yielded to analysis. ([28], p. 44)

Intuitively, the idea that some evidence is equivalent to one and only one probability is appealing. Suppose a prior distribution is to be assigned for an unknown future quantity  $Y$ , representing the number of systems that will fail during a specific period of time in a population comprising  $n$  such systems. We are to assign the probability distribution  $P(Y \leq y)$ . Instead of making a direct assignment of this distribution, we introduce a mind-constructed population of an infinite number of systems, and let  $p$  denote the failure fraction of this population. By conditioning on  $p$ , it follows that

$$P(Y \leq y) = \int P(Y \leq y|p)dF(p)$$

where  $F$  is the prior distribution of  $p$ . Given  $p$ , it follows, under reasonable conditions, that  $Y$  can be considered a binomial distributed random quantity with parameters  $n$  and  $p$ . Hence, it can be claimed that, if it is possible to establish a unique and objective distribution  $F$ , we have been able to derive a unique and objective distribution of  $Y$ . Now, suppose we have obtained some evidence of the type  $Z = z$ , where  $Z$  is the number of failed systems in a similar type of population comprising  $m$  systems. The problem is to assign a posterior probability distribution  $P(Y \leq y | Z = z) = P(Y \leq y | z)$ . Again, using a conditional probability argument, we obtain

$$P(Y \leq y|z) = \int P(Y \leq y|z, p)dF(p|z)$$

with obvious interpretations. We see that  $P(Y \leq y | z, p) = P(Y \leq y | p)$  and, using Bayes' formula,  $dF(p|z) = [f(z|p)f(p)/f(z)]dp$ , where  $f$  is used as a generic letter for a probability density function. We see that all terms can be determined in a unique and objective way, if we can do this also for  $F(p)$ .

The issue of specifying  $F$  has been subject to considerable discussion in the literature. The question addressed is this: Does a distribution  $F$  exist, which corresponds with the idea that we have no information or knowledge available about the relevant quantity, here  $p$ ? At a first grasp, the use of a uniform probability distribution for  $p$  over the interval  $[0,1]$  seems to provide a solution, but further reflections show that the issue is not that simple. The main problem using the uniform distribution as a non-informative prior is that this distribution is not invariant under reparameterization (e.g. [10,46]). In the case that we have no information about  $p$ , we should also have no information about, for example,  $1/p$ . However, a uniform prior on  $1/p$  does not correspond to a uniform distribution for  $p$ .

Many other approaches have been suggested; see, for example, discussions in Kass and Wasserman [24] and Syversveen [46]. It is concluded that no fully objective prior that represents ignorance can in general be derived. Under certain conditions, approaches exist that are independent of the parameterization and are impersonal, for example the so-called Jeffrey's priors or the maximum entropy priors, but something (information, principles, criteria) has to be added to establish the priors.

Thus, the concept of an objective logic probability has to be rejected. It cannot be justified, even for this simple case. In practice, we are not only facing situations where the knowledge basis can be traced to observations of relevant data. In general, the knowledge  $K$  is founded on combinations of data, information, argumentation, analysis, testing and assumptions. Then, there is subjectivity also related to assigning  $P(Y \leq y | K, p)$ , as there is not necessarily an obvious choice for the distribution of  $Y$  given  $K$  and  $p$ . Other knowledge aspects in  $K$  than  $p$  may be of relevance for  $Y$ .

In addition, the situation may not allow a meaningful probability model with parameters to be defined. We will discuss this issue in more detail in the coming section.

Finally in this section, a comment on metrics used to describe the

degree of uncertainties present. A common such metric is the Shannon entropy  $H$  introduced in information analysis. It is defined as:

$$H = - \sum_{i=1}^n p_i \log_2(p_i)$$

where  $p_i$  equals the probability that the discrete random quantity studied takes the value  $x_i$ . Now consider two cases, with the same probabilities. In the first, the probabilities refer to known frequentist probabilities and, in the second, subjective probabilities founded on a poor knowledge basis. The entropy  $H$  gives the same score, but clearly the situations are completely different. A probability distribution alone is, in general, a poor way of measuring the level of uncertainty. We also need to reflect the knowledge used to derive the probabilities; refer also to discussion in Aven ([3], p. 102).

### 3.2. Bayesian inference is based on probability models, but often these models are difficult to justify

A probability model is a collection of frequentist probabilities (chances), often associated with a parameter which is not specified. Bayesian inference builds on the existence of such models, as demonstrated in Section 2. In many cases, such models are easy to justify, for example the binomial distribution to model the success rate for a series of repeated types of similar experiments. However, in other cases, the model is not straightforward to establish. Consider the following example.

An availability performance study is to be conducted for a system in a design phase [6]. To simplify, suppose the system comprises two independent units in a series structure, labelled 1 and 2. Each unit is represented by an alternating renewal process corresponding to the unit being functioning and under repair, respectively. The uptimes have a common probability distribution  $F_i$ , whereas the downtimes have a common distribution  $G_i$ ,  $i=1, 2$ . To study the performance (availability) of the system, the distributions  $F_i$  and  $G_i$  need to be determined.

For this purpose, it is common practice in availability analysis to simply assume a parametric probability distribution, for example an exponential or Weibull distribution for the time to failures, and log-normal or triangular restoration times. For the sake of the present discussion, suppose exponential distributions with rates  $\lambda_i$  are used for the uptimes, and constants for the downtimes, and we can focus on the rationale and treatment in relation to  $F_i$ .

Let  $Y$  denote the performance of the system, and let  $\lambda = (\lambda_1, \lambda_2)$ . Then we have

$$P(Y \leq y) = \int P(Y \leq y|\lambda)dF(\lambda)$$

where  $F$  is the distribution of  $\lambda$ . Given the vector  $\lambda$ , we can find the distribution of  $Y$  by using the fact that the uptimes are independent with exponentially distributions with parameters  $\lambda_i$ ; see Aven and Jensen [6]. The distribution  $F$  can be prior or posterior, depending on the availability of information.

The issue raised here is the use of exponential distribution. In the above Bayesian analysis, the exponential distributions model variation within the population of consecutive lifetimes. Arguments can be provided for why a constant failure rate make sense for units comprising many components subject to maintenance ([6], p. 13); yet, using this particular model adds an assumption to the analysis that is important to be aware of when interpreting the result of the study. The knowledge  $K$  adopted for the analysis is built on this assumption (see discussions in Bergman [11], Aven and Bergman [5] and Flage and Askeland [17]).

This example again demonstrates the importance of adding judgements about the strength of knowledge supporting the probabilities' calculations. It matters a lot for how to understand and use the output probability results whether the models used have strong support or are more or less arbitrary chosen distributions.

When considering systems with a number of units with unknown

parameters for both uptimes and downtimes, the above methodology is challenging. In theory, a simultaneous distribution for all parameters is required. To simplify, the following approach is often used ([2], p. 93). Suppose we have rather strong background knowledge. Then, we fix the probability distributions, for example an exponential for the uptimes. The parameter  $\lambda$  is specified, based on the information available. Then, the focus is on using subjective probabilities to express uncertainties about the unknown quantities, first on unit level, then on system level. However, following this approach the Bayesian set-up is basically lost, as there is no probability model allowing us to use Bayes' formula to systematically update our knowledge, when new information on the unit level becomes available. Depending on the decision situation considered, this may, to varying degrees, be a problem. The assumption that we use the same distribution for the uptimes represents a strong simplification, as we ignore learning from observations of previous lifetimes. As discussed in Aven ([2], p. 93), the approach requires that the background knowledge is rather strong, so that we can, as an approximation, assume independence between consecutive lifetimes for the time period considered.

As another example of the challenges of using probability models in Bayesian risk analysis, consider the task of studying 'rare events' with extreme consequences (such as the September 11 attacks and the Fukushima event). To this end, a Bayesian framework is often formulated, based on the use of probability models. Concepts like heavy and fat distribution tails are referred to. However, this framework is seldom justified or questioned: is it in fact meaningful?

The key problem raised is: does it make sense to perform repeated experiments generating an infinite population of similar situations to the one studied, as is required to interpret the frequency-based probabilities of the probability model? Can we define a meaningful population characterizing the variation in relation to, for example, the September 11 events? No, these are rather unique types of events. A probability model representing similar situations to the one of September 11 cannot meaningfully be defined. If probability is to be used, it needs to be a subjective probability expressing someone's judgement about the uncertainties related to the occurrence of this type of events.

A scientific framework requires that all concepts introduced have precise meanings. Such a requirement is essential for ensuring that the uncertainties in a risk context can be adequately dealt with. If a quantity  $x$  is introduced and it lacks a clear interpretation, analysts would struggle to express uncertainties about  $x$ . For example, if a parametric probability model is introduced without having a proper interpretation, a Bayesian exercise deriving a subjective distribution for the parameters would be subject to considerable arbitrariness, as the assessors would not have a clear understanding of what the uncertainties are about. A subjective probability distribution for 'rare events' may have fat tails, but this would just reflect the assessor's beliefs; it is not an objective property of the world, which would be the case if a frequentist distribution was the point of reference.

### 3.3. The need to see beyond Bayesian decision analysis when making decisions involving risk

In practice, there will always be a need for a 'managerial review and judgement' which sees beyond analysis, so also with the use of the expected utility theory or any other formal decision analysis approach. There is a leap between formal analysis and actual decision-making, reflecting the fact that the analysis has limitations in capturing all aspects of interest for the decision makers, for example properly reflecting the potential for surprises and the unforeseen. Yet, applying a formal decision analysis process can provide structure and knowledge important for the decision-making but not a prescription for what to do. Scholars have discussed the role of formal decision analysis methods for years (e.g. [2,18,22,48]), yet current risk science literature to varying degrees acknowledges the importance of managerial review and

judgements [2].

A risk analyst can be a strong advocator of Bayesian probabilities and inference but prefer using other perspectives than the Bayesian decision analysis to support the decision-making. In most types of real-life problems, the present author would apply a type of multi-attribute analysis, aiming to show the pros and cons of the various decision alternatives considered, using both quantitative and qualitative approaches and methods to show activity performance, risks and uncertainties [2,4]. Some key arguments for not using a Bayesian decision analysis approach are discussed below. For some decision analysis perspectives and approaches challenging the expected utility theory (Prospective theory and rank-dependent utility theory); see Tversky and Kahneman [47] and Quiggin [36].

Quantitative decision analysis, including Bayesian expected utility theory, is based on the use of decision rules, which tell "decision-makers what to do, given what they believe about a particular problem and what they seek to achieve" [35]. The use of decision rules contributes to ensuring consistency and presuming that the rules have a strong rationale, good decisions. However, in many situations, particularly when facing large uncertainties, the specification of such rules is difficult. A case is discussed in Aven ([4], p. 179), where the issue concerns applying the precautionary principle; the consequences of the activity are subject to scientific uncertainties. If the decision rule is founded on probability assignments, as for Bayesian expected utility theory, it is obvious that care has to be shown when reading the recommendations produced by the decision analysis. The basis for the assigned probabilities would be poor, and a strict adherence to the results of the formal analysis would violate fundamental principles of risk management, giving weight to uncertainties and potential surprises.

Risk is not a concept used in Bayesian inference, but it is used in Bayesian decision analysis, as outlined in Section 2. Here, risk is defined as expected (dis)utility; in the following, without loss of generality, it is just referred to as the utility.

Following current definitions of risk [43], see also [1,4], the risk related to an activity can be conceptualized as (C,U), where C is the consequences of the activity and U, the associated uncertainties. Looking into the future of the activity, we do not know what C will be. Normally, C is seen in relation to a reference, for example a current level, a target or a goal. There will always be at least one outcome that is negative or undesirable. To characterize risk, we need to specify the consequences and represent, describe or measure the uncertainties. In general terms, this leads to a characterization of the form (C',Q,K), where C' is the specification of C, Q a measure (in a wide sense) of the uncertainties related to C', and K the knowledge that C' and Q are based on.

Different metrics can be defined on the basis of (C',Q,K), for example the expected consequences and a probability distribution P for C' (hence Q=P). The Bayesian risk is another example of such a metric. If  $u$  denotes the utility function, we have

$$\text{Bayesian risk} = E[u(C')] = \int u(C')dP$$

Thus, the Bayesian risk is not really a definition of risk but an approach for measuring or describing risk. An important issue is then how well this measure is able to actually represent or express risk.

A common risk metric is the expected value,  $E[C']$ . However, this measure suffers from serious weaknesses, as the potential for severe consequences is not properly reflected. Two distributions could have the same centre of gravities (expected values) but be completely different when it comes to the potential for extreme consequences. For the risk response and handling these situations are completely different, but, using the expected value as the risk metric, this type of difference is ignored.

The use of the utility function meets this critique of the expected value. By introducing the utility function, the preferences of the decision maker can be reflected. However, in practice, the derivation of the

**Table 1**  
Summary of critical issues of Bayesian analysis in a risk setting, with suggestions for how to deal with them

Issue	Explanation	How to deal with the issue
Subjective probability	The use of subjective probabilities represents a cornerstone of Bayesian analysis, but the term is commonly not given an interpretation or an unfortunate interpretation is used mixing uncertainty judgements and value judgments.	It is recommended to use the interpretation based on comparison with a standard approach [28], see Sections 2 and 3.1.
Bayesian probability	It is common to refer to Bayesian probability when probability is understood as a subjective probability	The concept of 'Bayesian probability' is confusing and should not be used, as its definition and meaning are not directly linked to Thomas Bayes or Bayes' theorem.
Subjective probability is a tool for expressing uncertainty	Bayesian analysis expresses epistemic uncertainty through subjective probabilities, but these probabilities are not sufficient for expressing uncertainty in a risk context	To express the uncertainties, the knowledge and judgements of the strength of the knowledge supporting the probabilities P need to be added. See Section 3.1.
Bayesian perspective on risk	Bayesian analysis has a 'narrow' perspective on risk, based on expected utility or probabilities. Important aspects of risk are not reflected in the characterizations	Broader characterizations and frameworks are needed as presented and discussed in Section 3.3. Specific processes need to be implemented to address potential surprises and the unforeseen, see Section 3.3, Aven [3] and Bjerga and Aven [13]
Probability models	Bayesian analysis is to large extent built on probability models, but in some risk settings such models can be difficult or impossible to justify. The models require that a large (in theory an infinite) population of similar situations to the one studied can be generated, but many events are unique and such populations cannot be meaningfully produced	Justification is needed for introducing and using a probability model. The strength of the knowledge that the model is built on (in particular the assumptions) need to be evaluated. Probabilistic analysis can also be conducted based on subjective probabilities without introducing probability models
Quantified Risk Assessment (QRA) Probabilistic Risk Assessment (PRA)	QRAs (PRAs) can be based on Bayesian analysis, using models and subjective probabilities. The assessments characterize risk by (C',P) and provide a basis for discussions about risk between different stakeholders – and they support the decision making. Assumptions are made explicit, but knowledge strength judgments are not commonly performed.	Knowledge strength judgments should be added, and specific processes implemented to address potential surprises and the unforeseen, see above. All models used need to be justified. The strength of the assumptions forming the models should be included in the overall strength of knowledge judgments for the probabilistic results produced.
Bayesian decision analysis	Bayesian decision analysis is a useful instrument and framework for decision analysis but has strong limitations in addressing all relevant risk issues	It needs to be supplemented with a management and review process, as described in Section 3.3. Practical methods exist based on different types of multi-attribute analysis, aiming to show the pros and cons of the various decision alternatives considered, using both quantitative and qualitative approaches and methods to show activity performance, risks and uncertainties

utility function is not straightforward – it requires a rather complex procedure, using lotteries (e.g. [26]), that is difficult for decision makers to carry out in practice. For some decision problems with many stakeholders, it can also be an issue to determine which utility function to use. As the Bayesian theory is applicable to only one decision maker, the solution is to consider the Bayesian decision analysis as a way of thinking and a tool to gain insights for different stakeholders and decision makers, by studying the implications of using various utility functions. A problem encountered in practice, however, is that many stakeholders and decision makers would hesitate to reveal their preferences as the expected utility approach requires.

Seeing risk as the expected utility is also restrictive for other reasons, the main one being that important aspects of risk are not adequately reflected by the probabilities. Probability here is a subjective probability, as explained in Section 3.2. The common risk management situation in practice is that some analysts perform a risk assessment, and then one or more stakeholders and decision makers are informed by the results of the risk assessment. The Bayesian decision analysis framework is designed for situations where the decision makers assign both probabilities and utilities. The more professional risk management setting, with analysts informing decision makers, is fundamentally different from the individual perspective used in traditional Bayesian decision analysis.

However, regardless of the setting, the problems discussed in Section 3.1 prevail. The knowledge could be more or less strong and even wrong. Consequently, a decision maker cannot simply limit the information and judgements to the expected utility, even though it captures the preferences of the decision maker. The decision maker also needs to be informed about the knowledge supporting the probabilities. A key question is how strong it is. Furthermore, the decision maker needs to take into account potential surprises relative to this knowledge – the unforeseen ('black swans'). Of special interest here are 'unknown knowns' (the event is unknown for the analysts, but known to others)

and known type of events ignored because of erroneous assumptions. This aspect of risk cannot meaningfully be quantified, but it certainly is an aspect of risk which needs to be given due attention in risk management, for example by designing systems which score highly on robustness and resilience. In recent years, considerable research and development has been conducted to develop concepts, principles, approaches, methods and models for assessing and managing this type of risk (e.g. [3,33]). Using the Bayesian decision perspective on risk, the importance of these enhancements is not sufficiently noticed, as the risk framing to a large extent ignores their contributions. In contrast, the current risk conceptualizations based on (C,U), seek to stimulate investigations into all types of risks, as a result of both weak knowledge and erroneous knowledge. See Bjerga and Aven [13].

The above discussion should not be interpreted as stating that Bayesian risk analysis is restricted to the expected utility calculations. Aspects of risk may also be quantified and expressed by probability distributions of C', as for example in Quantified Risk Assessment (QRAs) and Probabilistic Risk Assessments (PRAs). However, the discussion in the last paragraph also applies to the probability distributions. The strength of the knowledge is not addressed, nor the potential for surprises.

Paté-Cornell [31] provides an excellent exposition of the difference between risk analysis/assessment and decision analysis, including Bayesian decision analysis. A main goal of a risk assessment could be to enhance relevant stakeholders' understanding of risk. A specific decision may not even be on the agenda. The insights gained by the risk assessment could stimulate discussion and suggestions for completely different types of action than previously thought about. The results could also provide a basis for a political discussion on what to do next.

#### 4. Recommendations and conclusions

The main conclusions of the paper are summarized in Table 1.

Bayesian inference and Bayesian decision analysis are acknowledged as solid and important instruments ('instrument' interpreted broadly to also capture 'way of thinking') for prudent risk analysis and risk science. Although both refer to 'Bayesian', they are two completely different things. The former is about knowledge generation and coherent uncertainty judgements, whereas the latter is about how to make good decisions. We may value the usefulness of these two instruments differently. An analyst may apply Bayesian inference but not Bayesian decision analysis. Both instruments have their strengths and limitations, as summarized and discussed in previous sections. For many types of situations where risk analysis is conducted, particularly when the uncertainties are large, there is a need to see beyond the Bayesian approaches. A key problem is that the strength of the knowledge supporting the subjective probabilities is commonly not addressed in these approaches. It is, however, essential that this aspect of risk is looked into. Whereas the Bayesian approach highlights risks characterizations in form of  $E[u(C)]$  and  $(C, P)$ , risk science points to the more general formulation  $(C, Q, K)$ , which allows for also including judgments of the strength of knowledge supporting the probabilities. Methods have been developed for how to implement such an approach in practice (e.g. [3,4]). Also, the risk related to potential surprises needs to be considered, as discussed in the previous section.

Bayesian decision analysis may provide useful insights in many cases, but the results from this type of analysis cannot in general replace more overall considerations of the pros and cons of the alternatives considered, as the quantitative approach is not able to capture all relevant aspects of the decision-making situation, as discussed in Section 3.3.

The concept of 'Bayesian probability' is confusing and should not be used, as its definition and meaning are not directly linked to Thomas Bayes or Bayes' theorem. Analysts may be users and advocators of subjective probabilities but not of Bayesian inference or Bayesian decision analysis. A subjective probability should be interpreted as mentioned in Section 3.1, using the comparison with a standard approach [28].

#### Author statement

The work has been carried out fully by Terje Aven.

#### Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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