ON UNIVERSAL BLACK HOLES*

Sigbjørn Hervik

Faculty of Science and Technology, University of Stavanger
4036 Stavanger, Norway
sigbjorn.hervik@uis.no

Marcello Ortaggio

Institute of Mathematics of the Czech Academy of Sciences
Žitná 25, 115 67 Prague 1, Czech Republic
ortaggio@math.cas.cz

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Recent results on universal black holes in d dimensions are summarized. These are static metrics with an isotropy-irreducible homogeneous base space which can be consistently employed to construct solutions to virtually any metric theory of gravity in vacuum.

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1. Introduction

Let us consider the static black hole Ansatz

$$\boldsymbol{g} = \mathrm{e}^{a(r)}\Big(- f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)}\Big) + r^2 h_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j . \tag{1}$$

When $a = 0$, $f = 1 - \frac{\mu}{r}$ and $\boldsymbol{h} = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is the metric of a 2-dimensional round unit sphere, this represents the well-known spherical Schwarzschild black hole of four-dimensional general relativity.

Extensions to Einstein's gravity in $d = n + 2$ spacetime dimensions with a cosmological constant are readily obtained if one takes $f = K - \frac{\mu}{r^{d-3}} - \Lambda r^2$ and $\boldsymbol{h}$ is the metric of an n-dimensional Einstein space with Ricci scalar $\tilde{R} = n(n-1)K$ [2–4]. While $\boldsymbol{h}$ can be *any* Einstein space in Einstein's gravity, obstructions to the permitted geometries arise in more general higher dimensional theories such as Gauss–Bonnet and Lovelock gravity [5–8].

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In our recent work [1], we have studied the metric Ansatz (1) in higher-order vacuum gravity theories of the form of

$$S = \int \mathrm{d}^d x \sqrt{-g} \mathcal{L}(\boldsymbol{R}, \nabla \boldsymbol{R}, \ldots) \,, \tag{2}$$

where $\mathcal{L}$ is a scalar invariant constructed polynomially from the Riemann tensor $\boldsymbol{R}$ and its covariant derivatives of arbitrary order. We have obtained a sufficient condition on the metric $\boldsymbol{h}$ which enables Ansatz (1) to be consistently employed in any such theory, as we summarize in the following.

2. Black holes with universal horizons

First of all, let us recall the following geometric definition (quoted, for convenience, from [9]):

Definition 2.1 (IHS space) *An* isotropy-irreducible homogeneous space *(IHS)* $(M, \boldsymbol{h})$ *is a homogeneous space whose isotropy group at a point acts irreducibly on the tangent space of* M *at that point.*

For our purposes, it is important to observe that an IHS is necessarily Einstein (but not *vice versa*) and, more generally, for an IHS any symmetric 2-tensor on M possessing the symmetries of $\boldsymbol{h}$ must be proportional to $\boldsymbol{h}$ [10]. IHSs are equivalent to *universal* Riemannian spaces in the sense explained in [1]. Examples of IHS can be found in [11]. These include direct products of (identical) spaces of constant curvature and irreducible symmetric spaces. In particular, in $n = 4$ dimensions, an IHS must be symmetric and, therefore, locally one of the following: S^4, $S^2 \times S^2$, H^4, $H^2 \times H^2$, $\mathbb{C}P^2$, $H^2_{\mathbb{C}}$, or flat space (*cf.*, *e.g.*, [11] and references therein).

Now, we can quote the main result of [1]:

Proposition 2.2 *Consider any metric of the form of (1) where* $\boldsymbol{h}$ *is an IHS. Then, any symmetric 2-tensor* $\boldsymbol{E}$ *constructed from tensor products, sums and contractions from the metric* $\boldsymbol{g}$*, the Riemann tensor* $\boldsymbol{R}$*, and its covariant derivatives necessarily takes the form of*

$$\boldsymbol{E} = F(r)\mathrm{d}t^2 + G(r)\mathrm{d}r^2 + H(r)h_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j \,. \tag{3}$$

Let us now note that the field equations derived from (2) (neglecting boundary terms) are of the form of $\boldsymbol{E} = 0$, where $\boldsymbol{E}$ is a symmetric, conserved rank-2 tensor locally constructed out of $\boldsymbol{g}$ and its derivatives [12] (*cf.* also [13]). We can thus apply proposition (2.2) to observe that, in any theory of gravity (2), the tensorial field equation $\boldsymbol{E} = 0$ for metric (1) with $\boldsymbol{h}$ IHS reduces to three "scalar" equations $F(r) = 0$, $G(r) = 0$ and

$H(r) = 0$. Furthermore, the equation $H(r) = 0$ holds automatically once $F(r) = 0 = G(r)$ are satisfied, thanks to the fact that $\boldsymbol{E}$ is identically conserved. One is thus left with just *two ODEs* for the two metric functions $a(r)$ and $f(r)$ (their precise form will depend on the particular gravity theory under consideration — several examples can be found in [1] and references therein). This is a drastic simplification of the tensorial field equation $\boldsymbol{E} = 0$. These spacetimes will generically describe static black holes — we name them *universal black holes* because they possess a universal (IHS) horizon and because the construction described above works universally in any theory (2). The details (including the precise form of $a(r)$ and $f(r)$) and physical properties of the solutions depend on the specific theory one is interested in. Since for $n = 2, 3$ an n-dimensional Einstein space is necessarily of constant curvature, this result is of interest for dimension $d \geq 6$ (*i.e.*, $n \geq 4$).

Some comments on the near-horizon geometries associated with extremal limits of the universal black holes described above can be found in [1] (see also [14]).

3. Examples

Here, we illustrate the results of Section 2 by giving explicit examples of black holes solutions in certain gravity theories of the form of (2). Quantities with a tilde will refer to the transverse space geometry of $\boldsymbol{h}$ (taken to be IHS), so that

$$\tilde{R}_{ij} = (n-1)K h_{ij} \,, \tag{4}$$

and thus $\tilde{R} = n(n-1)K$.

3.1. Gauss–Bonnet gravity

This theory is defined by the Lagrangian density

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{\kappa}(R - 2\Lambda) + \gamma I_{\mathrm{GB}}\right], \qquad I_{\mathrm{GB}} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{5}$$

where κ, Λ and γ are constants.

With Ansatz (1), it possesses the black hole solution [5, 15–18]

$$a(r) = 0\,, \tag{6}$$

$$f(r) = K + \frac{r^2}{2\kappa\hat{\gamma}}\left[1 \pm \sqrt{1 + 4\kappa\hat{\gamma}\left(\frac{2\Lambda}{n(n+1)} + \frac{\mu}{r^{n+1}}\right) - \frac{4\kappa^2\hat{\gamma}^2\tilde{I}_{\mathrm{W}}^2}{r^4}}\,\right], \tag{7}$$

where μ is an integration constant and

$$\hat{\gamma} = (n-1)(n-2)\gamma\,, \qquad n(n-1)(n-2)(n-3)\tilde{I}_{\rm W}^2 = \tilde{C}_{ijkl}\tilde{C}^{ijkl}\,. \tag{8}$$

Equation (7) shows that the Weyl tensor of the geometry $\boldsymbol{h}$ affects the solution. The branch with the minus sign admits a GR limit by taking $\hat{\gamma} \to 0$. The non-negative constant $\tilde{I}_{\rm W}^2$ vanishes iff $\boldsymbol{h}$ is conformally flat (so necessarily when $n = 3$), in which case one recovers the well-known black holes with a constant curvature base space [19–21].

3.2. Pure cubic Lovelock gravity

In more than six dimensions, a natural extension of Gauss–Bonnet (and Einstein) gravity is given by Lovelock gravity [22]. The special *purely cubic* theory is defined by

$$\mathcal{L} = \sqrt{-g}\left(c_0 + c_3\mathcal{L}^{(3)}\right)\,, \qquad \mathcal{L}^{(3)} = \frac{1}{8}\delta^{\rho_1\sigma_1\rho_2\sigma_2\rho_3\sigma_3}_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}R^{\mu_1\nu_1}_{\rho_1\sigma_1}R^{\mu_2\nu_2}_{\rho_2\sigma_2}R^{\mu_3\nu_3}_{\rho_3\sigma_3}\,, \tag{9}$$

where $\delta^{\mu_1\ldots\mu_p}_{\rho_1\ldots\rho_p} = p!\delta^{\mu_1}_{[\rho_1}\ldots\delta^{\mu_p}_{\rho_p]}$ and c_0, c_3 are constants.

It possesses the solution

$$a(r) = 0\,, \tag{10}$$

$$\begin{aligned} f(r) - K = &\\ &\frac{1}{(2\hat{c}_3)^{1/3}}\left[c_0r^6 - \frac{\mu}{r^{n-5}} + \hat{c}_3\tilde{J}_{\rm W} + \sqrt{\left(c_0r^6 - \frac{\mu}{r^{n-5}} + \hat{c}_3\tilde{J}_{\rm W}\right)^2 + 4\hat{c}_3^2\tilde{I}_{\rm W}^6}\right]^{1/3}\\ &+\frac{1}{(2\hat{c}_3)^{1/3}}\left[c_0r^6 - \frac{\mu}{r^{n-5}} + \hat{c}_3\tilde{J}_{\rm W} - \sqrt{\left(c_0r^6 - \frac{\mu}{r^{n-5}} + \hat{c}_3\tilde{J}_{\rm W}\right)^2 + 4\hat{c}_3^2\tilde{I}_{\rm W}^6}\right]^{1/3}\,, \end{aligned} \tag{11}$$

where μ is an integration constant and we have defined $I_{\rm W}^2$ as in (8), and

$$\hat{c}_3 = (n+1)n(n-1)(n-2)(n-3)(n-4)c_3\,, \tag{12}$$

$$\begin{aligned} &(n-1)(n-2)(n-3)(n-4)(n-5)\tilde{J}_{\rm W} = \\ &4\tilde{C}_{ijkl}\tilde{C}^{klmn}\tilde{C}_{mn}{}^{ij} + 8\tilde{C}_{ijkl}\tilde{C}^{mjkn}\tilde{C}^{i}{}_{mn}{}^{l}\,. \end{aligned} \tag{13}$$

The above solution was obtained in [23] for the special case when $\boldsymbol{h}$ is a product of two identical spheres (a solution for cubic Lovelock theory including lower order curvature terms was obtained earlier in [6]). When $I_{\rm W}^6 = 0$ ($\Rightarrow J_{\rm W} = 0$), the base space is of constant curvature and one recovers the solution obtained in [24] (see also [7, 25]).

Comments about static black hole solutions in generic Lovelock gravity with a base space not of constant curvature can be found in [7, 8].

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