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Writer: Silje Kristin Larsen	Silje Kristin Larsen (Writer's signature)
Faculty supervisor: Eirik Eik Svanes	
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Differential forms, cohomology and topological field theory

Silje Kristin Larsen

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University of Stavanger

Abstract

In this thesis you will be introduced to higher-dimensional geometrical objects called manifolds. We will develop something called differential forms, which is a tool that is independent of coordinates. We will look at the topological properties, and topological invariants. These can be characterized by something called cohomology groups. These cohomology groups characterise the solution space to the equations of motion to a topological field theory. We will study some examples of this.

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Contents

1 Introduction

In physics we now have two big theories describing our nature. On one side we have the general relativity which describes the "big" things such as orbiting planets, galaxies and the dynamics of an expanding universe. Then on the other side we have the quantum mechanics which describe the "small" things, such as the particles and the forces between them. Unfortunately, the theories seem to contradict each other. Therefore scientist try to find one unifying theory for our nature. Both mathematicians and physicists are working together to develop an intersection of these theories. There are many ideas, and one of them is the string theory. But for string theory to work, we have to exist in higher dimensions than 3D. Evidence of a higher dimension does not yet exist, but the math is already worked out in higher dimensions.
We will not cover string theory, but we will get into some of the math that is used in higher dimensional spaces, and how this can be used to find solutions in physics in these dimensions. Hence we will see that there exists a connection between the math and the physics.

1.1 Focus and goal

In this thesis we will introduce some geometrical objects that exist in higher dimensions, and we will show how math is defined on these objects which is called *manifolds*. On these manifolds we will define multivariable calculus that is described by something called *differential forms*.

When looking at the manifolds and the differential forms we will see that there exists some topological properties that are invariant under smooth deformations. These topological invariants is the essence of topological field theory. Some of these topological invariants can be classified into characteristic classes. These will be introduced as cohomology groups.

In the end we will show some examples of different manifolds, calculate their cohomology and show that they give the solution space of the equation of motion to a topological field theory.

This thesis we will closely follow the books *An Introduction to Manifolds* by Loring W. Tu [4], *Geometry, topology and physics* by M. Nakahara [1] and *From Calculus to Cohomology* by Ib Madsen and Jørgen Tornehave [2]. The reader should be familiar with the concepts of topology and linear algebra. If not, some examples to read could be the appendices in [4] or chapter 2 in [1].

2 Manifolds

In our three dimensional space as we know it, curves and surfaces are not difficult to visualize or deal with using the mathematical tools we already have. But math has gone beyond the three dimensional space we can see and sense. We now look at spaces in higher dimension also described by mathematical tools. These curves and surfaces are called manifolds.

Think of a sphere. We are living on one. Unless your an astronaut in space looking down on the earth it doesn't look like we are living on a sphere, but more like a space in $\mathbb{R}^2$. This analogy we also see in the definition of a higher dimensional manifold. Locally on the manifold it would look like $\mathbb{R}^n$, but if you zoom out globally may not look the same.

A topological manifold M is a second countable, Hausdorff, locally Euclidean space [2]. If you take a point and look at the area around this point it looks Euclidean, but the space as a whole may not be Euclidean. In general the space is homeomorphic to $\mathbb{R}^n$ locally, but may differ from $\mathbb{R}^n$ globally. Because of this homeomorphism we can introduce local coordinates, giving each point in M a set of m numbers, and if the manifold is not globally homeomorphic to $\mathbb{R}^n$ we need to add several coordinate charts. Later we will see that we require the transition from one coordinate chart to another to be smooth.

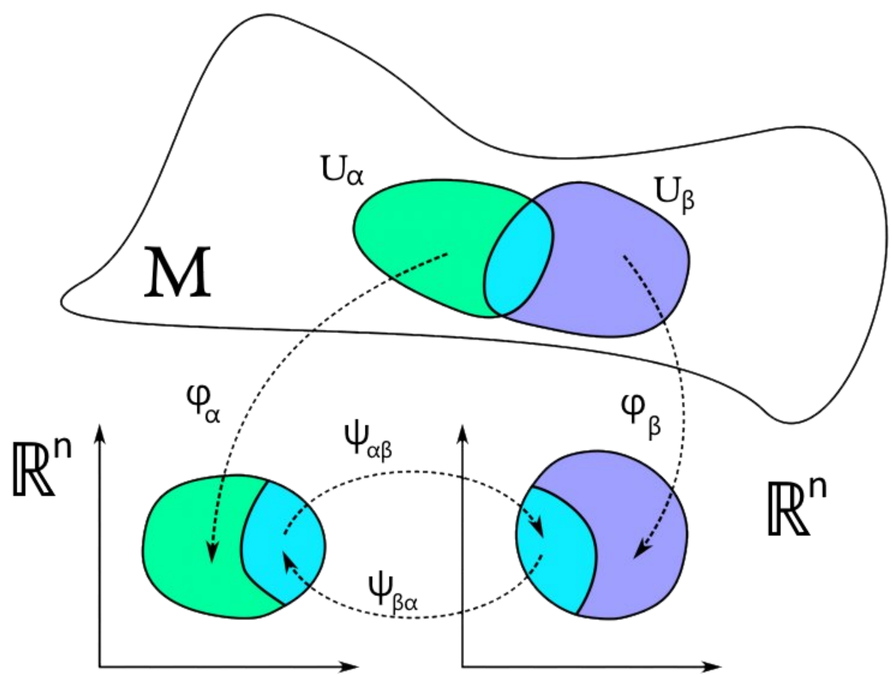


Figure 1: Original figure from [5] A manifold M with two subsets U_α and U_β

From the figure we have a manifold M which we have divided into subsets U_α and U_β. These we call the coordinate neighbourhoods, and the coordinate function is φ_α. The coordinate function is a homemorphism that maps U_α onto the open subset $U'_\alpha \subseteq \mathbb{R}^m$. This pair together $(U_\alpha, \varphi_\alpha)$ is a chart and the atlas is the whole family $\{(U_\alpha, \varphi_\alpha)\}$

We can assign any coordinate we like to a point $p \in M$. The homeomorpishm φ_1 can be denoted $x^1(p), ..., x^m(p)$ or just the set $\{x^\mu(p)\}$.

Definition 1. *M is a topological manifold if*

1. *M is a topological space, second countable, Hausdorff and locally Euclidean space.*
2. *You can provide M with a family of pairs* $(U_\alpha, \varphi_\alpha)$.
3. *The map* φ_α *is a homeomorphism from* U_α *onto the open subset* U'_i *in* $\mathbb{R}^m$. $\cup_\alpha U_\alpha = M$.

Example 1. *The surface of a two-sphere is a two-dimensional manifold. This can be defined as a subset of* $\mathbb{R}^3$.

$$S^2 = \{(x, y, z) \in R^3 \mid x^2 + y^2 + x^2 = 1\} \tag{1}$$

Since the manifold is two-dimensional the charts given will map parts of the sphere to an open subset of $\mathbb{R}^2$. *We can have a function mapping the open northern hemisphere to the open disc on the xy-plane. Same for the southern hemisphere. To cover the whole sphere we have to make six charts.*

This can be generalized to higher dimensions.

2.1 Differentiable and smooth manifolds

Recall that a point in the manifold can be given more than one coordinate. If the open sets U_α and U_β overlap there will be assigned two coordinate systems to one point in $U_\alpha \cap U_\beta$. For a smooth manifold the transition from one coordinate system to another must be infinitely differentiable, and therefore it is also smooth. The map $\psi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is the transition function as seen in figure 1.

Definition 2. *Differentiable smooth manifold*

1. *M has to be a topological space.*
2. *You can provide M with a family of pairs* $(U_\alpha, \varphi_\alpha)$.
3. *The map* φ_α *needs to be a homeomorphism from* U_α *onto an open subset* $U'_\alpha \subseteq \mathbb{R}^m$.

4. *Whenever the intersection of* U_α *and* U_β *is non-empty, the map* $\psi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ *is infinitely differentiable. These are the transition functions.*

These conditions must be met for M *to be a smooth manifold.*

From figure 1 we have two open subsets with different coordinate systems. For U_α we have that the coordinate is the set $\{y^\mu(p)\}$ but for U_β the coordinate is the set $\{x^\nu(p)\}$. The coordinate transformation from y to x is given by the map $\psi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$.

Example 2. *A circle* S^1 *is a one-dimensional manifold that is connected.*

We will look at the unit circle $x^2 + y^2 = 1$*. This we can divide in four charts, looking at the four arcs. The semi-circle, the top half, can be described by x-coordinates. Therefore by projecting this down we obtain a continuous mapping between the semicircle and the open interval* $(-1, 1)$*.*

The first chart of the circle will be denoted: $X_{top}(x, y) = x$*. All four charts will cover the whole circle and forms an atlas for* S^1*.*

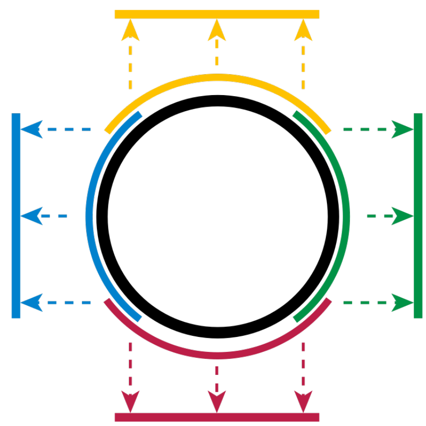


Figure 2: The different charts of S^1*. Figure taken from[6]*

But we also see that the charts will overlap at some points. The right and top charts overlap in the first quadrant of the xy-coordinate system. These two charts (X_{top} *and* X_{right}*) maps to the interval* $(0, 1)$*. We can construct a transition function T. a is any number in the interval* $(0, 1)$*:*

$$T(a) = X_{right}(X_{top}^{-1}(a)) = X_{right}(a, \sqrt{1 - a^2}) = \sqrt{1 - a^2}$$

This is smooth for $a \in (0, 1)$*. This example is one variant of an atlas. You can also use polar-coordinates to describe the circle.*

We will now need to collect some definitions and properties of manifolds that we will use later.

Definition 3. *Compact manifold*
If the manifold is compact as topological space then the manifold is a compact manifold.

Some geometrical objects you can deform continuously from one to the other. Two geometrical objects are homotopic if this is possible. See [2] for a precise definition of homotopy.

Definition 4. *Contractible Manifold*
A manifold M is contractible if it has homotopy type of a point.

Definition 5. *Paracompact Manifold*
If you have an open covering $\{U_i\}$ of the manifold M such that each point in the manifold is covered with a finite number of U_i the manifold is ***paracompact****.*

Manifolds and boundaries

If M is a manifold with a boundary, then the boundary is denoted ∂M. A compact manifold without boundary is also called a closed manifold.

Example 3. *Closed manifolds*
The circle is the only one dimensional compact closed manifold. The n-dimensional sphere and torus are other examples.

If M is a compact n-manifold with boundary then ∂M is a compact (n-1) manifold. The boundary ∂M of an oriented manifold itself has no boundary: $\partial(\partial M) = 0$. We will soon define orientation of a manifold.

Calculus on manifold

Because the mapping between the two overlapping sets are differentiable we can define calculus on manifolds. Later we will see that a differential form is a geometrical object on a manifold that can be integrated. Functions will be replaced by these more exotic objects called differential forms.

3 Differential forms

When we define manifolds we need a tool that is independent of coordinates. Differential forms are an approach to multivariable calculus which is not dependent of coordinates. We can use this tool over curves, surfaces, solids and higher-dimensional manifolds.

In this section we will look at differential forms in $\mathbb{R}^n$ as formal objects and show how we can do algebra with these objects. We will then generalise to manifolds.

3.1 Differential k-forms on $\mathbb{R}^n$

From calculus we are familiar with the notation dx, dy and dz which occur in derivation $\frac{dy}{dx}$ and integral $\int_M f(x,y)dxdy$. Without knowing we have worked with 0-forms, 1-forms, 2-forms and 3- forms, which are functions, line elements, surface element and volume elements. Also the known theorem such as Stokes theorem and Divergence theorem can be restated using differential forms. All these exists in three dimensions. When looking at higher dimensions we can have p-forms. From here we can boil down the fundamental theorems of vector calculus to one generalized theorem: Stokes theorem.
Here are some examples of how the differential forms is denoted.

Example 4. *A 0-form is a function*

Example 5. *A one form in* $\mathbb{R}^n$

$$\omega^1 = f_1 dx^1 + f_2 dx^2 + ... + f_n dx^n \tag{2}$$

Example 6. *A 2-form in* $\mathbb{R}^3$

$$\omega^2 = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy \tag{3}$$

Example 7. *A 3-form* $\mathbb{R}^3$

$$\omega^3 = f_1 dx \wedge dy \wedge dz \tag{4}$$

Example 8. *A k-form on* $\mathbb{R}^n$ *is denoted*

$$\omega^k = \frac{1}{k!} f_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k(\mathbb{R}^n) \tag{5}$$

where $\Omega^k(\mathbb{R}^n)$ is the space of k-forms. The $\frac{1}{k!}$ is for normalisation purposes. Note that we can add and substract forms, and multiply by real numbers, making $\Omega^k(\mathbb{R}^n)$ a vector space.

In the space $\mathbb{R}^3$ we are used to dx, dy, dx, but normally they are written as dx^1, dx^2, dx^3 when denoting differential forms. The 1-forms can be integrated over a curve, a 2-form can be integrated over a surface, etc.

3.2 Exterior product

Lets look at the algebra of differential forms. The multiplication operation is called the exterior multiplication and the symbol for this is the $\wedge$ as in $(dx \wedge dy)$. This is a anti-symmetric product.

Let $\omega \in \Omega^k(\mathbb{R}^n)$ and $\phi \in \Omega^l(\mathbb{R}^n)$. Then the product $\omega \wedge \phi \in \Omega^{k+l}(\mathbb{R}^n)$. The exterior product follows these properties:

1. Distributivity: $(dx^1 + dx^2) \wedge dx^3 = dx^1 \wedge dx^3 + dx^2 \wedge dx^3$
2. Assosciativity: $(dx^1 \wedge dx^2) \wedge dx^3 = dx^1 \wedge (dx^2 \wedge dx^3) = dx^1 \wedge dx^2 \wedge dx^3$
3. Anticommutativity: $\omega \wedge \phi = (-1)^{kl} \phi \wedge \omega$

Note that for any odd-order form of ω we have that

$$\omega \wedge \omega = -\omega \wedge \omega$$

$$\Rightarrow \omega \wedge \omega + \omega \wedge \omega = 0$$

$$\Rightarrow \omega \wedge \omega = 0$$

3.3 Exterior derivative

The exterior derivative is an extension of the derivative. The exterior derivative makes it possible to differentiate differential forms on manifolds in higher dimensions. It allows for derivatives to be expressed in coordinate-free form and is the basis of the generalized Stokes theorem. The exterior derivative maps a k-form to a (k+1)-form.

If we have an arbitrary k-form in n-dimensions

$$\omega = \frac{1}{k!} \omega_{i_1 ..., 1_k} dx^{i_1} \wedge ... \wedge dx^{i_k} \tag{6}$$

the derivative of ω is given by (k+1) form

$$d\omega = \frac{1}{k!} \partial_\mu \omega_{i_1 ... i_k} \wedge dx^\mu \wedge dx^{i_1} \wedge ... \wedge dx^{i_k} \in \Omega^{k+1}(\mathbb{R}^n) \tag{7}$$

The exterior derivative has the following properties:

1. For each function f, $df = \sum \frac{\partial f}{\partial x^i} dx^i$

2. $d(\alpha + \beta) = d\alpha + d\beta$

3. If α is a p-form and β is an q-form, then from the Leibniz rule, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p(\alpha \wedge d\beta)$ holds.

4. $d(d\alpha) = 0$.

Here we will introduce multi-index. Instead of writing the whole expression out we use the multi-index I:

$$dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

From the fourth property we say that $d^2 = 0$.

Proof. Will show that $d^2\omega = 0$ for $\omega = f_I dx^I$ (Multi index)

$$d^2(f_I dx^I) = d(\sum \frac{\partial f_I}{\partial x_i} dx_i \wedge dx^I)$$

$$= \sum \frac{\partial^2 f_I}{\partial x_j \partial x_i} dx_j \wedge dx_i \wedge dx^I \tag{8}$$

If $i = j$, then we know that $dx_i \wedge dx_j = 0$. But if $i \neq j$ the terms will cancel each other:

$$\frac{\partial^2 f_I}{\partial x_i \partial x_j} dx_i \wedge dx_j + \frac{\partial^2 f_I}{\partial x_j \partial x_i} dx_j \wedge dx_i$$

$$= \frac{\partial^2 f_I}{\partial x_i \partial x_j} dx_1 \wedge dx_j + \frac{\partial^2 f_I}{\partial x_i \partial x_j}(-dx_i \wedge dx_j) = 0$$

□

From this property, identities from vector calculus can be derived. For example on $\mathbb{R}^3$

$$\triangledown \times (\triangledown f) = 0$$

$$\triangledown \cdot (\triangledown \times \vec{v}) = 0$$

To connect with vectorcalulus on $\mathbb{R}^3$, consider the following example.

Example 9. *Let us see how the action of d works on a 1-form in three-dimensional space*

$$\omega_1 = \omega_x(x,y,z)dx + \omega_y(x,y,z)dy + \omega_z(x,y,z)dz$$

Then we take the exterior derivative of this 1-form

$$d\omega_1 = (\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y})dx \wedge dy + (\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z})dy \wedge dz + (\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x})dz \wedge dx$$

The exterior derivative acts as the curl on the vector-field given by the components of ω_1.

3.4 Differential forms on manifolds

Now we can look at differential forms on manifolds. The theory of integration on a manifold would not be possible without differential forms.

A more theoretical way of talking about manifolds and differential forms is to introduce the tangent and cotangent spaces. This is not relevant for this thesis and will therefore not be defined. We can still introduce the forms in another way.

What does a 0-form on a smooth manifold mean? We have a function f on M that assigns a unique number f(x) to each point in our manifold. Let us again go back to our Earth being an example of a manifold. Our function can then represent the temperature at the different coordinates. We know that our M is covered by open subsets U_α. If (U_α, ϕ_α) is a map of Stavanger, then the function f_α may represent the temperature in Stavanger. If we have one location (one point) where two charts U_α and U_β are overlapping we know from the transition functions, which are smooth, that they must give the same temperature: $f_\alpha = f_\beta$, if temperature is a well-defined function on the Earth.

If we have $(x^1, ..., x^n)$ as the coordinate system on the chart U_α on our manifold M, every k-form can be written

$$\omega = f_I dx^I \tag{9}$$

where $f_I \in C^\infty(U_\alpha)$ are smooth functions. In this way, we may view differential forms locally as forms on $\mathbb{R}^n$.

If we have the point in $(U_\alpha \cap U_\beta)$ we can do a change of coordinates. This will be introduced in section 3.5.

The collection of all k-forms on a manifold is denoted by $\Omega^p(M)$. The highest k-form we can have on a an manifold is the dimension of the manifold. Differential forms can be added, multiplied, differentiated and integrated. We can extend the exterior product and exterior derivative to a manifold.
The exterior derivative gives the mapping $\Omega^r(M) \to \Omega^{r+1}(M)$. This induces the sequence

$$0 \xrightarrow{i} \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} ... \xrightarrow{d_{n-1}} \Omega^n(M) \xrightarrow{d_n} 0$$

This sequence is called de Rham complex which we will come back to in section 4.4.

Definition 6. *Pullback*
From linear algebra we have vector spaces V and W with linear map: $f : V \to W$ and $g : W \to K$. The pullback of our function g is then $f^*(g) = g \circ f$. The pullback satisfy the following property:

$$f^* \circ d = d \circ f.$$

We also have a pullback of differential forms.

Definition 7. *Pullback of maps between manifold.*
We have manifold M and N. A smooth map $f : M \to N$ will naturally induce a map f^* called the differential map. $f^* : \Omega^p(N) \to \Omega^p(M)$. Explicitly, in local coordinates: $\omega \in \Omega^p(N)$

$$f^*(\omega) = \frac{1}{p!}\omega_{i_1 \dots i_p}(f(x))df^{i_1} \wedge \dots \wedge df^{i_p}$$

where $df^{i_k} = \frac{df^{i_k}}{dx^{j_k}}dx^{j_k}$ and $\{x^\mu\}$ are local coordinates on M.

3.5 Change of coordinates

From calculus we can change variables. We can do the same with differential forms. A form ω is defined without reference to any coordinate system. If we have a point in the intersection $p \in U_\alpha \cap U_\beta$ the one forms can be written like this:

$$\omega = \omega_\mu dx^\mu = \tilde{\omega}_\upsilon dy^\upsilon$$

where $x = \varphi_\alpha(p)$ and $y = \varphi_\beta(p)$. We use the fact that $dy^\upsilon = \frac{\partial y^\upsilon}{\partial x^\mu}dx^\mu$ then

$$\omega_\mu = \tilde{\omega}_\nu \frac{\partial y^\upsilon}{\partial x^\mu}$$

are the components of ω in the coordinates x of U_α, in terms of the components of $\tilde{\omega}_\nu$ in the coordinates of y of U_β. This can be generalised to higher degree forms.

We have two local parametrizations: $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ and $\phi_\beta : U_\beta \to \mathbb{R}^n$ of M at x. Then the local expression of the k-form ω are related by

$$\omega_\beta = (\phi_\alpha^{-1} \circ \phi_\beta)^*(\omega_\alpha) \tag{10}$$

Explicitly, we see how ω is written in coordinates x^μ on U_α and $\tilde{x}^\nu$ on U_β

$$\omega = \frac{1}{p!}\omega_{\mu_1 \dots \mu_p}dx^{\mu_1}\dots dx^{\mu_p} \Rightarrow \omega = \frac{1}{p!}\omega_{\mu_1 \dots \mu_p}\frac{\partial x^{\mu_1}}{\partial \tilde{x}^{\nu_1}}d\tilde{x}^\nu \wedge \dots \wedge \frac{\partial x^{\mu_p}}{\partial \tilde{x}^{\nu_p}}d\tilde{x}^{\nu^p}$$

where

$$\omega_{\mu_1 \dots \mu_p}\frac{\partial x^{\mu_1}}{\partial \tilde{x}^{\nu_1}}\dots\frac{\partial x^{\mu_p}}{\partial \tilde{x}^{\nu_p}} = \tilde{\omega}_{\nu_1 \dots \nu_p} \tag{11}$$

so then

$$\omega = \frac{1}{p!}\tilde{\omega}_{\nu_1 \dots \nu_p}d\tilde{x}^{\nu_1}\dots d\tilde{x}^{\nu_p}$$

Here $\tilde{\omega}_{\nu_1 \dots \nu_p}$ are the components of ω in the coordinates $\tilde{x}$ of U_β. Here $(\phi_\alpha^{-1} \circ \phi_\beta)^* : \Omega^p(U_\alpha) \to \Omega^p(U_\beta)$ is the pullback of $\phi_\alpha^{-1} \circ \phi_\beta$.

3.6 Closed and exact forms

Definition 8. *A k-form is closed if $d\omega = 0$. It is exact if there is a (k-1) form σ such that $d\sigma = \omega$.*

The first condition $d\omega = 0$ is local. A form on M is closed if it is closed at every point of M. The second condition of being exact is not local. Also we know $d^2 = 0$. Therefore every exact form must be closed, since $d(d\omega) = 0$.

Lemma 1. *Poincare lemma*
If M is contractible, every closed form is exact.

Example 10. *Counter example - punctured plane $\mathbb{R}^2 - \{(0,0)\}$*
$\omega = \frac{-ydx+xdy}{x^2+y^2}$ *has* $d\omega = 0$ *but* $\nexists\phi$ *s.t* $d\phi = \omega$ *on* $\mathbb{R}^2 - \{(0,0)\}$
It is a closed form, but not exact because of the hole in the space. In fact, ω is a explicit representative of $[\omega] \in H^1(\mathbb{R}^2 - \{(0,0)\}) \cong \mathbb{R}$, to be defined later.

3.7 Integral of forms

To integrate a p-form on a manifold the manifold needs to be orientable. The transition map $\phi_\beta \circ \phi_\alpha^{-1}$ has to have a positive Jacobian.

Definition 9. *Let M be a connected manifold covered by U_α. The manifold M is orientable if, for any overlapping charts U_α and U_β, there exist local coordinates x^μ for U_α and y^ν for U_β such that $J = det(\frac{\partial x^\mu}{\partial y^\nu}) > 0$.*

Definition 10. *If $J = det(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}) > 0$ on $(U_\alpha \cap U_\beta)$ then $\{\frac{\partial}{\partial x^\mu}\}$ and $\{\frac{\partial}{\partial \tilde{x}^\nu}\}$ have the same orientation.*

Then if we have a manifold with dimension p that is orientable there has to exist a p-form ω that will nowhere vanish. This form is usually called a volume element. This will be our choice of measure when we integrate a function f over M. If we have two volume elements ω and ω' that are equivalent there exist a positive function g, such that $\omega = g \cdot \omega'$.
Assume we have a p-form with a positive $g(p)$ on a chart (U, ϕ) with the coordinates $x = \phi(p)$.

$$\omega = g(p)dx^1 \wedge ... \wedge dx^p \tag{12}$$

If our manifold is orientable we can extend the form throughout M such that the component g is positive definite on any chart U_α. The form will therefore be a volume element. Under the change of coordinates between charts our form (12) becomes

$$\omega = g(p)\frac{\partial x^1}{\partial \tilde{x}^{\nu_1}}d\tilde{x}^{\nu_1} \wedge ... \wedge \frac{\partial x^p}{\partial \tilde{x}^{\nu_p}}d\tilde{x}^{\nu_p}$$

$$= g(p)det(\frac{\partial x^\mu}{\partial \tilde{x}^\nu})d\tilde{x}^1 \wedge ... \wedge d\tilde{x}^p \tag{13}$$

$$= \tilde{g}(p) d\tilde{x}^1 \wedge ... \wedge d\tilde{x}^p$$

As seen in (13) the determinant is the Jacobian of the transformation under coordinates and $\tilde{g}(p)$ can be assumed positive because M is orientable. Now can define a integration of a function $f : M \to \mathbb{R}$ over a manifold M of dimension p. We again take a volume element ω. Using the same coordinates we define the integration of an p-form $f\omega$ by

$$\int_{U_\alpha} f\omega = \int_{\phi(U_\alpha)} f(\phi_i^{-1}(x)) g(\phi_i^{-1}(x)) dx^1 ... dx^p = \int_{\phi(U_\alpha} (\phi^{-1})^* (f\omega) \tag{14}$$

On the right side we have an ordinary multiple integration of a function over a subset $\phi(U_\alpha) \subseteq \mathbb{R}^n$. When we can define the integral of f over U_α, the integral of $f\omega$ over the whole manifold M is given with the help of the partition of unity.

Definition 11. *We have a manifold M that is paracompact. If a family of differentiable functions $\epsilon_i(p)$ satisfy*

1. $0 \leq \epsilon_i \leq 1$
2. $\epsilon_i(p) = 0$ *if* $p \notin U_i$ *and*
3. $\epsilon_1(p) + \epsilon_2(p) + ... = 1$ *for any point* $p \in M$

the family $\{\epsilon_i(p)\}$ is called a partition of unity subordinate to the covering $\{U_i\}$.

From the third condition

$$f(p) = \sum_i f(p)\epsilon_i(p) = \sum_i f_i(p) \tag{15}$$

and because of condition 2, $f_i(p)$ vanishes outside U_i. The paracompactness ensures that there are only finite terms in the summation over i in (15). For each $f_i(p)$, we may define the integral over U_α according to (14). Then the integral of f on M is given by

$$\int_M f\omega = \sum_i \int_{U_i} f_i \omega \tag{16}$$

Example 11. *[1] Integration of 1-dimensional circle*
Let $S^1 : x^2 + y^2 = 1$. We will define two charts φ_1 and φ_2.
Here $\varphi_1^{-1} : (0, 2\pi) \to S^1$ is defined by $\theta \to (\cos\theta, \sin\theta)$. From this the image is $S^1 - \{(1,0)\}$
and $\varphi_2^{-1} : (-\pi, \pi) \to S^1$ is defined by $\theta \to (\cos\theta, \sin\theta)$. From this the image is $S^1 - \{(-1,0)\}$.

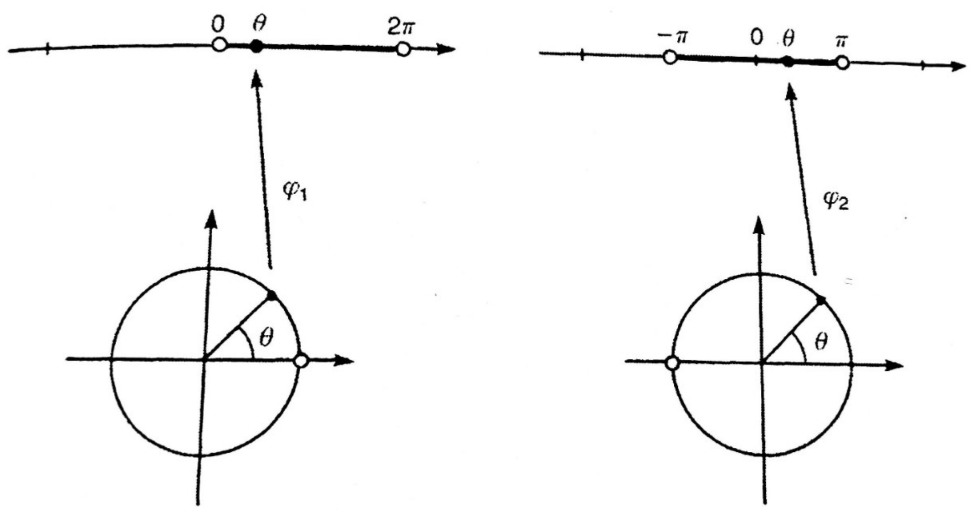


Figure 2: The two charts: φ_1 and φ_2. Figure from [1]

Let $U_1 = S^1 - (1,0)$, $U_2 = S^1 - (-1,0)$, $\epsilon_1(\theta) = sin^2\frac{\theta}{2}$ *and* $\epsilon_2(\theta) = cos^2\frac{\theta}{2}$

Let us integrate a function $f = cos^2\theta$ *using the partition of unity.*

$$\int_{S^1} d\theta cos^2\theta = \int_0^{2\pi} d\theta sin^2\frac{\theta}{2}cos^2\theta + \int_{-\pi}^{\pi} d\theta cos^2\frac{\theta}{2}cos^2\theta$$
$$= \frac{1}{2}\pi + \frac{1}{2}\pi = \pi.$$

This is the correct integral of $cos^2(\theta)$ *from* 0 *to* 2π.

Boundaries and orientation on manifolds

A manifold M with boundary is given by the atlas $\{(U_\alpha, \phi_\alpha)\}$. Here U_α is homeomorphic to either $\mathbb{R}^n$ or the upper half space $\mathbb{H}^n = \{(x_1, ..., x_n)|x_n \geq 0\}$. These are the charts on the boundary.

If the atlas is oriented this induces a oriented atlas also for ∂M. Then the induced orientation on its boundary $\partial\mathbb{H}^n = \{x_n = 0\}$ is the equivalence class of $(-1)^n dx_1...dx_{n-1}$. [3] This we will need in the next section when introducing the Stokes theorem.

3.8 Stokes theorem

From the fundamental theorem of calculus/vector calculus we learned about line integrals, Green's theorem, divergence/gauss theorem. These are in fact special case of the generalized Stoke's theorem.

Theorem 1. *Green's theorem*
Here $D \subseteq \mathbb{R}^2$ and ∂D is the boundary over the 2-dimensional domain.

$$\int_{\partial D} P(x,y)dx + Q(x,y)dy = \int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy \tag{17}$$

Theorem 2. *Divergence theorem*

$$\int_{D\subset\mathbb{R}^3} div \overrightarrow{\mathrm{F}} dxdydz = \int_{\partial D} \overrightarrow{\mathrm{F}} \cdot \overrightarrow{\mathrm{n}} \cdot dS \tag{18}$$

From these theorems we see that they have some similarity in the format. On one side you have a integral of a derivative over a boundary and on the other side a integral over the whole domain. This format comes from the Stokes theorem. It has a lot of different versions, but in this thesis we are looking at differential forms and manifolds. Therefore the integral of a differential form ω over the boundary of some orientable manifold equals the integral of its exterior derivative over the whole manifold.

Theorem 3. *Stokes theorem*
Let M be a compact oriented manifold with boundary, and let ω be an (k-1) form on M. Then

$$\int_{\partial M} \omega = \int_M d\omega \tag{19}$$

Proof. We will follow the proof from [3] and [7].

We have to look at three cases for proving the Generalized Stoke's theorem. First the special case with $\mathbb{R}^n$ with no boundaries. Then the second special case is the upper half plane. Then we combine these cases and look at the general case using the partition of unity.

For case 1 and 2, we assume ω to have compact local support contained in a U under a local diffeomorphism $f : U \to M$. The U is an open subset of $\mathbb{R}^k$ or $\mathbb{H}^k$.

Case 1: Since $\mathbb{R}^k$ has no boundaries we will expect both sides of the theorem to be 0.
We will assume that our subset U is open in $\mathbb{R}^k$. Then

$$\int_{\partial M} \omega = 0,$$

as ω is assumed to vanish on ∂M in this case. And

$$\int_M d\omega = \int_U f^*(d\omega) = \int_U d\mu$$

where $\mu = f^*\omega$. The pullback of ω under f. We take μ to be expressed as $\sum_{i=1}^{k}(-1)^{i-1} f_i dx_1 \wedge ... \wedge dx_k$, with dx^i removed (it is a (k-1)-form) and then $d\mu = (\sum_i \frac{\partial f_i}{\partial x_i}) dx_1 \wedge ... \wedge dx_k$

$$\int_U d\mu = \int_{\mathbb{R}^k} d\mu = \sum_i \int_{\mathbb{R}^k} \frac{\partial f_i}{\partial x_i} dx_i ... dx_k \tag{20}$$

The first equality follows as $d\mu$ has compact support in $U \subseteq \mathbb{R}^k$. But

$$\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i} dx_i = 0$$

by the fundamental theorem of calculus(FTC). Thus $\int_M d\omega = 0$.

Case 2 for $\mathbb{H}^k$.

Since $U \subset \mathbb{H}^k$ all the integrals on the right side of (20) will vanish except the last. Since boundary of $\mathbb{H}^k$ is the set where $x_k = 0$, the last integral will be

$$\int_{\mathbb{R}^{k-1}} (\int_0^{\infty} \frac{\partial f_k}{\partial x_k} dx_k) dx_1 ... dx_k$$

Again because of compact support f_k will vanish outside some large interval, so applying the FTC we obtain

$$\int_M d\omega = \int_{\mathbb{R}^{k-1}} -f_k(x_1, ..., x_{k-1}, 0) dx_1 ... dx_{k-1} \tag{21}$$

Also note that

$$\int_{\partial M} \omega = \int_{\partial H^k} \mu$$

on $\partial\mathbb{H}^k$ we know that $x_k = 0$ and $dx_k = 0$. If $i < k$ our form $(-1)^{i-1} f_i dx_1 \wedge ... \wedge dx_k$ restricts to 0 on the boundary of H. ∂H^k is diffeomorphic to $\mathbb{R}^{k-1}$. The induced orientation on $\partial\mathbb{H}^k$ changes orientation by the factor $(-1)^k$ so we then get

$$\begin{aligned} \int_{\partial M} \omega &= \int_{\partial H^k} (-1)^{k-1} f_k(x_1 ... x_{k-1}, 0) dx_1, ... dx_{k-1} \\ &= (-1)^k \int_{\mathbb{R}^{k-1}} (-1)^{k-1} f_k(x_1 ... x_{k-1}, 0) dx_1, ... dx_{k-1} \end{aligned}$$

The factors of (-1) will become -1, and therefore it becomes the same formula as (3). The theorem holds for the subset of $\mathbb{H}^k$.

Case 3: Using partition of unity to deduce the general case.

We choose an atlas $\{(U_\alpha, \phi_\alpha)\}$ for our manifold M where each subset U_α is diffeomorphic to either $\mathbb{R}^k$ or $\mathbb{H}^k$. We let $\{\rho_\alpha\}$ be a smooth partition of unity. We have showed that Stokes theorem holds for $\mathbb{R}^k$ or $\mathbb{H}^k$ and then it holds for all the charts in U_α in our atlas. We should note $(\partial M) \cap U_\alpha = \partial U_\alpha$

We get

$$\int_{\partial M} \omega = \int_{\partial M} \sum_\alpha \rho_i \omega$$

where $\sum_\alpha \rho_\alpha = 1$

$$= \sum_\alpha \int_{\partial M} \rho_\alpha \omega$$

$$= \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega$$

The support of $\rho_\alpha \omega$ is contained in U_α. Using Stokes theorem for U_α we get

$$\int_{\partial M} \omega = \sum_\alpha \int_{\partial M} d(\rho_\alpha \omega)$$

$$= \sum_\alpha \int_M d(\rho_\alpha \omega)$$

As also the support of $d(\rho_\alpha \omega) \subset U_\alpha$. We therefore get

$$\int_{\partial M} \omega = \int_M d(\sum \rho_\alpha \omega)$$
$$= \int_M d\omega$$

as $\sum_\alpha \rho_\alpha = 1$.

□

An example is applying the Stokes theorem to one dimension. We have a one dimensional manifold M, the line segment from $x = a$ to $x = b$ which also are the boundaries. Consider then

$$\int_a^b \frac{dF}{dx} dx = F(b) - F(a) \tag{22}$$

The 0-form ω^0 is $F(x)$ and its integral over the boundary is $[F(b) - F(a)]$. This is the fundamental theorem of calculus. Greens theorem is the two-dimension version, and the divergence theorem is the three-dimension version.

From Stokes theorem we can see that the integral won't change if we add a closed form $d\theta$ to ω because since $d(d\theta) = 0$. If ω is an exact form, then because $d^2 = 0$ the integral will be 0.

Example 12. *We have the form ω and the exact form $\varphi = d\phi$ on a closed and compact manifold M, then*

$$\int_M \omega + \varphi = \int_M \omega \tag{23}$$

because

$$\int_M \varphi = \int_M d\phi = \int_{\partial M} \phi = \int_\varnothing \phi = 0.$$

4 Topological field theory

As seen above when using Stokes theorem we can find changes to geometric quantities which leave other quantities invariant. For example $\omega \to \omega + d\phi$ leaves $\int_{\partial M} \omega$ invariant. This is the essence of topological field theory. To study invariants which only care about the topology of M. First we will introduce gauge symmetry and then come back to topological field theory.

4.1 Gauge symmetry

Gauge theory is about geometry and symmetry. This type of theory is important in modern quantum field theory. Gauge theories are field theories that have a certain kind of symmetry that gives rise to forces of nature. Some examples are the electromagnetic field and gravitational field. The forces between elementary particles is described by fields. The standard model of particle physics is a gauge theory.

We can do a transformation between these fields. These are called gauge transformations. If the transformation of a quantity gives the same quantity, then it is gauge invariant as seen in example 12.

For a global symmetry the redefinition is the same everywhere at all times. For a local gauge symmetry the redefinition varies from place to place and time. We are looking at differential forms locally on manifolds and this is the type of symmetry we will be most interested in. This is often called gauge symmetry.

Example 13. *An example from electromagnetism*
The magnetic field can be written in terms of vector potential, $B = \bigtriangledown \times A$. There are many potentials that gives rise to the same field. For example we can write $A \to A + \bigtriangledown\Lambda$. But remember that $\bigtriangledown \times (\bigtriangledown\Lambda) = 0$ Then

$$B = \bigtriangledown \times (A + \bigtriangledown\Lambda) = \bigtriangledown \times A + \bigtriangledown \times \bigtriangledown\Lambda = \bigtriangledown \times A.$$

The magnetic field B remains the same. $A \to A + \bigtriangledown\Lambda$ is a gauge-symmetry. Physics (B) is invariant under change $A \to A + \bigtriangledown\Lambda$.

4.2 Field theory

To specify a general theory in physics we look at two things. Geometry and topology. Without knowing it, when doing early physics we cared about the geometry and the topology of the space, but usually the topology of the space you were working on was trivial.

1) **Geometry:** Here we look at the scales and twist of the geometry. If we change the space by stretching it we also change the geometry. Also the distance matters. Not only the spacial distance but also distance in space-time. Changes to geometry are local. If we consider a large space we can change something at one

point in the space, but locally it wouldn't change the space somewhere else.

Change to geometry are described by the metric of space-time. In thesis we will not look at the properties of the metric and only at the topology where we don't need to define a metric. We also restrict to Euclidean geometries without time.

2) **Topology:** Topological quantities are insensitive to smooth geometric changes. By stretching you don't change the topology. Just look at the famous example of a doughnut and coffee cup. Geometrical they are not the same, but topological they are. So you can deform and stretch, but if you tear you change the topology.

An example of topological invariants are the characteristic classes of a manifold. So the coffee cup and the doughnut would have the same values of the various characteristic classes because they are topological equivalent. In section 4.4 we will introduce a class of these characteristics classes, namely cohomology groups.

So the physics cares about the geometry and the topology. Theories are usually complicated when looking at both, so we want to narrow it down to only looking at topological properties, and this is called *topological field theory*. These are theories where the physics only captures topological features. As you will see we can use the Stokes theorem to derive the equation of motion. The theory itself is often called *an action*.

4.3 Topological field theory

The action for a theory of differential forms we mentioned earlier will be denoted as $S(\omega)$. This is both denoted as the action or the theory. The equation of motion is the equation solved for all ω such that $\delta S(\omega) = 0$.

If $\omega \to \omega + d\theta$ is a gauge symmetry and the physics doesn't care about these symmetries, then our theory is a gauge theory. Let's see an example.

Example 14. *The E.O.M of three-dimensional abelian Chern-Simons theory.*

We have a 3-dimensional paracompact orientable manifold. Let $\omega \in \Omega^1(M^3)$ and $\partial M^3 = 0$. The action is

$$S(\omega) = \int_M \omega \wedge d\omega$$

We first need to show that the physics is invariant under gauge symmetry.

$$S(\omega + d\theta) = \int (\omega + d\theta) \wedge d(\omega + d\theta)$$

$$= \int \omega + d\theta \wedge d\omega$$

$$= \int \omega \wedge d\omega + \int d(\theta \wedge d\omega)$$

We can rewrite this as

$$= \int \omega \wedge d\omega + \int d\theta \wedge d\omega + \theta \wedge d(d\omega)$$

We know that $d^2 = 0$ *and* $\partial M^3 = 0$ *therefore*

$$= \int \omega \wedge d\omega.$$

The action is invariant under gauge symmetry since $S(\omega + d\theta) = S(\omega)$.

The E.O.M is given by

$$\delta S = \int_M (\delta\omega \wedge d\omega + \omega \wedge d\delta\omega) \tag{24}$$
$$= \int_M (\delta\omega \wedge d\omega + (d\delta\omega) \wedge \omega)$$

We know that

$$d(\delta\omega \wedge \omega) = (d\delta\omega) \wedge \omega - \partial\omega \wedge \omega$$

then the integral becomes

$$\delta S = \int_M 2\delta\omega \wedge d\omega + d(\delta\omega \wedge \omega)$$

Again, because $\partial M = 0$ *and Stokes*

$$\delta S = \int_M 2\delta\omega \wedge d\omega$$

The E.O.M is therefore

$$\int_{M^3} (2\delta\omega \wedge d\omega) = 0$$

If this is true for all $\delta\omega$ we see that $d\omega = 0$. Solutions of the equation of motion are given by closed form modulo exact form(the gauge symmetry):

$$\frac{\{\omega \in \Omega^1(M) | d\omega = 0\}}{\{\omega = df | f \in \Omega^0(M)\}}. \tag{25}$$

In the next section we will show that this is the same as calculating the de Rham cohomology groups for the manifold. The solution space of E.O.M for the example above is given by $H^1(M^3)$, the first cohomology of M. These are topological.

4.4 De Rham cohomology

In the end of section 3.4 we saw that the exterior derivative gave us the sequence

$$0 \xrightarrow{i} \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} ... \xrightarrow{d_n - 1} \Omega^n(M) \xrightarrow{d_n} 0 \tag{26}$$

This is the complex of differential forms and its called de Rham complex. We will use it to define de Rham cohomology.

In section 3.6 we looked at closed and exact forms. We know that all exact forms are closed, but are all closed forms exact? This depends on the topology of the manifold.
For an open subset of $\mathbb{R}^n$ we saw that a differential form is closed if $d\omega = 0$ and exact if $\omega = d\tau$. This also applies for differential forms on a manifold. We want to look for closed forms that are not exact. Let

$$Z^k(M) = \{\omega \in \Omega^k(M) | d\omega = 0\}$$

be the space of closed forms and

$$B^k(M) = \{\omega \in \Omega^k(M) | \omega \in d\Omega^{k-1}(M)\}$$

the space of exact forms.
Since every exact form is closed, B^k must be a subspace of Z^k. From this we can form the quotient space

$$H^k(M) = \frac{Z^k(M)}{B^k(M)} \tag{27}$$

This measures which closed k-forms fail to be exact.

Definition 12. *The k-th cohomology group of M is given by*

$$H^k(M) = \frac{Z^k(M)}{B^k(M)} \tag{28}$$

As we will see these groups are topological, invariant under smooth geometric changes.

Definition 13. *Two elements of $Z^k(M)$ are equivalent if they differ by an exact form*

$$\omega - \omega' = d\tau$$

ω and ω' are cohomologous. They give the same cohomology class.

Definition 14. *The dimension of $H^p(M)$ is called the p-th Betti number of M. This is denoted $h^p(M)$.*

Example 15. *Zeroth cohomology of M*
We have to look at 0-forms. These are just looking at $f \in C^\infty(M)$. Here we get trivial results. Elements of $Z^0(M)$ are closed. These are the locally constant functions.
However if M is disconnected the locally constants f need not have the same value. $Z^0(M)$ is hence a space with dimension equal to the number of disconnected components. $B^0(M) = 0$ ($\Leftarrow$ there exist no (-1)-form). Therefore

$$H^0(M) = \frac{Z^0}{B^0} = Z^0 = \mathbb{R}^n, \tag{29}$$

n being number of disconnected components.

Theorem 4. *[4] Let M be n-dimensional smooth orientable compact manifold. Then* $H^n(M) = \mathbb{R}$.

Example 16. *[9] The de Rham cohomology of the unit circle*
We know that S^1 *is connected. Therefore* $H^0(S^1) \cong \mathbb{R}$. *And our manifold is one-dimensional there is no k-forms for* $k \geq 2$. *Therefore for all* $k \geq 2$ *the* $H^k(S^1) = 0$. *Only thing we need to compute is the first cohomology group of* S^1

$$0 \xrightarrow{i} \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} 0$$

As the circle only has dimension 1, all 1-forms on S^1 *are closed so* $\Omega^1(S^1) = Z^1(S^1)$. *Therefore* $ker(d_1) = \Omega^1(S^1)$
Let us consider a map $\phi : Z^1(S^1) = \Omega^1(S^1) \to \mathbb{R}$ *which is defined by* $\phi(\omega) = \int_{S^1} \omega$. *It is possible to construct a global one form* α *s.t* $\alpha = \frac{d\theta}{2\pi}$. *Here* θ *is the angular coordinate on* S^1. *From this we see that* $\phi(\alpha) = 1$, *which implies that* $\phi(c\alpha) = c$ *for all* $c \in \mathbb{R}$. *Therefore* ϕ *is surjective and* $\frac{\Omega^1(S^1)}{ker(\phi)} \cong \mathbb{R}$

If $ker(\phi) = Im(d_0)$, *we see that* $H^1(S^1) \cong \mathbb{R}$. *For* $\omega \in ker(\phi)$ *we can define*

$$f(\theta) = \int_0^\theta \omega.$$

Since $\omega \in ker(\phi)$ *our function* f *is a smooth function on* S^1, *and* $d_0 f = \omega$. *There* $\omega \in im(d_0)$. *So* $ker(\phi) \subseteq im(d_0)$. *From Stokes it is clear that* $im(d_0) \subseteq ker(\phi)$, *so* $ker(\phi) = im(d_0)$.

Later we will introduce the *Mayer-Vietoris sequence*. This tool will do the computation of the cohomology of manifolds, including this example, much easier.

5 Mathematical tools

We have now looked at manifolds and differential forms on manifolds. We have also seen that there exists topological gauge theories where we can construct groups of cohomology which can be used in physics to classify solutions of E.O.Ms. We will need some mathematical tools to compute these groups, and we will see that they are topological, independent of smooth geometric changes.

5.1 Chain complexes

A *chain complex* $\mathcal{C}$ is a collection of vector spaces. Between these we have the linear maps $d_k : C^k \to C^{k+1}$ with the property $d_k \circ d_{k-1} = 0$ for all k.

We introduced the sequence (26) above where $\Omega^*(M)$ is the vector space of differential forms on a manifold M. Together with the exterior derivative we get a chain complex, the *de Rham complex* of M.

Definition 15. *A sequence of vector spaces and linear maps $A^* = (A^i, d^i)$,*

$$... \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \tag{30}$$

is called a chain complex if $d^{i+1} \circ d^i = 0$ for all i.

When we look at these complexes we have to look at the algebraic properties and we will need to introduce some of these.

Definition 16. *A sequence of vector spaces and linear maps is exact when $Im(f) = Ker(g)$.*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

In this sequence the kernel of g is given by: $Kerg = \{b \in B | g(b) = 0\}$ and the image of f is given by $Imf = \{f(a) | a \in A\}$. Note that the linear map $A \xrightarrow{f} B \longrightarrow 0$ is exact when the map f is surjective and the linear map $0 \longrightarrow B \xrightarrow{g} C$ is exact when the map g is injective.

Theorem 5. *If $f : A \to B$ is a linear map, and A and B are finite dimensional, then*

$$dimA = dim(ker(f)) + dim(im(f)) \tag{31}$$

If we require f to be injective, g to be surjective and $Im(f) = Ker(g)$ we get an shot exact sequence

Definition 17. *A short exact sequence (S.E.S) is of the form:*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

5.2 Cohomology of chain Complexes

From our chain complex $\mathcal{C}$ we have that $imd_{k-1} \subset kerd_k$. Because of this we can form a quotient space. This quotient space is the *kth cohomology vector space.*

Definition 18. *For a chain complex like equation (30) we can define the k-th cohomology vector space to be*

$$H^k(\mathcal{C}) = \frac{Kerd_k}{Imd_{k-1}}.$$

The elements of $Ker(d_k)$ are called a $k-cocycle$, which are the closed forms in the de Rham complex, and the elements of $Im(d_{k-1})$ are called k-coboundary which are the exact forms. We have the equivalence class $[c] \in H^k(\mathcal{C})$ of a k-cocycle $c \in ker(d_k)$. This is called a cohomology class.

We also define a chain map between two chain complexes. If we have $\mathcal{A}$ and $\mathcal{B}$ with the differentials d and d'. Consider a map $f : \mathcal{A} \to \mathcal{B}$, where we have the linear maps $f_k : A^k \to B^k$. We require these to commute with d and d' so that $d' \circ f_k = f_{k+1} \circ d$. This gives us a commutative diagram:

$$\begin{tikzcd}
... \arrow[r] & A^{k-1} \arrow[r, "d_{k-1}"] \arrow[d, "f_{k-1}"] & A^k \arrow[r, "d_k"] \arrow[d, "f_k"] & A^{k+1} \arrow[r] \arrow[d, "f_{k+1}"] & ... \\
... \arrow[r] & B^{k-1} \arrow[r, "d'_{k-1}"] & B^k \arrow[r, "d'_k"] & B^{k+1} \arrow[r] & ...
\end{tikzcd}$$

Note that this chain map induces a linear map in cohomology.

$$f^* : H^k(\mathcal{A}) \to H^k(\mathcal{B}) \tag{32}$$

where $f^*[a] = [f(a)]$.

Example 17. *A pullback of $f : M \to N$ is a chain map $f^* : \Omega^*(N) \to \Omega^*(M)$.*

Let us look at a sequence of chain complexes.

$$0 \longrightarrow \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \longrightarrow 0$$

This is a short exact sequence if the maps i and j are chain maps and we get the a short exact sequence of vector spaces:

$$0 \longrightarrow A^k \xrightarrow{i_k} B^k \xrightarrow{j_k} C^k \longrightarrow 0 \ . \tag{33}$$

Theorem 6. *Long exact sequence in cohomology*

A short exact sequence of chain complexes

$$0 \longrightarrow \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \longrightarrow 0$$

will give rise to a long exact sequence in cohomology

$$... \longrightarrow H^{k-1}(\mathcal{C}) \xrightarrow{\partial} H^k(\mathcal{A}) \longrightarrow H^k(\mathcal{B}) \longrightarrow H^k(\mathcal{C}) \longrightarrow H^{k+1}(\mathcal{A}) \longrightarrow ...$$

Proof. The whole proof is long. You have to prove the connecting homomorphism, exactness and that the mapping is well-defined. In this thesis we will only define the connecting homomorphism. The whole proof can be read for example in [2].

$$\begin{tikzcd} 0 \arrow[r] & A^k \arrow[r, "i"] \arrow[d, "f"] & B^k \arrow[r, "i"] \arrow[d, "g"] & C^k \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A^{k+1} \arrow[r, "i"] & B^{k+1} \arrow[r, "j"] & C^{k+1} \arrow[r] & 0 \end{tikzcd}$$

If we start with $[c] \in H^k(C)$. We know that the map j is onto, therefore there exist an element $b \in B^k s.t j'(b) = c$. Then $g(b) \in B^{k+1}$ is the kernel of the map j. Because of the commutativity of the diagram we have that $j(g(b)) = h(j(b))$ which is $h(c) = 0$ since $c \in Ker(h)$ (c is a cocycle). Also by the exactness of the sequence in the $(k+1)$ dimension we know that $ker(j) = im(i)$. Then $i(a) = g(b)$ for some $a \in A^{k+1}$. Once we have chosen b, a unique because i is injective. This also implies that $f(a) = 0$ since $i(f(a)) = g(i(a)) = g(g(b)) = 0$.

We can therefore say that a is a cocycle and this defines a cohomology class $[a]$. Set $\partial[c] = [a] \in H^{k+1}(A)$. □

5.3 The Mayer-Vietoris sequence

One way to calculate the de Rham cohomology is using the Mayer-Vietoris sequence. This is very useful tool. Instead of looking at the manifold as one whole piece you divide it up into subsets U_α. The cohomology for these subsets should be easier to compute. From this we get a sequence. This sequence calculates $H^*(U_1 \cup U_2)$ as a "function" of $H^*(U_1)$, $H^*(U_2)$ and $H^*(U_1 \cap U_2)$. We are still working with smooth manifolds. Suppose we have two open sets, U_1 and U_2, in M such that $M = (U_1 \cup U_2)$.
As show in e.g. [4] we for every integer $k \geq 0$ we will get an exact sequence.

$$0 \longrightarrow \Omega^k(M) \xrightarrow{i} \Omega^k(U_1) \oplus \Omega^k(U_2) \xrightarrow{j} \Omega^k(U_1 \cap U_2) \longrightarrow 0$$

Because of theorem 6 this short exact sequence will give us a long exact sequence in cohomology which is called the *Mayer-Vietoris sequence.*

Theorem 7. *Mayer-Vietoris*
Let U_1 and U_2 be open sets in M so that $U_1 \cup U_2 = M$, then from the connecting homomorphism

$$\partial_k : H^k(U_1 \cap U_2) \to H^{k+1}(M)$$

we will get a long exact sequence

$$... \xrightarrow{\partial_{k-1}} H^k(M) \xrightarrow{i} H^k(U_1) \oplus H^k(U_2) \xrightarrow{j} H^k(U_1 \cap U_2) \xrightarrow{\partial_k} H^{k+1}(M) \xrightarrow{i} ...$$

5.4 Homotopy

As stated earlier the topology doesn't change when you deform a space. Only if you tear the space. From this one could say that the cohomology for two different manifolds are the same if you can deform one of the figures to the other figures. Just as you can do a continuous deformation of the torus to a cup.

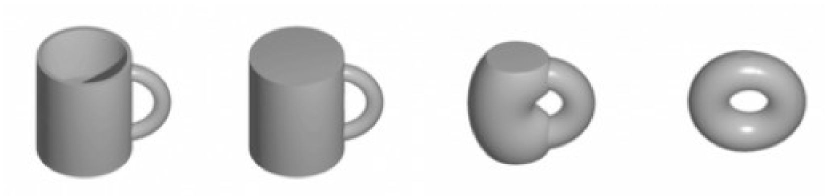


Figure 3[8]: The cup is the same homptopy as the torus.

The reader should be familiar with the concept of smooth homotopy and homotopy invariance. In this thesis we will only introduce *The homotopy axiom for de Rham cohomology.*

Theorem 8 (Homotopy axiom for de Rham cohomology)**.** *See for example [4] for proof.*
If you have manifolds M and N with the map $f : M \to N$ and it is homotopy equivalence then we get the isomorphism

$$f^* : H^*(N) \to H^*(M)$$

In the above example the cup could be manifold N and the doughnut would be manifold M such that the deformation would be the map f. Because of this, these two manifolds have the same cohomology groups.

Theorem 9. ***Poincare Lemma****. See for example [1] for proof.*
If the manifold is contractible it will have same cohomology as a point.

$$H^p(M) = H^*(point) = \begin{cases} \mathbb{R} & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases} \tag{34}$$

5.5 Künneth formula

One more useful tool is the Künneth formula. If we have a manifold that is the direct product of other manifolds we already have the cohomology group for, we can use this to calculate the groups for your original manifold. If you have two manifolds, the product will be $M_1 \times M_2 = M$. An easy example of this is the torus.

Example 18. *The torus*
The n-torus is a product of n 1-spheres.

$$T^n = S^1 \times ...S^1$$

Theorem 10. ***The Künneth formula***. *See e.g Nakahara*

$$H^p(M_1 \times M_2) = \oplus_{r+q=p}[H^r(M_1) \otimes H^q(M_2)]$$

If we rewrite it terms of Betti numbers, we get

$$h^r(M) = \sum_{p+q=r} h^p(M_1)h^q(M_2). \tag{35}$$

For the torus T^n the dimension becomes

$$h^r = dim(H^r(T^n)) = \binom{n}{r}. \tag{36}$$

6 Examples of cohomology

Now we should have enough tools to compute some examples of the cohomology of different manifolds. First we will look at the circle again, computing the cohomology using Mayer-Vietoris.

Example 19 (Example of Mayer Vietoris for $H^k(S^1)$)**.** .

We want to compute the cohomology for the circle S^1. As mentioned in section 5.3, if we can divide it in to subsections U_1 and U_2. The computation should be easier, and that is what we want to to. We divide the S^1 into to the subsets U_1 and U_2

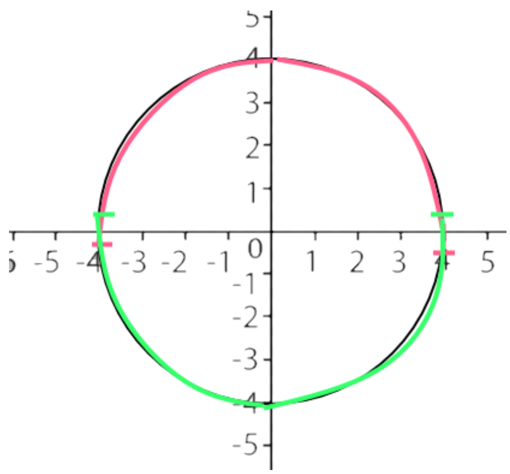


Figure 4: The two subsets U_1 and U_2.

We define U_1 to be the pink semicircle in figure 4 and U_2 to be the green semicircle.

$$U_1 : \{x^2 + y^2 = 16 \backslash (y < -0,5)\}$$
$$U_2 : \{x^2 + y^2 = 16 \backslash (y > 0,5)\}$$
$$U_1 \cup U_2 = S^1$$

Now we can use the Mayer-Vietoris sequence. Then first we set up the short exact sequence.
S.E.S:

$$0 \longrightarrow \Omega^p(U_1 \cup U_2) \longrightarrow \Omega^p(U_1) \oplus \Omega^p(U_2) \longrightarrow \Omega^p(U_1 \cap U_2) \longrightarrow 0$$

From the short exact sequence we extend it to the long exact sequence.
L.E.S:

$$0 \longrightarrow H^0(S^1) \longrightarrow H^0(U_1) \oplus H^0(U_2) \longrightarrow H^0(U_1 \cap U_2)$$

$$\longrightarrow H^1(S^1) \longrightarrow H^1(U_1) \oplus H^1(U_2) \longrightarrow H^1(U_1 \cap U_2) \longrightarrow 0$$

The subsets U_1 and U_2 has the same homotopy type as a point. Therefore, from the Poincare Lemma we get:

$$H^p(U_1) \cong H^p(U_2) \cong H^p(point) \cong \begin{cases} \mathbb{R} & \textit{if } p = 0 \\ 0 & \textit{if } p > 0 \end{cases}$$

From this we can then write out the long exact sequence as:

$$0 \longrightarrow \mathbb{R} \xrightarrow{\alpha} \mathbb{R}^2 \xrightarrow{\beta} \mathbb{R}^2 \xrightarrow{\gamma} H^1(S^1) \longrightarrow 0$$

We want do find the dimension for $H^1(S^1)$ denoted by $h^1(S^1)$. The map α is injective and the map γ is surjective. From the sequence we know that $|im(\alpha)| = 1$. From theorem 5,

$$|im(\beta)| + |ker(\beta)| = 2. \tag{37}$$

From exactness we know that $|im(\alpha)| = |ker(\beta)| = 1$. Since $|Ker(\beta)| = 1$, then from (37), $|Im(\beta)| = 1$.

Further, we see that $|im(\beta)| = |ker(\gamma)| = 1$. Next we see that

$$|im(\gamma)| + |ker(\gamma)| = 2.$$

Then $|im(\gamma)|$ must be 1. Therefore $|im(\gamma)| = h^1(S^1) = 1$.
From this we have shown the cohomology of all dimensions:

$$H^p(S^1) = \begin{cases} \mathbb{R} & \textit{if } p = 0,1 \\ 0 & \textit{if } p > 1 \end{cases}$$

Now we can move up to a higher dimensional circle. The 2-sphere.

Example 20. *Example $H^p(S^2)$*
We will again divide the manifold into smaller subsets. Here the result of the previous example will help us. Define the upper half of the sphere as U_1 and the the other half to be U_2. From this we see that $U_1 \cup U_2 = S^2$ and $U_1 \cap U_2 = S^1$. Then again from the Poincare lemma $U_1 \cong U_2 \cong \{point\}$.

We again use Mayer-Vietoris, but directly write the long exact sequence.

$$0 \longrightarrow H^0(S^2) \longrightarrow H^0(U_1) \oplus H^0(U_2) \longrightarrow H^0(S^1)$$

$$\longrightarrow H^1(S^2) \longrightarrow H^1(U_1) \oplus H^1(U_2) \longrightarrow H^1(S^1)$$

$$\longrightarrow H^2(S^2) \longrightarrow H^2(U_1) \oplus H^2(U_2) \longrightarrow H^2(S^1) \longrightarrow 0$$

From this we get the sequence we get an exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\alpha} \mathbb{R}^2 \xrightarrow{\beta} \mathbb{R} \xrightarrow{\gamma} H^1(S^2) \longrightarrow 0,$$

and $H^2(S^2) \cong H^1(S^1) \cong \mathbb{R}$.

Again, we have to do the same procedure as last example finding out the dimension of $H^1(S^2)$.
From the sequence we see that $|im(\alpha)| = 1$. *Then from theorem 5,*

$$|im(\beta)| + |ker(\beta)| = 2. \tag{38}$$

From exactness we know that $|im(\alpha)| = |ker(\beta)| = 1$. *Since* $|Ker(\beta)| = 1$, *then from* (38), $|Im(\beta)| = 1$.

Further, we know that $|im(\beta)| = |ker(\gamma)| = 1$. *Next we see that*

$$|im(\gamma)| + |ker(\gamma)| = 1.$$

Then $|im(\gamma)|$ *must be 0. Therefore* $|im(\gamma)| = h^1(S^1) = 0$.
We then have the solution for $H^p(S^2)$.

$$H^p(S^2) = \begin{cases} \mathbb{R} & \textit{if } p = 0 \\ 0 & \textit{if } p = 1 \\ \mathbb{R} & \textit{if } p = 2 \end{cases}$$

Example 21. *Cohomology for* S^3
Here we define U_1 *and* U_2 *as upper and lower hemispheres, so that* $U_1 \cup U_2 = S^3$ *and* $U_1 \cap U_2 = S^2$. *Then we get the long exact sequence*

$$\begin{aligned} 0 \longrightarrow H^0(S^3) &\longrightarrow H^0(U_1) \oplus H^0(U_2) \longrightarrow H^0(S^2) \\ \longrightarrow H^1(S^3) &\longrightarrow H^1(U_1) \oplus H^1(U_2) \longrightarrow H^1(S^3) \\ \longrightarrow H^2(S^3) &\longrightarrow H^2(U_1) \oplus H^2(U_2) \longrightarrow H^2(S^3) \\ \longrightarrow H^3(S^3) &\longrightarrow H^3(U_1) \oplus H^3(U_2) \longrightarrow H^3(S^2) \longrightarrow 0 \end{aligned}$$

From this we get the sequence we get an exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\alpha} \mathbb{R}^2 \xrightarrow{\beta} \mathbb{R} \xrightarrow{\gamma} H^1(S^3) \longrightarrow 0,$$

and $H^1(S^1) \cong H^2(S^2) \cong H^3(S^3) \cong \mathbb{R}$*.*

If we follow the same procedure as the two last example you will end up seeing that the dimension of $H^1(S^3)$ *must be 0. We also see that* $H^2(S^3) = 0$*. From this we have shown the cohomology of all dimensions:*

$$H^p(S^3) = \begin{cases} H^0(S^3) = \mathbb{R} \\ H^1(S^3) = 0 \\ H^2(S^3) = 0 \\ H^3(S^3) = \mathbb{R} \end{cases}$$

Whatever the dimension of the sphere is, when following the Mayer-Vietoris and the mathematical tools, a pattern emerges in the cohomology and it is possible to see what the solution for the n-sphere should be.

$$H^p(S^n) = \begin{cases} \mathbb{R} & \text{if } p = 0 \text{ or n} \\ 0 & \text{if } 0 < p < n \end{cases}$$

We know have found the cohomology of the same type of manifold but in different dimensions. Now lets look at another manifold, the torus. As mentioned in example 18 the torus is the product of circles. We then use the result we got from example 19 and apply the Künneth formula from section 5.

Example 22. ***The cohomology of a 2-Torus*** (T^2)
We write the 2-torus as the product of S^1*:* $T^2 = S^1 \times S^1$*.*

$$H^p(T^2) = \begin{cases} H^0(T^2) = H^0(S^1) \otimes H^0(S^1) \cong \mathbb{R} \\ H^1(T^2) = H^0(S^1) \otimes H^1(S^1) \oplus H^1(S^1) \otimes H^0(S^1) \cong \mathbb{R}^2 \\ H^2(T^2) = H^1(S^1) \otimes H^1(S^1) \cong \mathbb{R} \end{cases}$$

Example 23. ***The cohomology of a 3-Torus*** (T^3)

As seen earlier we can define $T^3 = S^1 \times S^1 \times S^1$ *Again using the Künneth formula and the Betti number from equation (35) and (36)*

$$H^p(T^n) \simeq \mathbb{R}^{\binom{n}{k}}$$

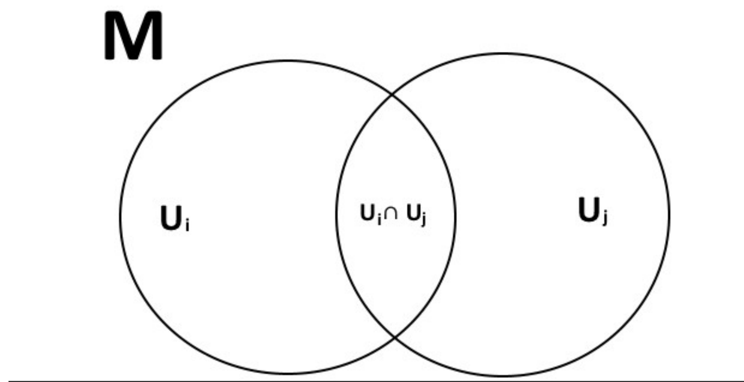
so therefore

$$H^p(T^3) = \begin{cases} H^0(T^3) = \mathbb{R}^{\binom{3}{0}} \cong \mathbb{R} \\ H^1(T^3) = \mathbb{R}^{\binom{3}{1}} \cong \mathbb{R}^3 \\ H^2(T^3) = \mathbb{R}^{\binom{3}{2}} \cong \mathbb{R}^3 \\ H^3(T^3) = \mathbb{R}^{\binom{3}{2}} \cong \mathbb{R} \end{cases}$$

In section 4.3 we said that the solutions to equation of motion of 3d Chern-Simons theory was given by $H^1(M^3)$. In example (21) and (23) we calculated this group for two different three-dimensional manifolds. From the first example where the manifold was S^3 we see that there is no solution. But from example (23) we found out that $H^1(T^3) = \mathbb{R}^3$. We then know that the solutions span a three-dimensional space.

Finally, let's consider an example of a gauge theory without an action.

Example 24. *Conservative vector-field on a manifold*
Let us look at a physical theory locally described of a conservative vector-field on a manifold. This could for example be wind or heat-flow.



The manifold M given by to subsets U_i and U_j

On U_1 we have the physics given by $\vec{v} = \nabla f_i$. Locally it is $v = df_i$ (1-form). We see that we can change f, $f \to f + c$ without changing the physic because of gauge symmetry, $(dc = 0)$.
Note that on $U_i \cap U_j$ we have that $v_i = v_j$.

$$\Rightarrow d(f_i - f_j) = 0$$

$$f_i - f_j = c_{ij}$$

$$f_i = f_j + c_{ij}$$

We think of c_{ij} as the difference in how we measure the pressure/temperature in U_i and U_j. It is a local gauge transformation. ***Given our v on M, is it possible to find a global function f such that $v = df$?*** *That is, is v described by a global pressure/temperature field?*

Note that $[v] \in H^1(M)$ because $dv = 0$. If $H^1(M) = \{0\}$ then we must have $v = df$ for a global f. If we have a compact and closed manifold and $v = df$, then f will reach a maximum or minimum on M where $v = df = 0$.
$\Rightarrow$ The vector-field v (wind/heat-flow) vanishes somewhere!

From example 20 we get a trivial cohomology, $(H^1(S^2) = 0)$, and thus it must be windless somewhere on Earth.

7 Summary

In this thesis we have encountered geometrical objects called manifolds which describe higher dimensional spaces. We have seen that we can take a point in our space and look at the area around it such that it looks like a Euclidean space even thought it may not be the case globally. We saw that two subsets can have different coordinates, but if the manifold is smooth there exists a transition function that is infinitely differentiable.

Since manifolds exist in higher dimension we had to introduce mathematical objects that made it possible to do math on these geometrical figures. The differential forms made it possible to do calculus, using tools such as the exterior product, exterior derivative, and integration.

After introducing the generalized Stokes theorem we saw that there exists some invariants of manifolds. These are topological. These topological properties can be classified into characteristic classes such as cohomology. This topological property is the essence of a topological field theory. Because of gauge symmetry the solution space of the E.O.M to the topological field theory where given by closed form modulo exact form, which is the De Rham cohomology groups of the manifolds. To compute these groups we introduced some mathematical tools such as chain complexes, Mayer-Vietoris sequence, homotopy and the Künneth formula.

As we saw in the last section, two examples of two different 3-dimensional manifolds where calculated, but only the cohomology of the torus gave rise to an answer where the solution space to Chern-Simons theory exist in the three-dimensional space. For the last example we have shown that there exists a windless place on Earth.

We have shown that the De Rham cohomology groups describes the topology of a higher dimensional geometrical object, but these groups also characterise the solution space to the equations of motion to a topological field theory. Thus we have found a connection between physics and geometry.

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