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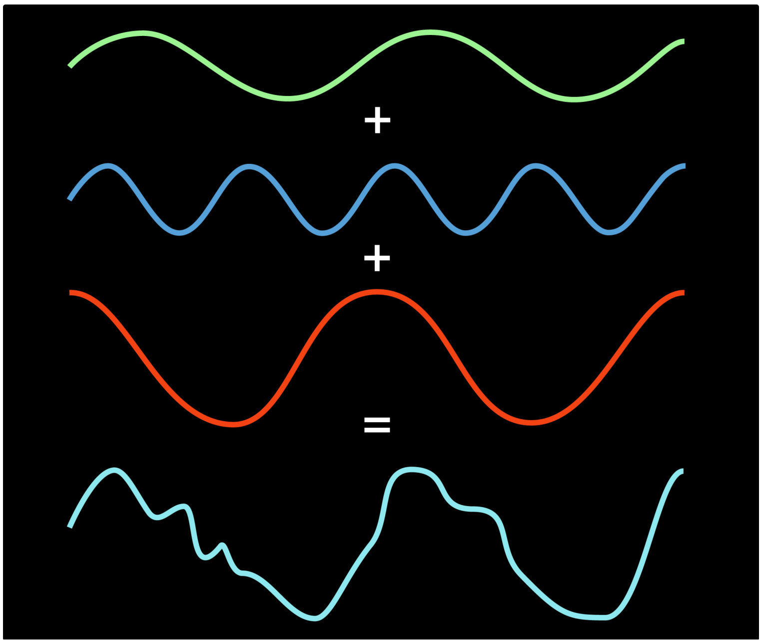
THE FOURIER TRANSFORM

and its applications to partial differential equations



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Chapter 1

Introduction

In this master's thesis I will introduce a way to solve partial differential equations and boundary value problems by transforming signals from a time domain to a frequency domain, and back. This algorithm is called the Fourier transform and is widely used in signal-processing and wave studies, quantum mechanics and in spectropy, for example nuclear magnetic resonance. These areas of applications will not be further discussed in this thesis, but as a future teacher, I think that it is important to say something about the purpose and practical use before introducing new and difficult mathematical content. I have no doubt that this helps as a motivation in the learning process, as well as preparing the reader for what to come.

The Fourier transform is, as we will see, defined as an improper Riemann-integral. Even though it may seem as a detour to transform a problem back and forth, my goal is to show that the Fourier transform helps to decompose an "insoluble" mathematical problem into smaller steps that are easier to solve.

In Chapter 2, we will derive the Fourier transform by generalizing the Fourier series. Here both the originial Fourier transform and its inverse will be defined, as well as alternative definitions and notations. Following comes basic properties of the transform, differentiation and convolution.

In Chapter 5 the Gaussian function will be defined, and the proof of its Fourier transform will be given. Later you will see a proof of the inverse Fourier transform, the Fourier transform method and other solving strategies, such as the Gauss' kernel. In Chapter 9 we introduce the cosine and sine Fourier transforms and their properties.

The last part of is dedicated to the Fourier transform of generalized functions, or distributions. In Chapter 11 one will see how to solve the non-homogeneous heat equation using the so-called incomplete Gamma-function, which completes my thesis.

1.0.1 Sources and other comments

When it comes to my sources, I have mainly used chapter 7 in Nakhlé Asmar's book *Partial Differential Equations and Boundary Value Problems* and the Wikipedia-site *Fourier transform*. I have also watched the video *But what is the Fourier transform*, the webpage *fouriertransform.com* and the website-post *An interactive guide to the Fourier transform*. For a more detailed source-list, have a look at the reference-list in the end of this document. Last but not least, I have recieved the most welcome help from my supervisor, Alexander Rashkovskii, which have been available at all times needed, both in person and online. Thank you so much for that!

Throughout this thesis, many examples and some proofs will be given to substantiate the definitions and properties that are being stated. The main part of these examples are problems that I solved myself , but one can also find examples that are more or less taken directly from its source. These "given" examples are used as a supplement to approve my own work and are often modified a bit to make it fit better to what I try to show. It is also a way to show that my example is well thought out and not just placed randomly. I guess it is also about taking control over other's work and make it my own by trying to give it a customized touch.

I hope that my readers achieves some more knowledge on Fourier transforms after reading this thesis and receive some of the tools to solve even more complicated boundary value problems in the future. Have a good reading!

Chapter 2

Deriving the Fourier Transform

2.1 Motivation

How can we take a microphone-signal and decompose it into pure, single frequencies given by the curves of sines and cosines? You guessed it, the answer is to use the Fourier transform.

Think of the operation in a practical way, like a "unmixing"-machine for paint colors that have already been stirred - or like finding the recipe of a cake that has already been baked. Why? Because it is much easier to analyse each ingredient separately and especially to find and modify the recipe. Fascinating, right? As far as I know, such machines do not exist for neither paint or cakes, but are widely used for filtering out noise in sound modification.

Mathematically speaking, all waveforms are actually just the sum of simple sinusoids of different frequencies. In simple terms, that means that any pattern or function can be described by circles.

2.2 The "unwinding" principle

The Fourier transform takes a time-based pattern, measures every possible cycle and returns the "recipe" given by amplitude, offset and rotation speed for each cycle. This machine is called a "frequency unmixing machine", which unwinds the given curve around a circle. Then it gives a spike when the winding frequency is the same as the signal frequency.

A pure frequency has a transform close to zero except for a spike around the frequency itself. Even if there are several different frequencies, we still get the different spikes distributed around the frequencies itself. So if a music-producer wants to filter out a high-frequency noise from a soundtrack, this method can be used to reveal its frequency and then get rid of it.

It is important that each "filter" is independent, which means that it measures only one ingredient or signal, and that it is complete (can extract all possible signals). Last, but not least, the filters has to be combineable: the sum of each ingredients must add up to the same results as before (a smoothie, a melody etc.).

Put in short, the Fourier transform gives us another way to represent a waveform. In general, waveforms are made up of a continuous range of frequency, not only discrete ones. Therefore, as we will see, the sum ($\sum$) will be replaced by an integral ($\int$).

2.2.1 Circular paths

What if, indeed, any signal could be filtered into many circular paths? Then each path we would need a given size, speed and starting angle - namely amplitude, frequency and phase. The Fourier transform can be described as building a recipe frequency by frequency with a whole lot of time-spikes (individual frequencies). We can write up the following equations:

$$X_k = \sum_{n=0}^{N-1} X_n e^{-\frac{i2\pi kn}{N}},$$

which states that the frequency recipe (X_k) adds up to the sum of contributions from each individual time spike. The time-point is given by

$$X_n = \frac{1}{N} \sum_{n=0}^{N-1} X_k e^{\frac{i2\pi kn}{N}},$$

where each time spike contributes to the time-point X_n.

The idea behind the Fourier transform is well stated by the software engineer Stuart Riffle:

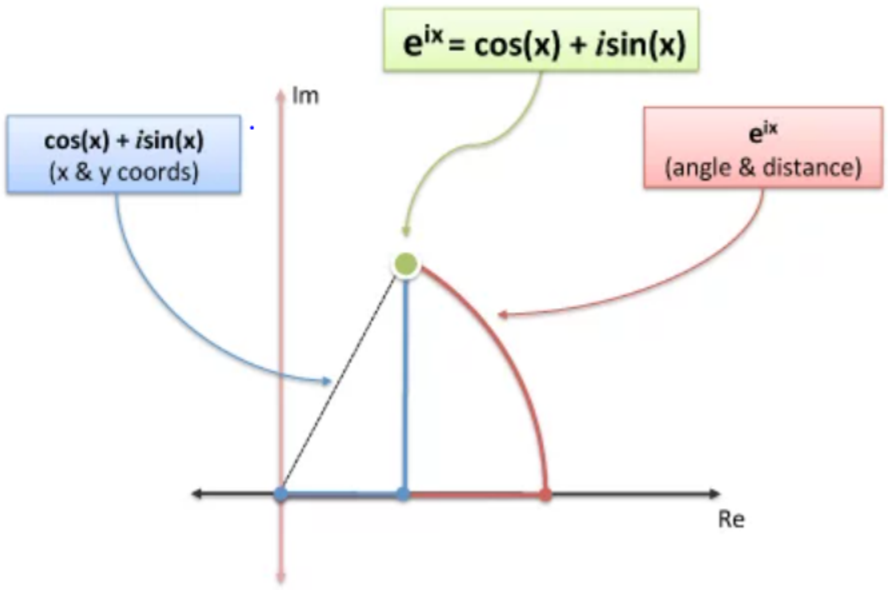
> *To find the energy at a particular frequency, spin your signal around a circle at that frequency, and average a bunch of points along that path.* (Unknown 2, 2021)

2.3 A generalization of the Fourier series

As we know, periodic functions like sines and cosines are written as a sum of waves. These sums are called Fourier series. The Fourier transform is therefore an extension, or a generalization, of these sums as the period of these wave-functions is approaching infinity.

For simplicity, it is often appropriate to express the Fourier series by using Euler's formula:

$$e^{2\pi i\theta} = \cos 2\pi\theta + i \sin 2\pi\theta$$



This way of writing also makes it easier to connect the formulas to the expression of the Fourier transform, denoted $\hat{f}$. It is important to remark that the Fourier coefficients now can be negative and complex valued. Therefore, the frequency can no longer be given as number of cycles per unit of time, since it makes no sense to talk about negative frequencies.

2.3.1 Fourier series representation theorem

Real problems are often given on unbounded regions. An example is the distribution of the temperature in a long insulated wire that would give us a boundary value problem over an infinite line. So when Fourier series where useful on bounded regions, like intervals or disks, we have to use the generalized version, namely the Fourier transform, on the unbounded ones.

Given a 2p-periodic function $f(x)$, consider its Fourier series

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x) \tag{2.1}$$

where

$$a_0 = \frac{1}{2p} \int_{-p}^{p} f(t)dt,$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(t) \cos \frac{n\pi}{p} tdt,$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(t) \sin \frac{n + pi}{p} tdt.$$

From the Fourier series representation theorem, we have that if f is continuous at x, $f(x)$ is given by

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

What if $f(x)$ is not periodic, but is defined on the whole real line ($\mathbb{R}$)? Then we get a **Fourier integral representation**.

2.3.2 Fourier integral representation

Theorem 1

Suppose $f(x)$ is a piecewise smooth function on every finite interval, satisfying

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty$$

Then $f(x)$ has the Fourier integral representation

$$\int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega \quad (-\infty < x < \infty) \quad \forall\omega \geq 0 \tag{2.2}$$

at any continuity point x, where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t dt,$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t dt.$$

The proof of this will be given later.

As for the Fourier series: when $f(x)$ is even, $A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t dt$ and $B(\omega) = 0$, and when $f(x)$ is odd, $A(\omega) = 0$ and $B(\omega) = \frac{2}{\pi}\int_0^\infty \sin\omega t dt$. The integral in equation (2) converges to $f(x)$ if it is continuous at x, and to $\frac{f(x+)+f(x-)}{2}$ otherwise.

What have been changed from equation (2.1) to (2.2)?

- Sum has been changed to integral.
- The boundaries have gone from bounded ($-p$ to p) to unbounded ($-\infty$ to ∞).
- Discrete range of values of n has gone to continuous range of ω.

$f(x)$ has to be integrable on the whole $\mathbb{R}$, namely that

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty$$

2.3.3 Examples

Here we find the Fourier integral representation of some piecewise continuous functions and use it for computing some interesting integrals.

Example 1

Let

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $\omega \neq 0$,

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) cos\omega t dt = \frac{1}{\pi} \int_{-1}^{1} cos\omega t dt =$$

$$= \frac{1}{\pi} \left[\frac{sin\omega t}{\omega} \right]_{-1}^{1} = \frac{1}{\pi} \left[\frac{sin\omega}{\omega} - (-\frac{sin\omega}{\omega}) \right]$$

$$= \frac{2sin\omega}{\pi\omega}.$$

If $\omega = 0$, $A(0)$ is just the value of $\int \frac{1}{\pi} \cos \omega t$ when $\omega = 0$.

$$\Longrightarrow \frac{1}{\pi} \int_{-1}^{1} \cos 0 dt = \frac{1}{\pi} \int_{-1}^{1} dt = \frac{2}{\pi}.$$

Since f is even, we know that $B(\omega) = 0$. It can be shown as following: When $\omega \neq 0$,

$$B(\omega) = \frac{1}{\pi} \int_{0}^{\infty} f(t) \sin \omega t dt = \frac{1}{\pi} \int_{-1}^{1} \sin \omega t dt =$$

$$= \frac{1}{\pi} \left[-\frac{\cos \omega}{\omega} \right]_{-1}^{1} = \frac{1}{\pi} \left[-\frac{\cos \omega}{\omega} - (-\frac{\cos \omega}{\omega}) \right] =$$

$$= \frac{1}{\pi} \left[-\frac{\cos \omega}{\omega} + \frac{\cos \omega}{\omega} \right] = 0.$$

For $|x| \neq 1$, the function is continuous and Theorem 1 gives

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{sin\omega x}{\omega} d\omega.$$

At the points of discontinuity, $x = \pm 1$, the same theorem gives

$$\frac{f(x+) + f(x-)}{2} = \frac{1+0}{2} = 1/2$$

for $x = -1$ and

$$\frac{0+1}{2} = 1/2$$

for $x = 1$.

Thus,

$$\frac{2}{\pi} \int_0^\infty \frac{\sin \omega x}{\omega} dw = \begin{cases} 1 & \text{if } |x| < 1 \\ 1/2 & \text{if } |x| = 1 \\ 0 & \text{if } |x| > 1 \end{cases} \quad \blacksquare$$

Example 2

By setting $x = 1$ in the example above, we get the Dirichlet integral

$$\int_0^\infty \frac{\sin \omega}{\omega} d\omega = \frac{\pi}{2}$$

Example 3

a) Now we can show that

$$\int_0^\infty \frac{\sin \omega \cos \omega}{\omega} d\omega = \frac{\pi}{4}$$

From Example 1, we see that

$$\frac{2}{\pi} \int_0^\infty \frac{\sin \omega \cos \omega x}{\omega} d\omega = \frac{1}{2}$$

when $|x| = 1$. By multiplying by $\frac{\pi}{2}$ we get

$$\int_0^\infty \frac{\sin\omega\cos\omega}{\omega} d\omega = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

which is what we should show.

b) Use integration by parts and a) to obtain

$$\int_0^\infty \frac{\sin^2\omega}{\omega^2} d\omega = \frac{\pi}{2}$$

Set $u = sin^2\omega$, $v' = \frac{1}{\omega^2}$, $u' = 2\sin\omega\cos\omega$ and $v = -\frac{1}{\omega}$. Then we get

$$-\frac{1}{\omega} sin^2\omega + \int \frac{2\sin\omega\cos\omega}{\omega} d\omega =$$

$$= -\frac{1}{\omega}\sin^2\omega + \frac{\pi}{2}$$

(same integral as in Example 3). Setting the limits, we then obtain

$$\left[-\frac{1}{\omega}\sin^2\omega\right]_0^\infty + \frac{\pi}{2} = 0 + \frac{\pi}{2} = \frac{\pi}{2}.$$

Example 4

Given

$$f(x) = e^{-|x|},$$

we get that

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty e^{-|t|}\cos\omega t dt = \frac{1}{\pi}\int_{-\infty}^0 e^t \cos\omega t dt + \frac{1}{\pi}\int_0^\infty e^{-t}\cos\omega t dt =$$

$$= \frac{2}{\pi}\int_{-\infty}^0 e^t \cos\omega t dt$$

since $f(t)$ is even. We solve this by integrating by parts twice. The formula is given by $\int uv' = uv - \int u'v$.

$$\int_{-\infty}^0 e^t \cos\omega t dt$$

Setting $u = \cos\omega t$, $v' = e^t$, $u' = -\omega\sin\omega t$ and $v = e^t$, we get

$$\int e^t \cos\omega t dt = \cos\omega t e^t - \int \omega \sin\omega t e^t dt$$

Now we integrate by parts again for this integral by setting $u = -\omega\sin\omega t$, $v' = e^t$, $u' = -\omega^2\cos\omega t$ and $v' = e^t$. We get

$$\int e^t \cos\omega t dt =$$

$$= \cos\omega t e^t - (-\omega\sin\omega t e^t + \int \omega^2 \cos\omega t e^t) dt =$$

$$= e^t\cos\omega t + \omega e^t \sin\omega t - \omega^2 \int e^t \cos\omega t dt$$

Now we have obtained the integral $\int e^t \cos\omega t dt$ on both sides of the equations. Let us solve for it:

$$\int \cos\omega t dt = e^t\cos\omega t + \omega e^t \sin\omega t - \omega^2 \int e^t \cos\omega t dt$$

$$\implies \int e^t \cos\omega t dt + \omega^2 \int e^t \cos\omega t dt = e^t\cos\omega t + \omega e^t \sin\omega t$$

$$\implies (\omega^2+1)\int e^t \cos\omega t dt = e^t\cos\omega t + \omega e^t \sin\omega t$$

$$\implies \int e^t \cos\omega t dt = \frac{e^t\cos\omega t + \omega e^t \sin\omega t}{\omega^2+1} = \frac{e^t(\omega\sin\omega t + \cos\omega t)}{\omega^2+1}.$$

Remembering our limits, we write $A(\omega)$ as

$$A(\omega) = \frac{2}{\pi}\left[\frac{e^t(\omega\sin\omega t + \cos\omega t)}{\omega^2+1}\right]_{-\infty}^{0} =$$

$$= \frac{2}{\pi}\left[\frac{1}{\omega^2+1} - 0\right] = \frac{2}{\pi(\omega^2+1)}.$$

We know that $B(\omega) = 0$. Then we get the following solution:

$$e^{-|x|} = \frac{2}{\pi}\int_0^\infty \left[\frac{1}{\omega^2+1}\right] \cos\omega x \; d\omega.$$

2.4 The Fourier Transform

Writing the Fourier integral representation in complex form using the exponential function, we derive the Fourier transform and the inverse Fourier transform. These are very helpful when solving boundary value problems, as we will see later in this thesis.

Assume $f(x)$ is a continuous piecewise smooth integrable function. From the Fourier integral representation, we have that

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)(\cos\omega t \cos\omega x + \sin\omega t \sin\omega x)dtd\omega$$

Using that $\cos(a-b) = \cos a \cos b + \sin a \sin b$, we get

$$\frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\omega(x-t)dtd\omega$$

Then we have that $\cos u = 1/2(e^{iu} + e^{-iu})$, which gives

$$\frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(t)(e^{i\omega(x-t} + e^{-i\omega(x-t)})dtd\omega$$

By splitting it into two integrals, we get

$$\frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(t)e^{i\omega(x-t)}dtd\omega + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(t)e^{-i\omega(x-t)}dtd\omega$$

If we know change the sign of omega ($\omega \to -\omega$) and adjust the limits on ω from $-\infty$ to 0, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)e^{i\omega(x-t)}dtd\omega$$

by adding the integrals, which equals

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{i\omega x}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(t)e^{-i\omega t}dtd\omega.$$

This is the complex form of the Fourier integral representation, where the last integral is $\hat{f}(\omega)$. Now we can write the following transform pair:

2.4.1 The Fourier Transform ($\mathcal{F}$)

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \qquad (-\infty < \omega < \infty) \tag{2.3}$$

and

2.4.2 The Inverse Fourier Transform ($\mathcal{F}^{-1}$)

$$\mathcal{F}^{-1}(g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} g(x) d\omega \qquad (-\infty < x < \infty) \tag{2.4}$$

The reason for calling this $\mathcal{F}^{-1}$ is that we get the original function back when taking the inverse Fourier transform of the Fourier transform: $\mathcal{F}^{-1}(\mathcal{F}f) = f(x)$. In other words, we can say that these transform are "opposite" operations.

As before, if $f(x)$ is not continuous at x, the left side of equation (2.4) should be replaced by $\frac{f(x+)+f(x-)}{2}$. Setting $\omega = 0$ in equation (2.3), we get

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) dx$$

Which is just the signed area between the graph of $f(x)$ and the x-axis, multiplied by the constant $\frac{1}{\sqrt{2\pi}}$.

2.4.3 Example 5

a) Find the Fourier transform of the piecewise continuous function

$$f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{if } |x| > a \end{cases}$$

Let us first look at the transform when $\omega \neq 0$. We have

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} d = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-i\omega x} d\omega =$$

$$= \left[\frac{-1}{\sqrt{2\pi}i\omega}e^{-\omega x}\right]_{-a}^{a} = \sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega}.$$

For $\omega = 0$, we have

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} dx = a\sqrt{\frac{2}{\pi}}$$

From L'Hopital's rule, we have that $\frac{\sin a\omega}{\omega} \to a$ as $\omega \to 0$. This is because $\frac{(\sin a\omega)'}{\omega'} = \frac{a\cos a\omega}{1} = 0$ as $\omega \to 0$. Therefore,

$$\lim_{\omega\to 0} \hat{f}(\omega) = \hat{f}(0)$$

so $\hat{f}(\omega)$ is continuous at 0. Then we can write

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega}, \forall\omega$$

.

b) Express $f(x)$ as an inverse Fourier transform.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x}\sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega} = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\omega x}\frac{\sin a\omega}{\omega}d\omega =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}(\cos\omega x + i\sin\omega x)\frac{\sin\omega}{\omega}d\omega =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos\omega x \sin a\omega}{\omega}d\omega$$

since $\sin\omega x\frac{\sin a\omega}{\omega}$ is an odd function of ω, and its integral is zero. ■

2.4.4 Variations in the definition of the Fourier transform

There are several other definitions for the Fourier transform as well. An example is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx, \quad \forall\xi \in \mathbb{R}$$

which often is used if x is given in time and ξ represents the frequency. So if the time is given in seconds, ξ is given in Hertz (cycles per second). The inverse Fourier transform is then given by:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi, \quad \forall x \in \mathbb{R}$$

If t is given in seconds and ξ in angular frequency, it is common to to use the variable $\omega = 2\pi\xi$. Then we get what we call a forward Fourier transform. This is given by the analysis equation

$$\hat{x_1}(\omega) = \hat{x}\left(\frac{\omega}{2\pi}\right) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

and the syntesis equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{x_1}(\omega) e^{it\omega} d\omega$$

If the frequency is given in radians we have a so-called symmetric form. The Fourier integral pair is then given by:

$$\hat{x_2}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

and

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{x_2}(\omega) e^{it\omega} d\omega$$

2.4.5 Units and duality

The Fourier transform goes from one space of functions to a different space of functions, for which often have different domain of definition.
This means that we have to deal with two copies of the real line, one where the original t ranges (for f) and one where the inverse units of t ranges (for $\hat{f}$). In most cases, the former is given in time and the latter as frequency.

These must not be compared or mistakenly identified with each other. The new real line we obtain in the transform is called the ”dual space” of the original one.

Chapter 3

Basic properties of Fourier transform

Assuming $f(x)$, $g(x)$ and $h(x)$ integrable, satisfying the following condition:

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty$$

Now, let $\hat{f}(\omega)$, $\hat{g}(\omega)$ and $\hat{h}(\omega)$ be the Fourier transforms of the given functions respectively. Then we have the following properties for the Fourier transform.

3.1 Linearity

$\forall a, b \in \mathbb{C}$, if

$$h(x) = af(x) + bg(x)$$

then

$$\hat{h}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (af(x) + bg(x))e^{-i\omega x}dx =$$

$$= \frac{1}{\sqrt{2\pi}} \left[a \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx + b \int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx \right] =$$

$$= a\hat{f}(\omega) + b\hat{g}(\omega).$$

Thus, the Fourier transform is linear.

3.2 Translation (time-shifting)

$\forall x_0 \in \mathbb{R}$, if

$$h(x) = f(x - x_0),$$

then

$$\hat{h}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - x_0) e^{-i\omega x} dx.$$

Let $u = x - x_0$, $x = u + x_0$ and $dx = du$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\omega(u+x_0)} du = \frac{1}{\sqrt{2\pi}} e^{-i\omega x_0} \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du =$$

$$= e^{-ix_0\omega} \hat{f}(\omega),$$

so $\hat{h}(\omega) = e^{-ix_0\omega} \hat{f}(\omega)$.

If a function is delayed in time, we get a phase shift of $-\omega x_0$ radians by multiplying by a complex exponential.

3.3 Modulation (frequency-shifting)

If we run time-shifting backwards, we have that $\forall \omega_0 \in \mathbb{R}$, if

$$h(x) = e^{ix\omega_0} f(x)$$

then

$$\hat{h}(\omega) = \hat{f}(\omega - \omega_0).$$

3.4 Time-scaling

$\forall a \in \mathbb{R} \neq 0$, if

$$h(x) = f(ax)$$

then

$$\hat{h}(\omega) = \frac{1}{\mid a \mid}\hat{f}(\frac{\omega}{a}).$$

Indeed, let $t = ax$. Then we have, if $a > 0$,

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{-i\omega x}dx =$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a}\int_{-\infty}^{\infty} f(t)e^{-i\frac{\omega}{a}t}dt\right] =$$
$$= \frac{1}{a}\hat{f}(\frac{\omega}{a}).$$

If $a < 0$, then we get $\hat{h}(\omega) = -\frac{1}{a}\hat{f}(\frac{\omega}{a})$.

The time-scaling can be put into words as follows:
The Fourier Transform is compressed in frequency by the same amount as the original function is expanded in time.

When a=-1, we derive what is called the "time-reversal property":
If

$$h(x) = f(-x)$$

then

$$\hat{h}(\omega) = \hat{f}(-\omega)$$

3.5 Conjugation

If

$$h(x) = \overline{f(x)}$$

then

$$\hat{h}(\omega) = \overline{\hat{f}(-\omega)}.$$

This can be shown as following:

$$\hat{h}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} \overline{f(x)} dx = \overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} f(x) dx}.$$

Now, we write $e^{i\omega x}$ as $e^{-i(-\omega)x}$ and get

$$\overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(-\omega)x} f(x) dx} = \overline{\hat{f}(-\omega)}.$$

When f is real (in $\mathbb{R}$), we have the reality condition:

$$\hat{f}(-\omega) = \overline{\hat{f}(\omega)}$$

Then $\hat{f}$ is called a "Hermitian function" (the conjugate of a complex function equals the original function with the sign of the variable changed).
If f is imaginary, then

$$\hat{f}(-\omega) = -\overline{\hat{f}(\omega)}$$

3.6 Real and imaginary part in time

Let $\Re(z)$ and $\Im(z)$ stand for the real and imaginary part of z, respectively.
If

$$h(x) = \Re(f(x))$$

then

$$\hat{h}(\omega) = \frac{1}{2}(\hat{f}(\omega) + \overline{\hat{f}(-\omega)})$$

If

$$h(x) = \Im(f(x))$$

then

$$\hat{h}(\omega) = \frac{1}{2i}(\hat{f}(\omega) - \overline{\hat{f}(-\omega)})$$

3.7 Integration

Setting $\omega = 0$ in the definition, we get:

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)dx$$

That means that the Fourier transform at the origin is just the integral of f over its domain.

Chapter 4

Differentiation and convolution

4.1 Derivatives of Fourier transform

Let $f(x)$ and $xf(x)$ be integrable functions, then $\hat{f}(\omega)$ is differentiable and the first derivative is given by

$$\hat{f}'(\omega) = -i\mathcal{F}(xf(x)(\omega).$$

Indeed,

$$\begin{aligned}
\hat{f}'(\omega) &= \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\right)' = \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(e^{-i\omega x}\right)' dx = \\
&= -\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xf(x)e^{-i\omega x}dx = \\
&= -i\mathcal{F}(xf(x))(\omega).
\end{aligned}$$

By reading this backwards, we get the formula

$$\mathcal{F}(xf(x))(\omega) = i\hat{f}'(\omega). \tag{4.1}$$

If $x^2 f(x)$ is integrable, we can apply this to $xf(x)$ instead of $f(x)$:

$$\mathcal{F}(x^2 f(x)) = \mathcal{F}(x \times xf(x)) = i(\mathcal{F}(xf(x))' = -\hat{f}(\omega)''.$$

For the general case, if $f(x)$ and $x^n f(x)$ are integrable, the n-th derivative is given by

$$\mathcal{F}(x^n f(x)) = i^n \hat{f}^{(n)}(\omega).$$

These formulas are used when solving ordinary differential equations by transforming them into a a simpler problem involving algebraic equations.

From this we can extract a rule of thumb for the Fourier transform:

f(x) is smooth if and only if $\hat{f}(\omega)$ quickly falls to zero for $\mid \omega \mid \to \infty$.

And conversely,

f(x) quickly falls to 0 for $\mid x \mid \to \infty$ if and only if $f(\hat{\omega})$ is smooth.

4.2 Fourier transform of derivatives

i) If $f(x)$ and $f'(x)$ integrable and $f(x) \to 0$ as $|x| \to \infty$, then

$$\mathcal{F}(f') = i\omega\mathcal{F}(f) = i\omega\hat{f}.$$

We prove this by integrating by parts:

$$\mathcal{F}(f') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-i\omega x} dx$$

We set $u' = f'(x)$, $v = e^{-i\omega x}$, $u = f(x)$ and $v' = -i\omega e^{-i\omega x}$.

$$\implies \frac{1}{\sqrt{2\pi}} \left[f(x) e^{-i\omega x}|_{-\infty}^{\infty} - (-i\omega) \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \right] =$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + i\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \right] =$$

$$= \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx =$$

$$= i\omega \mathcal{F}(f) \quad \blacksquare$$

ii) If, in addition, $f''(x)$ is integrable and $f'(x) \to 0$ as $|x| \to \infty$, then

$$\mathcal{F}(f'') = i\omega \mathcal{F}(f') = -\omega^2 \mathcal{F}(f).$$

iii) In general, if $f^(k)$ are integrable, $f^{(k)} \to 0$ as $|x| \to \infty$ for $k = 0, 1, ..., n-1$, then

$$\mathcal{F}(f^{(n)}) = (i\omega)^n \mathcal{F}(f)$$

The second and third part is derived by applying integration by parts repeatedly, as shown in part one.

4.2.1 Example 1

Given

$$F(x) = \begin{cases} x & \text{if } |x| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

let us find its Fourier transform.

In example 2.4.3 a), we have the same problem, just that x is replaced by 1 and the limit is a constant a. Setting $a = 1$ in the example, we get that

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega}.$$

Now, multiplying this by x and using the formula 4.1, we have that

$$\mathcal{F}(F(x)) = \mathcal{F}(xf(x))(\omega) = i\hat{f}'(\omega)$$

$$\Longrightarrow i\left[\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}\right]'$$

$$\Longrightarrow \hat{F}(\omega) = i\sqrt{\frac{2}{\pi}}\left(\frac{\cos\omega}{\omega} - \frac{\sin\omega}{\omega^2}\right).$$

4.3 Convolution

A convolution is an operation to merge functions and to make discontinuous functions smooth. For example taking the convolution $f(x)$, which can be any function, and the Delta-function, we get the constant given at f(0). This is because the Delta-function is only defined as a spike at x=0. On the other hand, if we take the convolution of $f(x)$ with a rounded, symmetric graph around the Delta-function's spike, we get a smooth function which is an approximation of $f(x)$. This function is now continuous for all values of x and can be described as a Gaussian function.

The convolution of $f * g$ of integrable functions $f(x)$ and $g(x)$ is defined as

$$h(x) = (f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)g(x-y)dy.$$

By substituting $s = (x - y)$, we get

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g((x-y)dy = \int_{-\infty}^{\infty} f(x-s)g(s)ds = (g * f)(x).$$

So $(f * g) = (g * f)$. The convolution theorem states that

$$\hat{h}(\omega) = \hat{f}(\omega)\cdot\hat{g}(\omega).$$

In other words, the Fourier transform of the convolution of two functions is simply the product of the Fourier transform of individual functions.

Proof

$$\mathcal{F}(g * f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-y)e^{-i\omega x} dx f(y) dy =$$

Setting $u = (x - y)$ and $du = dx$

$$\implies \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(u)e^{-i\omega u} du e^{-i\omega y} f(t) dy =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(u)e^{-i\omega u} du \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{-i\omega y} dy =$$

$$= \mathcal{F}(g)(\omega) \cdot \mathcal{F}(f)(\omega) \quad \blacksquare$$

4.4 Example 2

Let $f(x)$ equal 1 if $|x| < 1$ and 0 otherwise. This is called a step function (or box function). The Fourier transform of the convolution of $f(x)$ with itself, is given by

$$\mathcal{F}(f * f) = \hat{f}(\omega)^2 = \frac{2}{\pi} \frac{\sin^2 \omega}{\omega^2}$$

This can be shown both by using the definition of a convolution and by using tables for the inverse Fourier transform.

4.4.1 Solving by using the definition of a convolution

We have that

$$(f * f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y-x) f(y) dy.$$

Since f is 1 for $x \in [-1, 1]$ and 0 otherwise, we can set $f(y) = 1$ in the integral. We also know that $|y - x| < 1$. Then we have

$$\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} f(x-y)dy = \int_{x-1}^{x+1}$$

which can be divided into two separate integrals:

$$\int_{-1}^{x+1} dy = [y]_{-1}^{x+1} = x + 1 - (-1) = x + 2.$$

and

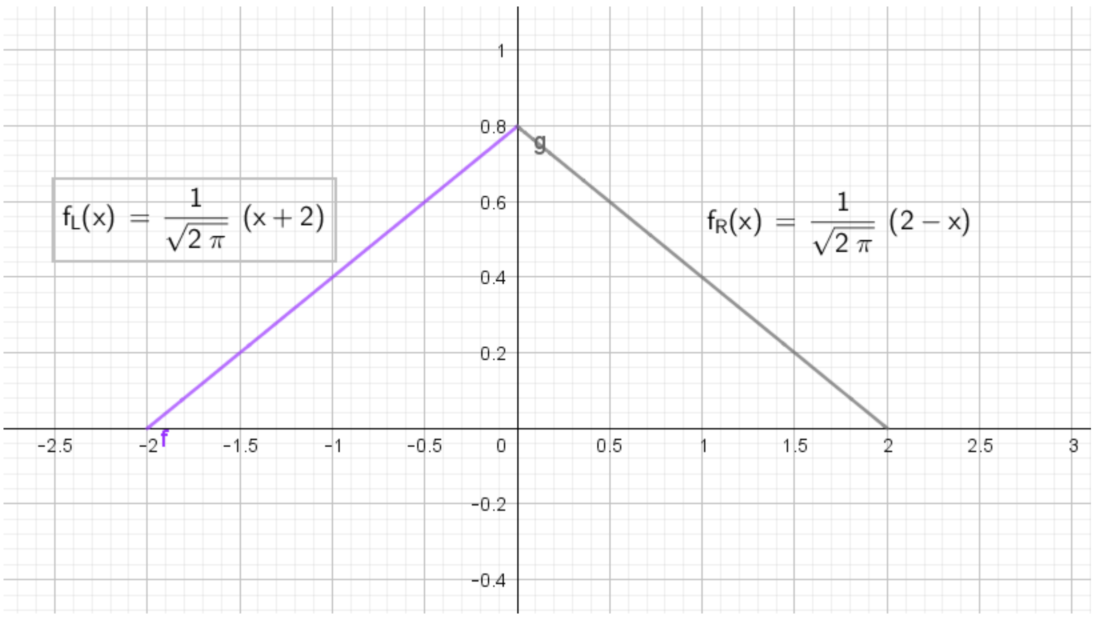
$$\int_{x-1}^{1} dy = [y]_{x-1}^{1} = 1 - (x - 1) = 2 - x.$$

Multiplying both by the factor $\frac{1}{\sqrt{2\pi}}$, we get the following functions:

$$f_{left} = \frac{1}{\sqrt{2\pi}}(x + 2)$$

$$f_{right} = \frac{1}{\sqrt{2\pi}}(2 - x)$$

which results in the following graph:



4.4.2 Solving using a table

The definition of a convolution states that:

$$(f * f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)f(y)dy$$

Taking the Fourier transform of this, we obtain

$$\mathcal{F}(f * f) = \mathcal{F}(f) \cdot \mathcal{F}(f) = \hat{f}(\omega)^2$$

Now, let us calculate $\hat{f}(\omega)$ and deal with the square later:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 \cdot e^{-i\omega x} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega x}}{i\omega}\right]_{-1}^{1} = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega} - e^{i\omega}}{i\omega}\right]$$

Multiplying by a factor of -2, we see that we obtain the complex exponential form of sine:

$$-\frac{2}{\sqrt{2\pi}} \left[\frac{e^{i\omega} - e^{-i\omega}}{2i\omega}\right] = -\frac{2}{\sqrt{2\pi}} \frac{\sin\omega}{\omega}.$$

Now we must remember to square the whole expression:

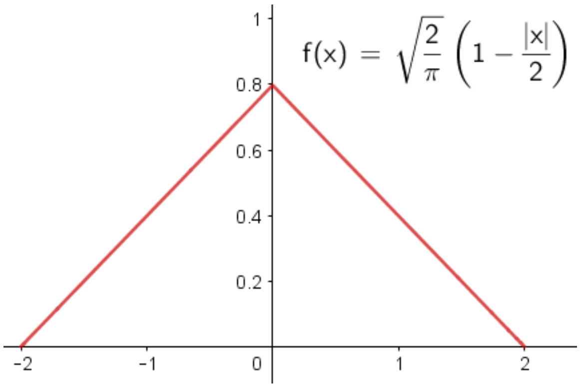
$$\left(-\frac{2}{\sqrt{2\pi}} \frac{\sin\omega}{\omega}\right)^2 = \frac{4}{2\pi} \frac{\sin\omega^2}{\omega^2} =$$

$$= \frac{2}{\pi} \frac{\sin^2\omega}{\omega^2}.$$

By using the inverse transform and given tables, we get

$$f * f(x) = \mathcal{F}^{-1}\left(\frac{2}{\pi} \frac{\sin^2\omega}{\omega^2}\right) = \begin{cases} \sqrt{\frac{2}{\pi}}(1 - \frac{|x|}{2}) & \text{if } |x| < 2 \\ 0 & \text{if } |x| \geq 2 \end{cases}$$

This gives us the following graph:



which shows us that $f * f$ is continuous even though f is not.

Chapter 5

The Gaussian function

The function

$$f(x) = e^{-x^2}$$

is called the Gaussian function and is used to solve the heat equation in one dimension. Its integral over the whole real line given by

$$I = \int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$$

which is a very well-known improper integral.

5.1 Proof

To prove this fact, we use polar coordinates and square the integral. Now $r^2 = x^2 + y^2$ and $dxdy = rdrd\theta$.

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy =$$

$$= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} rdrd\theta =$$

$$= \int_0^{2\pi} -\frac{1}{2}e^{-r^2}|_0^\infty d\theta =$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta = \pi$$

Then we get that

$$I^2 = \pi \implies I = \sqrt{\pi}$$

by taking the square root. We reject the negative squareroot since $I > 0$. ■

5.2 Fourier transform of the Gaussian

For any real contant $a > 0$:

$$\mathcal{F}(e^{-\frac{ax^2}{2}})(\omega) = \frac{1}{\sqrt{a}}e^{-\frac{\omega^2}{2a}}$$

5.2.1 Proof

Let $f(x) = e^{-\frac{ax^2}{2}}$. It can easily be shown that this function satisfies the following first order linear differential equation:

$$f'(x) + axf(x) = 0$$

By taking the Fourier transform of the different parts of the equation, we get by using the formulas from chapter 3,

$$i\omega\hat{f}(\omega) + ai\hat{f}'(\omega) = 0$$

Solving this equation for $\hat{f}$, we start by getting the derivative alone on the left hand side:

$$i\omega\hat{f}'(\omega) = -i\omega\hat{f}(\omega)$$

$$\implies \hat{f}'(\omega) = -\frac{\omega}{a}\hat{f}(\omega)$$

$$\implies \int \frac{1}{\hat{f}(\omega)} d\omega = \int -\frac{\omega}{a} d\omega$$

$$\implies ln|\hat{f}(\omega)| = -\frac{\omega^2}{2a}$$

$$\implies \hat{f}(\omega) = \pm e^{-\frac{\omega^2}{2a}} = Ae^{-\frac{\omega^2}{2a}}.$$

for A being a constant. Now we must derive that $A = \frac{1}{\sqrt{a}}$:

$$A = \hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} dx$$

Setting $u = \sqrt{\frac{a}{2}}x$ and $dx = \sqrt{\frac{2}{a}}du$:

$$\implies \frac{1}{\sqrt{a\pi}} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{1}{\sqrt{a\pi}}\sqrt{\pi} = \frac{1}{\sqrt{a}} \quad \blacksquare$$

by the famous improper integral (I) stated in the beginning of this chapter.

By replacing a with $2a$, we get

$$\mathcal{F}(e^{-ax^2})(\omega) = \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}}$$

which implies that $e^{-\frac{x^2}{2}}$ is its own Fourier transform.

5.3 Examples

Here we use the operational properties and a known Fourier transform to compute the Fourier transform of the given functions.

5.3.1 Example 1

Given

$$f(x) = xe^{-x^2},$$

we know from section 4.1 that

$$\mathcal{F}(xe^{-x^2}) = i[\hat{f}]'(e^{-x^2})(\omega).$$

Then we have to calculate the following:

$$i\left(\hat{f}(e^{-x^2})\right)'$$

In section 5.2 we have already calculated this transform, so we have to find the following derivative:

$$i\left[\frac{1}{\sqrt{a}}e^{-\frac{\omega^2}{2a}}\right]'$$

We see that $a = 2$ in this case, which gives us

$$i\left[\frac{1}{\sqrt{2}}e^{-\frac{\omega^2}{4}}\right]' = i\left(\frac{1}{\sqrt{2}}(-\frac{2}{4}\omega)e^{-\frac{\omega^2}{4}}\right) =$$

$$= -\frac{i\omega e^{-\frac{\omega^2}{4}}}{2\sqrt{2}}.$$

5.3.2 Example 2

Let $f(x) = xe^{-\frac{x^2}{2}}$ and $g(x) = e^{-x^2}$.

a) What are the Fourier transforms of f and g?

Fourier transform of $f(x)$

For $f(x)$, we have the same as in Example 1, but now with a=1. That gives us that

$$\mathcal{F}(f(x))(\omega) = i\left[\frac{1}{\sqrt{a}}e^{-\frac{\omega^2}{2a}}\right]' =$$

$$= i\left[e^{-\frac{\omega^2}{2}}\right]' = i\left(-\frac{1}{2}2\omega\right)e^{-\frac{\omega^2}{2}} =$$

$$= -i\omega e^{-\frac{\omega^2}{2}}$$

Fourier transform of $g(x)$

In Example 1 we have already found that

$$\mathcal{F}(g(x))(\omega) = \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}}.$$

b) What is the Fourier transform of $f * g$?

From section 4.3 we have that the Fourier transform of a convolution $f * g$ is given as the product of the Fourier transform of each function. Then we have that

$$\mathcal{F}(f * g)(\omega) = \hat{f} \cdot \hat{g} = \left(-i\omega e^{-\frac{\omega^2}{2}}\right) \cdot \left(\frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}}\right) =$$

$$= -\frac{i\omega}{\sqrt{2}} \left(e^{-\frac{\omega^2}{2} - \frac{\omega^2}{4}}\right) = -\frac{i\omega}{\sqrt{2}} e^{-\frac{3\omega^2}{4}}.$$

c) What is $f * g$? From section 5.2, we have that, $\forall a > 0$,

$$\mathcal{F}(e^{-\frac{ax^2}{2}})(\omega) = \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{2a}}.$$

In our case, $a = 2$, therefore $f * g$ is given by

$$\frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}}$$

multiplied by a constant $i\omega$. Then we get that

$$f * g = i\omega \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}}.$$

Chapter 6

Invertibility and periodicity

Let $\mathcal{F}$ be the Fourier transform operator, such that $\mathcal{F}(f(x)) = \hat{f}(x)$. Then, under suitable conditions, we can obtain the function f itself from the transform $\hat{f}$. The reason why this is so helpful in solving boundary value problems and differential equations, is that the solving process often is much more easy for the transformed problem than for the original one. Then we get a transformed solution, which we can transform back to the original solution by using the inverse Fourier transform. This idea can be shown as following table:

$$
\begin{tikzcd}
\text{(problem)} \arrow[r, "\text{Fourier transform}"] \arrow[d, dashed, "\text{solve}"', "\text{(hard)}"] & \text{(transformed problem)} \arrow[d, "\text{solve}"', "\text{(easy)}"] \\
\text{(solution)} & \text{(transformed solution)} \arrow[l, "\text{inverse Fourier transform}"']
\end{tikzcd}
$$

As we know, we obtain the transform $\hat{f}$ by applying the Fourier transform method once:

$$\mathcal{F}(f(x)) = \hat{f}(x)$$

If we do the Fourier transform twice, the function is "flipped":

$$\mathcal{F}^2(f(x)) = f(-x)$$

This can be seen as reversing time, which is a two-periodic operation. That means if we do this twice, we will recover the function itself.

$$\mathcal{F}^4(f(x)) = f(x)$$

Hence, the Fourier transform is said to be four-periodic. In practise this can be explained as rotating the plane by 90° for each iteration.
The inverse Fourier transform can also be derived by applying the transform three times.

$$\mathcal{F}^3(\hat{f}(x)) = f(x)$$

6.0.1 Example 1

Now, let us compute $\mathcal{F}(\frac{\sin x}{x})$ using one of the above properties.

From example 2.4.3 a), we have that

$$\mathcal{F}(f)(x) = \sqrt{\frac{2}{\pi}}\frac{\sin x}{x}$$

Then

$$\mathcal{F}(\sqrt{\frac{2}{\pi}}\frac{\sin x}{x}) = \mathcal{F}^2(f) = f(-x)$$

So, by applying the Fourier transform twice, we get the box-function back (since it is symmetric).

This is an appropriate time to state the proof for the inverse Fourier transform.

6.1 Proof of the inverse Fourier transform

6.1.1 Step 1

We have that, for any g and f integrable,

$$\int_{-\infty}^{\infty} \hat{f}(\omega)g(\omega)d\omega = \int_{-\infty}^{\infty} f(y)\hat{g}(y)dy$$

This can be shown using Fubini's theorem, which states that the integration order does not matter:

$$\int\int e^{-i\omega y} f(y)dy \cdot g(\omega)d\omega = \int\int e^{-i\omega y} g(\omega)d\omega \cdot f(y)dy.$$

6.1.2 Step 2

$$\int_{-\infty}^{\infty} e^{i\omega x}\hat{f}(\omega)d\omega = \lim_{a\to 0}\int_{-\infty}^{\infty} e^{-\frac{a}{2}\omega^2+i\omega x}\hat{f}(\omega)d\omega.$$

Let us write $e^{-\frac{a}{2}\omega^2+i\omega x}$ as $g_{a,x}(\omega)$. From Step 1, we can write

$$\lim_{a\to\infty}\int_{-\infty}^{\infty} \mathcal{F}[g_{a,x}](y)f(y)dy.$$

6.1.3 Step 3

By the modulation-property, we can write

$$\mathcal{F}[g_{a,x}](y)$$

as

$$\mathcal{F}[g_{a,0}](y-x)$$

where $g_{a,0}(\omega) = e^{-\frac{a^2}{2}\omega^2} = e^{-\frac{(a\omega)^2}{2}}$.

By time-scaling,

$$\mathcal{F}[g_{a,0}](t) = \frac{1}{|a|}\mathcal{F}[g_{1,0}](\frac{t}{a}) = \frac{1}{|a|}g_{1,0}(\frac{t}{a})$$

since $\mathcal{F}[e^{-\frac{\omega^2}{2}}] = e^{-\frac{t^2}{2}}$.

Therefore,

$$\mathcal{F}[g_{a,x}](y) = \frac{1}{|a|}e^{-\frac{(y-x)^2}{2a^2}}.$$

6.1.4 Step 4

By step 2 and step 3,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_{a,x}(\omega)\hat{f}(\omega)d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{|a|}e^{-\frac{(y-x)^2}{2a^2}}f(y)dy.$$

Now, let $\phi_a(y-x) = e^{-\frac{(y-x)^2}{2a^2}}$. Then we can write the integral above as the convolution

$$(\phi_a * f)(x)$$

It is known that the functions $\phi_a(t) \to_{a\to 0} \delta_0(t)$. This implies that

$$\int \phi_a(t)f(t)dt \to f(0)$$

$$\implies (\phi_a * f)(x) \to_{a\to 0} f(x).$$

Hence, we have shown that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x}\hat{f}(\omega)d\omega = f(x).$$

Chapter 7

The Fourier transform method

The Fourier transform method is used on differential equations on infinite $(-\infty, \infty)$ and semi-infinite (x_0, ∞) regions. In the latter, x_0 is often set to the origin (0). This method is mostly used on the wave- and heat equation in one dimension.

7.1 Partial derivatives

Let $u(x,t)$ be a function of two variables. Assume $-\infty < x < \infty$ and $t > 0$. Now we have to take the Fourier transform with respect to one variable, for example x. Then we get $\hat{u}(\omega, t)$:

$$1) \qquad \mathcal{F}(u(x,t))(\omega) = \hat{u}(\omega,t) = \frac{1}{\sqrt{2}\pi}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx$$

For partial derivatives, we have the following transforms:

$$2) \qquad \mathcal{F}(\frac{\partial}{\partial t}u(x,t))(\omega) = \frac{d}{dt}\hat{u}(\omega,t)$$

$$3) \qquad \mathcal{F}(\frac{\partial^n}{\partial t^n}u(x,t)(\omega) = \frac{d^n}{dt^n}\hat{u}(\omega,t)$$

where $n \in \mathbb{N}$ (n=1, 2, ..., k).

$$4) \qquad \mathcal{F}(\frac{\partial}{\partial x}u(x,t))(\omega) = i\omega\hat{u}(\omega,t)$$

$$5) \qquad \mathcal{F}(\frac{\partial^n}{\partial x^n}u(x,t))(\omega) = (i\omega)^n\hat{u}(\omega,t)$$

where $n \in \mathbb{N}$. The property used for the derivatives is found in chapter 4.2.

7.1.1 Example: Wave equation

Find the solution $u(x,t)$ given the second order differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{d^2 u}{dx^2}$$

with boundary conditions

$$u(x,0) = f(x)$$

as the initial displacement and

$$u(0,t) = g(x)$$

as the initial velocity. Here $f(x)$ and $g(x)$ are integrable on $(-\infty,\infty)$.

Solution

Start by taking the Fourier transform on both sides of the equation and its boundary conditions. Using equation 3) and 5), we get

$$\frac{d^2}{dt^2}\hat{u}(\omega,t) = -c^2\omega^2\hat{u}(\omega,t)$$

$$\hat{u}(\omega,0) = \hat{f}(\omega)$$

$$\frac{d}{dt}\hat{u}(\omega,0) = \hat{g}(\omega)$$

This implies the equation

$$\frac{d^2}{dt^2}\hat{u}(\omega,t) + c^2\omega^2\hat{u}(\omega,t) = 0$$

which has the general solution

$$\hat{u}(\omega,t) = A(\omega)\cos c\omega t + B(\omega)\sin c\omega t$$

where $A(\omega)$ and $B(\omega)$ are constant in time (t). For the boundary conditions, we have

$$\hat{u}(\omega,0) = A(\omega) = \hat{f}(\omega)$$

$$\frac{d}{dt}\hat{u}(\omega,0) = c\omega B(\omega) = \hat{g}(\omega) \implies B(\omega) = \frac{1}{c\omega}\hat{g}(\omega)$$

Then the solution is given by

$$\hat{u}(\omega,t) = \hat{f}(\omega)\cos c\omega t + \frac{1}{c\omega}\hat{g}(\omega)\sin c\omega t$$

Remember that this is the transformed solution. By applying the inverse transform method, we derive the original solution:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\hat{f}(\omega)\cos c\omega t + \frac{1}{c\omega}\hat{g}(\omega)\sin c\omega t]e^{i\omega x}d\omega \qquad \blacksquare$$

For some cases, the integral can be computed explicitly.

7.2 Recipe for the Fourier transform method

Given a boundary value problem, we go via the Fourier transform to make the computations easier. This is done by the following steps:

1) Take the Fourier transform of the given BVP.

2) Solve the derived differential equation and find its solution, $\hat{u}(\omega,t)$.

3) Get the original solution, $u(x,t)$, back by applying the inverse transform method.

Let us try the same recipe on a heat-equation-problem.

7.2.1 Example: Heat equation

Given

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with boundary condition

$$u(x,0) = f(x)$$

find $u(x,t)$.

Solution

By taking the Fourier transform, we get

$$\frac{d}{dt}\hat{u}(\omega,t) = -c^2\omega^2\hat{u}(\omega,t)$$

$$\hat{u}(\omega,0) = \hat{f}(\omega)$$

$$\implies \hat{u}(\omega,t) = A(\omega)e^{-c^2\omega^2 t}$$

$$\hat{u}(\omega,0) = A(\omega) = \hat{f}(\omega)$$

This gives the transformed solution

$$\hat{u}(\omega,t) = \hat{f}(\omega)e^{-c^2\omega^2 t}$$

By applying the inverse transform, we derive the solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-c^2\omega^2 t}e^{i\omega x}d\omega \qquad \blacksquare$$

7.3 Examples

Determine the solution of the given wave or heat problems.

7.3.1 Example 1

Given the following boundary value with wave-equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = \frac{1}{1+x^2}, \quad \frac{\partial u}{\partial t}(x,0) = 0$$

Fourier transforming the problem, we get

$$\frac{d^2}{dt^2}\hat{u}(\omega,t) = -\omega^2 \hat{u}(\omega,t)$$

$$\hat{u}(\omega,0) = \mathcal{F}(\frac{1}{1+x^2}) = \sqrt{\frac{\pi}{2}}e^{-|w|}$$

$$\frac{d}{dt}\hat{u}(\omega,0) = 0.$$

Now we write the ordinary differential equation in the standard form

$$\frac{d^2}{dt^2}\hat{\omega, t} + \omega\hat{u}(\omega,t) = 0$$

The general solution is then given by

$$\hat{u}(\omega,t) = A(\omega)\cos\omega t + B(\omega)\sin\omega t$$

From the initial conditions, we determine $A(\omega)$ and $B(\omega)$:

$$\hat{u}(\omega,0) = A(\omega) = \sqrt{\frac{\pi}{2}}e^{-|w|}, \quad \frac{d}{dt}\hat{u}(\omega,0) = \omega B(\omega) = 0.$$

That gives us the following transformed solution

$$\hat{u}(x,t) = \sqrt{\frac{\pi}{2}}\cos\omega t$$

Taking the inverse transform of this, we obtain our solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\sqrt{\frac{\pi}{2}}e^{-|w|}\cos\omega t\right]e^{i\omega x}d\omega.$$

7.3.2 Example 2

Given the boundary value problem with heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}$$

with initial temperature distribution

$$u(x,0) = e^{-x^2}$$

First, we take the Fourier transform of both equations:

$$\implies \frac{d}{dt}\hat{u}(\omega,t) = -\frac{1}{4}\omega^2\hat{u}(\omega,t)$$

$$\hat{u}(\omega,0) = \sqrt{\pi}$$

since the initial condition is given by the Gaussian function. Then we have the general transformed solution

$$\hat{u}(\omega,t) = \sqrt{\pi}e^{-\frac{1}{4}\omega^2 t}$$

Taking the inverse transform of this, we get our solution

$$u(x,t) = \frac{\sqrt{\pi}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{4}\omega^2 t}e^{i\omega x}d\omega =$$

$$= \frac{1}{\sqrt{2}}\mathcal{F}(e^{-\frac{1}{4}\omega^2 t}) = \frac{1}{\sqrt{2}}\left[\frac{\sqrt{2}e^{-}t}{\sqrt{t}}\right]$$

$$= \frac{2e^{-t}}{\sqrt{t}}.$$

7.3.3 Example 3

Given boundary value problem with wave-equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

with initial displacement

$$u(x,0) = \sqrt{\frac{2}{\pi}}\frac{\sin x}{x}$$

and initial velocity

$$\frac{du}{dt}(x,0) = 0.$$

Taking the Fourier transform of this, we get

$$\frac{d^2}{dt^2}(\hat{u}(\omega,t) = -c^2\omega^2\hat{u}(\omega,t)$$

$$\hat{u}(\omega,0) = \mathcal{F}(\sqrt{\frac{2}{\pi}}\frac{\sin x}{x}) =$$

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

which is the box-function derived in example 6.0.1.

$$\frac{d}{dt}\hat{u}(\omega,0) = 0.$$

Now we reorder the first equation to obtain the standard form of an ordinary differential equation with 0 on the right hand side:

$$\frac{d^2}{dt^2}(\hat{u}(\omega,t) + c^2\omega^2\hat{u}(\omega,t) = 0$$

with general solution

$$\hat{u}(\omega,t) = A(\omega)\cos c\omega t + B(\omega)\sin c\omega t$$

Now, we have that

$$\hat{u}(\omega,0) = A(\omega) = f(x)$$

and

$$\frac{d}{dt}\hat{u}(\omega,0) = 0$$

Then the transformed solution is given by

$$\hat{u}(\omega, t) = f(x) \cos c\omega t$$

Taking the inverse transform of this, we obtain the solution

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x) \cos c\omega t] e^{i\omega x} d\omega.$$

Chapter 8

Other solving strategies

8.1 Gauss's kernel

So far we have expressed the solution as an inverse transform after using the Fourier transform method of some given initial data. This is a tedious process, so in practise we want to find the inverse transform directly and give the solution as the initial data itself.

8.1.1 Solving the heat equation as a convolution

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{{}^2}$$

where $-\infty < x < \infty$ and $t > 0$, with initial condition

$$u(x, 0) = f(x)$$

We know from the previous example that

$$\hat{u}(\omega, t) = \hat{f}(\omega) e^{-c^2 \omega^2 t}$$

By applying the inverse transform to this, we get the solution

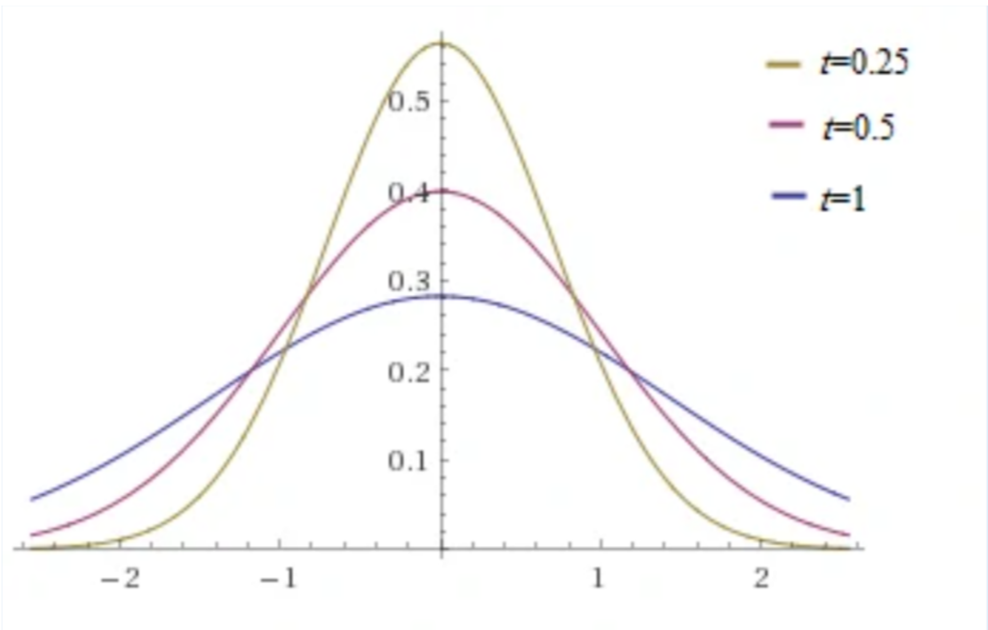
$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-c^2\omega^2 t} e^{i\omega x} d\omega$$

Now, remember our goal expressed in the beginning of this chapter, namely to find the inverse Fourier transform in terms of f. We see that the inverse solution $\hat{u}(\omega, t)$ is given as the product of two Fourier transforms: $\hat{f}(\omega)$ and $e^{-c^2\omega^2 t}$. Therefore, we can say that u is the convolution of f with the function that has $e^{-c^2\omega^2 t}$ as its Fourier transform.

This function is known as the heat kernel, or Gauss's kernel. It is given by the following equation:

$$g_t(x) = \frac{1}{c\sqrt{2t}} e^{-x^2/4c^2 t}$$

Plotting this results in different Gaussian curves, where the area under each curve is constant.



Solution

The solution of the heat equation given as a convolution is given by

$$u(x,t) = \frac{1}{c\sqrt{2t}} e^{-x^2/4c^2t} * f =$$

$$= \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(s) e^{-(x-s)^2/4c^2t} ds$$

8.1.2 Proof

One must show that $e^{-c^2\omega^2 t}$ is the Fourier transform of $g_t(x)$. Using the Fourier transform of the Gaussian from chapter 5, we immediately get

$$\hat{g}(\omega) = \frac{1}{c\sqrt{2t}} \sqrt{2c^2t} e^{-\omega^2c^2t} = e^{-\omega^2c^2t} \qquad \blacksquare$$

by setting $\frac{a}{2} = \frac{1}{4c^2t}$.

8.2 Example 1

Use convolutions, the error function, and other operational properties of the Fourier transform to solve the following boundary problem. The heat equation is given by

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}$$

with initial condition

$$u(x,0) = \begin{cases} 20, & -1 < x < 1 \\ 0, & otherwise \end{cases}$$

Applying the solution for the Gauss's kernel, we have that

$$u(x,t) = \frac{20}{\sqrt{\pi t}} \int_{-1}^{1} e^{\frac{-(x-s)^2}{t}} ds$$

$$\Longrightarrow \frac{20}{\sqrt{t}} \int_{\frac{x+1}{\sqrt{t}}}^{\frac{x-1}{\sqrt{t}}} e^{-z^2} dz = \frac{20}{\sqrt{t}} \left(\int_0^{\frac{x+1}{\sqrt{t}}} e^{-z^2} dz - \int_0^{\frac{x-1}{\sqrt{t}}} e^{-z^2} dz \right) =$$

$$= 20 \left[erf(\frac{x+1}{\sqrt{t}}) - erf(\frac{x-1}{\sqrt{t}}) \right],$$

where

$$erf(\omega) = \frac{2}{\sqrt{\pi}} \int_0^{\omega} e^{-z^2} dz, \forall \omega.$$

8.2.1 Remark

The so-called "error-function" is used to simplify the expressions in the solutions of boundary value problems. The function is given as the integral of a Gaussian function and its values is given in tables (N. Asmar (2000), p. 355).

Chapter 9

The cosine and sine transform

Consider a Dirichlet problem in the first quadrant. Then the boundary is given by the x- and y-axis. Let us now introduce the Fourier cosine and sine transforms. How can we use the Fourier transform when $f(x)$ is only defined for $x > 0$?

From the idea of half-range expansions in Fourier series, we have a clue that the extension of f will be non-periodic (even and odd). An even function is symmetric with respect to the y-axis, while an odd function is symmetric with respect to the origin.

9.1 Cosine and sine integral representation

If f is even on the whole $\mathbb{R}$, we get

$$f(x) = \int_0^\infty A(\omega) cos\omega x d\omega$$

where $x > 0$, when applying the Fourier integral representation. $A(\omega)$ is given by

$$\frac{2}{\pi}\int_0^\infty f(t)\cos\omega t dt$$

This is called a Fourier cosine integral representation.

For an odd funciton, we get the Fourier sine integral representation of f:

$$f(x) = \int_0^\infty B(\omega)\sin\omega x d\omega$$

where $x > 0$.

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t dt$$

9.2 Cosine and sine transforms

The Fourier cosine transform of f is given by

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\omega t dt$$

where $\omega \geq 0$. By taking the inverse transform, we get

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\omega)\cos\omega x d\omega$$

for $x > 0$.

For the Fourier sine transform, we get

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\omega t dt$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\omega)\sin\omega x d\omega$$

9.2.1 Remarks

$\hat{f}_c$ and $\hat{f}_s$ can sometimes be written as $\mathcal{F}_c(f)$ and $\mathcal{F}_s(f)$. As before, $f(x)$ is replaced by $\frac{f(x+)+f(x-)}{2}$ at points of discontinuity.

If $f(x)$ $(x \geq 0)$ is restriction of an even function, f_e, then

$$\hat{f}_c(\omega) = \mathcal{F}(f_e)(\omega) \quad \forall \omega \geq 0$$

If $f(x)$ $(x \geq 0)$ is restriction of an odd function, f_o, then

$$\hat{f}_s(\omega) = i\mathcal{F}(f_o)(\omega) \quad \forall \omega \geq 0$$

9.2.2 Example 1

$$f(x) = e^{-ax}, \quad a > 0, x > 0$$

Let us find the Fourier cosine transform of f(x).

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at} \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \omega^2} [e^{-at}(\frac{\omega}{a} \sin \omega t - \cos \omega t)]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \omega^2}$$

The values in the square bracket are found in a table.

Now we calculate the inverse Fourier transform of $f(x)$.

$$\mathcal{F}(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega x d\omega = \frac{2}{\pi} \int_0^\infty \frac{a \cos \omega x}{a^2 + \omega^2} d\omega.$$

9.2.3 Example 2

Given

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < b \\ 0 & \text{otherwise} \end{cases}$$

we want to find its Fourier sine transform. Using the given formula for $\hat{f}_s(\omega)$ we get

$$\sqrt{\frac{2}{\pi}}\int_0^b \sin\omega t dt = \sqrt{\frac{2}{\pi}}\left[-\frac{\cos\omega t}{\omega}\right]_0^b =$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{\omega} - \frac{\cos b\omega}{\omega}\right].$$

Hence,

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos b\omega}{\omega}\right).$$

9.3 Operational properties

The properties for the Fourier cosine and sine transforms are closely related to the ones for the original transform.

9.3.1 Linearity

Let f, g be functions and a, b numbers.

$$\mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$$

and

$$\mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g)$$

9.3.2 Transforms of derivatives

Suppose $f(x) \to 0$ as $x \to \infty$. Then

$$\mathcal{F}_c(f') = \omega\mathcal{F}_s(f) - \sqrt{\frac{2}{\pi}}f(0)$$

$$\mathcal{F}_s(f'') = -\omega\mathcal{F}_c(f)$$

If also $f'(x) \to 0$ as $x \to \infty$, then

$$\mathcal{F}_c(f'') = -\omega^2 \mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}} f'(0)$$

$$\mathcal{F}_s(f'') = -\omega^2 \mathcal{F}_s(f) + \sqrt{\frac{2}{\pi}} \omega f(0)$$

9.3.3 Derivatives of transforms

$$\mathcal{F}_c(xf(x)) = \frac{d}{d\omega} \mathcal{F}_s(f(x))$$
$$\mathcal{F}_s(xf(x)) = -\frac{d}{d\omega} \mathcal{F}_c(f(x))$$

Fourier sine and cosine transforms are used for solving boundary value problems on $(0, \infty$, so-called "semi-infinite intervals". '

9.3.4 Example 3

$$f(x) = \begin{cases} T_0 & \text{if } 0 < x < b \\ 0 & \text{otherwise} \end{cases}$$

Now we solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x, 0) = f(x)$$

$$u(0, t) = 0$$

Given that

$$\mathcal{F}_s(\frac{\partial u}{\partial t}) = \frac{d}{dt} \hat{u}$$

and

$$\mathcal{F}_s(\frac{\partial^2 u}{dx^2}) = -\omega^2 \hat{u}_s(\omega, t) + \sqrt{\frac{2}{\pi}} \omega, u(0, t) = -\omega^2 \hat{u}_s(\omega, t)$$

we get the following transforms:

$$\frac{d}{dt}\hat{u_s}(\omega,t) = -\omega^2\hat{u_s}(\omega,t)$$
$$\hat{u_s}(\omega,0) = \hat{f_s}(\omega) = T_0\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos b\omega}{\omega}\right)$$

from example 9.2.3, where T_0 is a constant.

The general transformed solution is then given by

$$\hat{u_s}(\omega,t) = A(\omega)e^{-\omega^2 t}$$

with

$$\hat{u_s}(\omega,0) = A(\omega) = \hat{f_s}(\omega) = T_0\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos b\omega}{\omega}\right)$$
$$\implies \hat{u_s}(\omega,t) = T_0\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos b\omega}{\omega}\right)e^{-\omega^2 t}.$$

By taking the inverse transform, we obtain our original solution

$$u(x,t) = T_0\sqrt{\frac{2}{\pi}}\int_0^b(\frac{1-\cos b\omega}{\omega})e^{-\omega^2 t}\sin\omega x d\omega.$$

Chapter 10

$\mathcal{F}$ on generalized functions

The real life often involve impulsive phenomenons, which can be described as applying a strong force in a short time interval. An example would be to kick a fotball or chopping wood. The strengt of this force can be written as

$$\int_a^b F(t)dt,$$

where $F(t)$ is the impulse and $[a, b]$ is the time interval in which the impulse is active. $k \in \mathbb{N}$.

Given

$$F_k(t) = \begin{cases} k, & \text{if } -\frac{1}{2k} \leq t \leq \frac{1}{2k} \\ 0, & \text{if } \qquad |t| \geq \frac{1}{2k}. \end{cases}$$

Then, if $f(x)$ continuous on the interval $[a, b]$,

$$\lim_{h \to \infty} \int_a^b f(x) F_k(x) dx = \begin{cases} f(0), & \text{if } a \leq 0 \leq b \\ 0, & \text{if } 0 \text{ is not in } [a, b]. \end{cases}$$

10.1 The Dirac Delta function

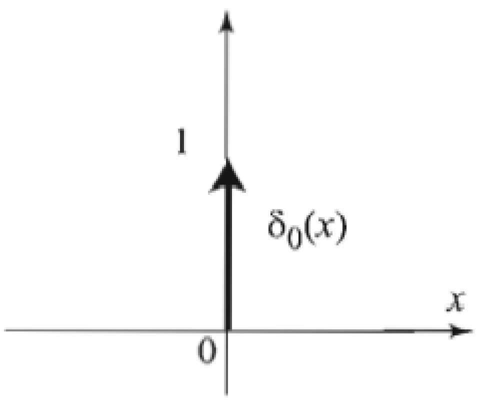
The δ-function is a generalized function, or a distribution, defined as the limit of the F_k's such that $\forall$ continuous functions $f \in [a, b]$,

$$\int_a^b f(x)\delta_0(x)dx = \lim_{x\to\infty} \int_a^b f(x)F_k(x)dx.$$

It follows that

$$\int_{-\infty}^{\infty} f(x)\delta_0(x)dx = \begin{cases} f(0), & \text{if } a \le 0 \le b \\ 0, & \text{if } 0 \text{ is not in } [a, b]. \end{cases}$$

We can not define $\delta_0(x)$ like a normal function, namely as its values at each point x. Nevertheless, it may be helpful to think of the Delta-function as a function of x. Then $\delta_0(x) = 0 \ \forall x \neq 0$ and $\delta_0(0) = \infty$ (a spike or impulse at $x = 0$), as shown in the figure below.



Such distributions are often defined as "the values of the intervals against other functions". For a generalized function φ, it is sufficient to to define how it acts on test functions, which symbolically can be written as

$$< \varphi, f >= \int_{-\infty}^{\infty} f(x)\varphi(x)dx,$$

where f is a test function. Test functions have derivatives of all order and goes to zero rapidly at $\pm\infty$ or equals zero outside a given bounded interval.

Example 1

Consider the Heaviside function.

Remember the property if f is differentiable, then

$$\mathcal{F}(f')(\omega) = i\omega\mathcal{F}(f)(\omega).$$

Now, since $\varphi'(x) = 0\ \forall x \neq \pm 0$ and $\mathcal{F}(0) = 0$, we get that $\mathcal{F}(\varphi')(\omega) = 0 \forall \omega$. Therefore, the above property does not hold for φ. If we consider it as a usual function and not as distribution, a generalized function will help us to apply the operational properties to step functions and piecewise linear functions. Later we will compute the derivative of the Heaviside function considered as a generalized function (Section 10.3, Example 2).

10.2 Translation

We say that φ and ψ, both generalized functions, are equal if

$$< \varphi, f >=< \psi, f >,$$

$\forall$ test functions f.

Given $\alpha \in \mathbb{R}$, the translate of δ_0 is given by

$$< \delta_\alpha, f >= \int_{-\infty}^{\infty} f(x)\delta_\alpha(x)dx = f(\alpha).$$

In other words, we only have to compute the value of f at α to find the translation $\delta_0(x - a)$.

For the Heaviside function, H_0, being a generalized function,

$$< H_0, f >= \int_{-\infty}^{\infty} f(x)H_0(x)dx = \int_{-\infty}^{\infty} f(x)dx,$$

f being a test function. The translate of H is given by

$$H_\alpha(x) = \begin{cases} 0, & \text{if } x < \alpha \\ 1, & \text{if} x \geq \alpha. \end{cases}$$

$$\implies H_\alpha(x) = H_0(x - \alpha).$$

10.3 Derivatives

We have that if φ is differentiable, then

$$< \varphi', f >= \int_{-\infty}^{\infty} f(x)\varphi'(x)dx =$$

$$= f(x)\varphi(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\varphi(x)dx,$$

by integrating by parts. Since $f(x) = 0$ when $x \to \pm\infty$, we obtain

$$- \int_{-\infty}^{\infty} f'(x)\varphi(x)dx =< \varphi, -f' > .$$

This motivates the definition of the derivatives of generalized functions: φ' is the distribution that acts on test functions f as φ acts on $-f'$:

$$< \varphi', f >= - < \varphi, f' > .$$

10.3.1 Example 2

Finding the derivative of the Heaviside function, we derive the Delta-function:

$$< H_0', f >=< H_0, f' >=$$

$$= - \int_0^{\infty} f'(x)dx = -f(x)|_0^{\infty} = f(x)$$

$$\implies < H_0', f >=< \delta_0, f > .$$

10.4 Fourier transforms

We define the Fourier transform of a generalized function φ as the one that acts on f as φ acts on the Fourier transform of f:

$$< \hat{\varphi}, f >=< \varphi, \hat{f} >= \int_{-\infty}^{\infty} \hat{f}(x)\varphi(x)dx.$$

10.4.1 Example 3

Fouirer transforming the Delta-function, we get

$$< \hat{\delta_0}, f >=< \delta_0, \hat{f} >= \int_{-\infty}^{\infty} \hat{f}(x)\delta_0 dx =$$

$$= \hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)dx =< \frac{1}{\sqrt{2\pi}}, f > .$$

Thus,

$$\mathcal{F}(\delta_0(x))(\omega) = \frac{1}{\sqrt{2\pi}}.$$

This can also be shown using the definition of the Fourier transform:

$$\hat{\delta_0}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x}\delta_0(x)dx = \frac{1}{\sqrt{2\pi}} e^{-i\omega \cdot 0} = \frac{1}{\sqrt{2\pi}}.$$

Further, we have the following Fourier transforms for the Delta- and Heaviside-function:

$$\mathcal{F}(\delta_\alpha)(\omega) = \frac{1}{\sqrt{2\pi}} e^{-i\alpha\omega}$$

$$\mathcal{F}(H_0)(\omega) = -\frac{i}{\sqrt{2\pi}\omega}$$

$$\mathcal{F}(H_\alpha)(\omega) = -\frac{i}{\sqrt{2\pi}\omega} e^{-i\alpha\omega}$$

One can show that, with this a definition of the derivative of generalized functions, the formulas for the derivative of the Fourier transform (Section 4.1) and the Fourier transform (Section 4.2) remain true for generalized functions. The proof of this will not be included in this thesis.

10.5 Convolution

Let the convolution $\varphi * \psi$ be defined as

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(t)\psi(x-t)dt.$$

Taking the convolution of δ with φ, we get

$$\delta_\alpha * \varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta_\alpha(t)\varphi(x-t)dt =$$

$$= \frac{1}{\sqrt{2\pi}}\varphi(x-\alpha) = \frac{1}{\sqrt{2\pi}}\varphi_\alpha(x).$$

In other words, we multiply by $\frac{1}{\sqrt{2\pi}}$ and translate by α. That implies that $\sqrt{2\pi}\delta_0$ is an identity with respect to the distribution, since

$$(\sqrt{2\pi}\delta_0) * f(x) = f(x).$$

So, in general, we have that

$$(\sqrt{2\pi}\delta_\alpha) * f(x) = f(x-\alpha).$$

10.5.1 Convolution of the Delta-functions

If we take the convolution of two Delta-functions, we get

$$\delta_a * \delta_b = \frac{1}{\sqrt{2\pi}}\delta_{a+b}.$$

10.5.2 Differentiation of a convolution

Differentiating a convolution, we obtain

$$(\varphi * \psi)' = \varphi' * \psi = \varphi * \psi'.$$

This can be shown by using the properties: $\mathcal{F}(\varphi * \psi) = \mathcal{F}(\varphi) \cdot \mathcal{F}(\psi)$ and $i\omega\hat{f} = \mathcal{F}(f')$.

Let

$$h(t) = (\varphi * \psi)'$$

$$\implies \hat{h}(t) = \mathcal{F}(h(t)) = \mathcal{F}((\varphi * \psi)') = i\omega\mathcal{F}(\varphi * \psi) =$$

$$= i\omega\hat{\varphi}(\omega)\hat{\psi}(\omega).$$

Since this commute, we either can write the expression as

$$\hat{\varphi}'(\omega)\hat{\psi}(\omega)$$

or

$$\hat{\varphi}(\omega)\hat{\psi}'(\omega).$$

by following the derivative-property above. Now, by taking the inverse transform we obtain

$$(\varphi * \psi)' = \varphi' * \psi = \varphi * \psi' \qquad \blacksquare$$

By taking the derivative again, we obtain

$$(\varphi * \psi)'' = \varphi' * \psi'.$$

As for the Fourier transform and its derivative, it can be shown that the convolution theorem (Section 4.3) holds true for generalized functions as well. This will not be a part of this thesis.

10.6 More examples

10.6.1 Example 4

Given

$$\varphi(x) = (H_0(x) - H_1(x))',$$

Let us find its Fourier transform. Given $(cH_\alpha)' = c\delta_\alpha$, we have that

$$(H_0(x) - H_1(x))' = (H_0(x))' - (H_1(x))' = \delta_0 - \delta_1.$$

Then

$$\begin{aligned}\mathcal{F}(\delta_0 - \delta_1)(\omega) = \mathcal{F}(\delta_0)(\omega) - \mathcal{F}(\delta_1)(\omega) = \\ = \frac{1}{\sqrt{2\pi}} e^{-i\cdot 0\omega} - \frac{1}{\sqrt{2\pi}} e^{-i\cdot 1\omega} = \\ \frac{1}{\sqrt{2\pi}}(1 - e^{-i\omega}).\end{aligned}$$

10.6.2 Example 5

Let

$$\varphi(x) = -\delta_{-1}(x) + \delta_1(x) + H_0(x) - H_1(x),$$

then its Fourier transform is given by

$$\begin{aligned}\mathcal{F}(\varphi)(\omega) = \mathcal{F}(-\delta_{-1}) + \mathcal{F}(\delta_1) + \mathcal{F}(H_0) - \mathcal{F}(H_1) = \\ = -\frac{1}{\sqrt{2\pi}} e^{i\omega} + \frac{1}{\sqrt{2\pi}} e^{-i\omega} - \frac{i}{\sqrt{2\pi}\omega} + \frac{i}{\sqrt{2\pi}\omega} e^{-i\omega} = \\ = \frac{1}{\sqrt{2\pi}}(e^{-i\omega} - e^{i\omega}) + \frac{i}{\sqrt{2\pi}\omega}(e^{-i\omega} - 1) \\ = \frac{1}{\sqrt{2\pi}}\left[e^{-i\omega} - e^{i\omega} + \frac{i}{\omega}(e^{-i\omega} - 1)\right].\end{aligned}$$

10.6.3 Example 6

Now, let us find the convolution of φ and ψ, where

$$\varphi(x) = 3\delta_{-1}(x)$$

and

$$\psi(x) = \delta_2(x) - \delta_1(x).$$

Remember that

$$f * (g + h) = f * g + f * h,$$

so $(\varphi * \psi)$ can be written as

$$3\delta_{-1} * (\delta_2 - \delta_1) = (3\delta_{-1} * \delta_2) + (3\delta_{-1} * (-\delta_1)) =$$

$$= \frac{3}{\sqrt{2\pi}}\delta_{(-1+2)} - \frac{3}{\sqrt{2\pi}}\delta_{(-1+2)} =$$

$$\implies (\varphi * \psi) = \frac{3}{\sqrt{2\pi}}(\delta_1 - \delta_0).$$

Chapter 11

Solving the non-homogeneous heat equation

Non-homogeneous heat equations arise in the heat spreading processes involving heat and cooling sources. An example may be an instantaneous heat impulse represented by the delta-function. Knowing the properties of this function, we can now look at boundary value problems that have functions that are concentrated at isolated points.

"Two generalized functions are equal if their integrals against test functions are equal". So, if the solution of a boundary value problem is given by $\varphi(x)$ and it verifies the initial condition as a generalized function, $\varphi(x,t)$ is said to be a weak solution. In other words, a weak solution solves a problem that includes the Delta-function.

The weak solution is given as a constant multiple of the heat kernel, namely $\varphi(x) = \frac{1}{\sqrt{2\pi}} g_t(x)$, where $g_t(x) = \frac{1}{c\sqrt{2t}} e^{-x^2/4c^2t}$. Thus, this solution is fundamental, which means that it can be used to express the general solution to the heat equation.

When solving the non-homogenuous heat equation, we first find the weak solution and then us it to find the general solution.

11.0.1 Fundamental solution

Now, we have to extend the the heat kernel for $t \leq 0$: $\forall -\infty < x < \infty$,

$$\varphi(x,t) = \begin{cases} \frac{1}{2c\sqrt{\pi t}} e^{-x^2/(4c^2 t)} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

In words, that means that $\varphi(x) = \frac{1}{\sqrt{2\pi}} g_t(x)$ if $t > 0$ and 0 otherwise. Given this, $\varphi(x,t)$ is a weak solution of the non-homogenuous heat equation $u_t = x^2 u_{xx} + \delta_0(x)\delta_0(t)$. If the initial condition is given by $u(x,o) = 0$, the solution is given by

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \varphi(x-y, t-s) f(y,s) dy ds,$$

where $\varphi(x,t) = \frac{1}{\sqrt{2\pi}} g_t(x) = \frac{1}{2c\sqrt{\pi t}} e^{-x^2/(4c^2 t)}$.

11.0.2 The incomplete gamma-function

For simplicity, we use the so-called incomplete gamma-function, Γ, to express our solution. This function is defined as

$$\Gamma(a,x) = \int_x^{\infty} e^{-t} t^{a-t} dt.$$

If $x = 0$, $\Gamma(a,o) = \Gamma(a)$.

If $a \neq 0$, we integrate by parts and get

$$\int_x^{\infty} e^{-t} t^{a-1} dt = \frac{1}{a} e^{-t} t^a|_x^{\infty} + \frac{1}{a} \int_x^{\infty} e^{-t} t^a dt =$$

$$= -\frac{e^{-x}x^a}{a} + \frac{1}{a}\int_x^\infty e^{-t}dt = -\frac{e^{-x}x^a}{a} + \frac{1}{a}[\Gamma(a+1,x)].$$

If $-1 < a < 0$ and $x > 0$,

$$\int_x^\infty e^{-t}t^{a-1}dt = \frac{1}{a}[-e^{-x}x^a + \Gamma(a+1,x)].$$

11.0.3 General solution

Given $u(x,0) = h(x)$, we have that the general solution is given by

$$u(x,t) = \int_0^t \int_{-\infty}^\infty \varphi(x-y,t-s)f(y,s)dyds + \int_{-\infty}^\infty \varphi(x-y,t)h(y)dy.$$

Superposition principle

If $u_1(x,t)$ solves $u_t = c^2u_{xx} + f_1(x,t)$ with initial condition $u(x,0) = 0$, and $u_2(x,t)$ solves $u_t = c^2u_{xx}f_2(x,t)$ with initial condition $u(x,0) = 0$. Then $u = u_1+u_2$ is a solution to $u_t = c^2u_{xx}+f_1(x,t)+f_2(x,t)$ with initial condition $u(x,0) = 0$.

Translation principle

If $\varphi(x,t)$ solves $u_1 = c^2u_{xx} + f(x,t)$ with initial condition $u(x,0=0)$, then $u(x,t) = \varphi(x-a,t)$ solves $u_t = c^2u_{xx} + f(x-a)$ with initial condition $u(x,0) = 0$.

11.1 Examples

11.1.1 Example 1

Let us solve the heat problem

$$u_t = \frac{1}{4}u_{xx} + \delta_a(x), \qquad u(x,0) = 0,$$

where a is arbitrary. Let us first solve for $a = 0$.

We have that $f(x,t) = \delta_0(x)$ and $c = \frac{1}{2}$. This gives us the fundamental solution

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_0^t \int_{\infty}^{\infty} \frac{1}{\sqrt{t-s}} e^{-(x-y)^2/(t-s)} \delta_0(y) dy ds$$

$$\implies u(x,t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-s}} e^{-x^2/(t-s)} ds$$

when integrating against $\delta_0(y)$.

Now, we make a change of variables to simplify the exponential function: $z = \frac{x^2}{t-s}$, $ds = \frac{x^2}{x^2} dz$ and $\frac{1}{\sqrt{t-s}} = \frac{\sqrt{z}}{|x|}$. Then we have that

$$u(x,t) = \frac{|x|}{\sqrt{\pi}} \int_{\frac{x^2}{t}}^{\infty} e^{-z} z^{-\frac{3}{2}} dz$$

If we now apply the incomplete Gamma-function

$$\frac{1}{a}[-e^{-x}x^a + \Gamma(a+1, x)]$$

with $a = -\frac{1}{2}$ and x replaced with $\frac{x^2}{t}$, we obtain

$$\frac{|x|}{\sqrt{\pi}} \left[\frac{1}{-\frac{1}{2}} (-e^{-x^2/t} (\frac{x^2}{t})^{-\frac{1}{2}} + \Gamma(-\frac{1}{2} + 1, \frac{x^2}{t}) \right] =$$

$$= \frac{|x|}{\sqrt{\pi}} \left[2e^{-\frac{x^2}{t}} \frac{\sqrt{t}}{|x|} - 2\Gamma(\frac{1}{2}, \frac{x^2}{t}) \right]$$

$$\implies u(x,t) = 2\frac{\sqrt{t}}{\sqrt{\pi}} e^{-x^2/t} - 2\frac{|x|}{\sqrt{\pi}} \Gamma(\frac{1}{2}, \frac{x^2}{t}).$$

Now, having that $f(x,t) = \delta_a$, we use the translation principle, we have that our problem is translated a units to the right ($a > 0$) or to the left ($a < 0$) in the x-variable. That means that x is replaced by $\frac{(x-a)^2}{t}$. Then we have our solution

$$\varphi(x,t) = u(x-a,t) =$$

$$= 2\frac{\sqrt{t}}{\sqrt{\pi}} e^{-(x-a)2/t} - 2\frac{|x-a|}{\sqrt{\pi}} \Gamma(\frac{1}{2}, \frac{(x-a)^2}{t}).$$

11.1.2 Example 2

If

$$u_t = \frac{1}{4}u_{xx} + \delta_0(x) + \delta_1(x), \qquad\qquad u(x,0) = 0$$

From the superposition principle, we have that our solution $\varphi(x,t)$ is the sum of the solutions $u(x,t)$ and $u(x-1,t)$. That implies

$$2\frac{\sqrt{t}}{\sqrt{\pi}}e^{-x^2/t} - 2\frac{|x|}{\sqrt{\pi}}\Gamma(\frac{1}{2},\frac{x^2}{t}) + 2\frac{\sqrt{t}}{\sqrt{\pi}}e^{-(x-1)^2/t} - 2\frac{|x-1|}{\sqrt{\pi}}\Gamma(\frac{1}{2},\frac{(x-1)^2}{t}).$$

11.1.3 Example 3

Let

$$U_t = \frac{1}{4}u_{xx} + \delta_0(x)(U_0(t) - U_1(t))$$

be a heat source at $(0,0)$ that is ON for $0 < t \leq 1$ and OFF for $t > 1$.

When t is between 0 and 1, we have the same problem as in Example 1, since the conditions are the same. So, if our solution is given by $v(x,t)$ in this problem, for $0 < t < 1$

$$v(x,t) = u(x,t).$$

When $t > 1$ and s varies in the interval $(0,t)$, $U_0(s) - U_1(s) = 1$ if s is in $(0,1)$ and 0 if s is in $(1,t)$. So, if $t > 1$

$$v(x,t) = \frac{1}{\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{t-s}}e^{-x^2/(t-s)}ds = \frac{|x|}{\sqrt{\pi}}\int_{\frac{x^2}{t}}^{\frac{x^2}{t-1}} e^{-z}z^{-3/2}dz.$$

This can be written as

$$v(x,t) = \frac{|x|}{\sqrt{\pi}}\int_{\frac{x^2}{t}}^{\infty} e^{-z}z^{-3/2}dz - \frac{|x|}{\sqrt{\pi}}\int_{\frac{x^2}{t-1}}^{\infty} e^{-z}z^{-3/2}dz =$$

$$= u(x,t) - u(x,t-1).$$

$$\implies v(x,t) = \begin{cases} u(x,t), & \text{if } 0 < t < 1 \\ u(x,t) - u(x,t-1), & \text{if } t > 1 \end{cases}$$

11.1.4 Example 4

Now, let us take this one step further:

$$u_t = \frac{1}{4}u_{xx} + (\delta_0(x) + \delta_1(x))(U_0(t) - U_1(t)), \qquad u(x,0) = 0.$$

Now, we have a heat source at the origin that is turned ON for $0 < t < 1$ and OFF for $t > 1$ (as in Example 3) and a heat source at $x = 1$ that is turned ON for $0 < t < 1$ and OFF for $t > 1$. Therefore, we can write

$$(\delta_0(x) + \delta_1(x))(U_0(t) - U_1(t))$$

as

$$\delta_0(x)[U_0(t) - U_1(t)] + \delta_1(x)[U_0(t) - U_1(t)].$$

When $t \in (0,1)$, we have that the solution $w(x,t)$ is composed of $v(x,t) = u(x,t)$ and the same solution just shiftet by one unit: $u(x-1,t)$. Thus,

$$w(x,t) = u(x,t) + u(x-1,t),$$

for $0 < t < 1$.

For $t > 1$, we use the same idea as above: we have the same solution as before, in addition to a translated version of the same solution, like this:

$$w(x,t) = u(x,t) - u(x,t-1) + u(x-t,t) - u(x-1,t-1),$$

for $t > 1$. This implies that our solution is given by

$$\implies w(x,t) = \begin{cases} u(x,t) + u(x-1,t), & \text{if } 0 < t < 1 \\ u(x,t) - u(x,t-1) + u(x-1,t) - u(x-1,t-1), & \text{if } t > 1 \end{cases}$$

Chapter 12

Conclusion

Now we have seen some of the most important properties and applications of the Fourier transform to partial differential equations and boundary value problems. My idea at the beginning of this thesis was to start with the "birth" of the Fourier transform, namely from Fourier series. From there, we have built stone by stone by adding definitions, proof and examples to the main properties and different solving methods.

I was kind of nervous about the last part of my assignment about generalized functions and the non-homogeneous heat equation, so I am glad that my supervisor did not give me all the details of the thesis in the beginning, but part by part along the way. Then I could focus on the present material without concerning about what comes next. I also think that helps me to understand more of the material than if I knew what was to come.

My Achilles heel is with no doubt the writing in English, but I think it have worked out quite well throughout the spring. Also, the writing in Latex gets easier each time I open my document, so in the end I did not have to look up all the commands every time. Therefore, my writing has become not so time-consuming as it was in February.

As an end to my thesis, I once again want to thank my supervisor, Alexander Rashkovskii, for guidence along the way. Even though a master's thesis include more independent work than the bachelor's thesis, I feel that I have received sufficient help along the way. I really appreciate that, so thank you very much!

References

Books

- Asmar, N. (2000). The Fourier Transform and its Applications. Partial differential equations and boundary value problems (Chapter 7, p. 325-375). New Jersey: Prentice-Hall, Inc.

- Asmar, N. (2005). The Fourier Transform and its Applications. Partial differential equations with Fourier series and boundary value problems, Second Edition (Chapter 7.8-7.9, p. 445-470). Pearson Education International.

Websites

- Wikipedia. (2021, 29. January). Fourier transform. `https://en.wikipedia.org/wiki/Fourier_transform`

- 3Blue1Brown. (2018, 26. January). But what is the Fourier Transform? A visual introduction. Youtube. `https://www.youtube.com/watch?v=spUNpyF58BY`

- Cheever, E. (2021, 12. February). Introduction to the Fourier Transform. `https://lpsa.swarthmore.edu/Fourier/Xforms/FXformIntro.html`

- Weisstein, Eric W. Fourier Transform. WolframMathworld. `https://mathworld.wolfram.com/FourierTransform.html`

- Unknown 1. (2021, 12. February). Fourier Transforms. `https://www.cv.nrao.edu/course/astr534/FourierTransforms.html`

- Unknown 2. (2021, 12. February). An Interactive Guide To The Fourier Transform. https://betterexplained.com/articles/an-interactive-guide-t
- Bevel, P.J, (2010).Fourier Transforms. https://www.thefouriertransform.com/
- Wilkes, M. (2013, 10 March). Convolution of a Rectangular "Pulse" With Itself. http://mikewilkes-irsc.weebly.com/uploads/2/3/1/7/23176182/rectangular_pulse_convolution-update.pdf
- Wikipedia. (2020, 25. June). Fourier inversion theorem. https://en.wikipedia.org/wiki/Fourier_inversion_theorem