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ORIGINAL ARTICLE

Axiomatizing additive multi-effort contests

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Abstract
A rent seeking model is axiomatized where players exert multiple additive efforts which are substitutable in the contest success function. The axioms assume the sufficiency of exerting one effort, and that adding an amount to one effort and subtracting the same amount from a second equivalent substitutable effort keeps the winning probabilities unchanged. In contrast, the multiplicative Cobb–Douglas production function in the earlier literature requires players to exert all their complementary efforts. The requirement follows from assuming a homogeneity axiom where an equiproportionate change in two players' matched efforts does not affect the winning probabilities. This article abandons the homogeneity axiom and assumes an alternative axiom where the winning probabilities remain unchanged when a fixed positive amount is added to all players' efforts. This article also assumes a so-called summation axiom where the winning probabilities remain unchanged when a player substitutes an amount of effort from one effort into another effort. The summation axiom excludes multiplicative production functions, and furnishes a foundation for additive production functions.

Keywords Rent seeking · Additive efforts · Axioms · Contest success function

JEL Classification C70 · C72 · D72 · D74

Introduction

Background

Axiomatization within the contest literature has progressed gradually. Axioms ensure solid, consistent, theoretical foundations from which other statements are logically derived. Whereas multiplicative efforts have been axiomatized in the literature, multiple additive efforts have not been axiomatized which is the objective of

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this article. Multiple additive efforts are relevant when contending players have multiple non-overlapping available efforts. An example is competition for elected office applying political strategists, campaigners, media operatives, speech writers, advertisements, ground troops speaking with people, etc. These efforts operate additively, not multiplicatively. The absence of one or a few efforts does not obliterate the overall effort. Rents can be procured through influence campaigns, lobbying, advertisements, product developments, fighting, etc. (Hirshleifer 1995, p. 28). Examples of rents are Research and Development budgets, licenses, privileges, election opportunities, and competition for budgets by interest groups.

Contribution

Whereas multiplicative efforts complement each other, additive efforts substitute each other in the contest success function. When multiplying the contest success function with the rent, and subtracting the expenditure of exerting efforts, a new function arises where multiple efforts can be substitutes, complements, or neither substitutes nor complements.

Influenced by Tullock (1980) and Krueger (1974), Arbatskaya and Mialon (2010, p. 24) state for multiplicative efforts that firms may obtain rents from the government by improved efficiency, or by lobbying or bribery. This presupposes that both efficiency and lobbying are required to succeed in rent seeking. Demanding that all efforts, out of many efforts, are strictly positive can be realistic for some phenomena such as career promotions to senior positions. However, for some phenomena, Cobb–Douglas type multiplicative efforts become increasingly unrealistic as the number of efforts increases since each effort must be strictly positive for success to follow.

The axiomatization of additive efforts in this article allows improved efficiency without lobbying, lobbying without improved efficiency, and, of course, both improved efficiency and lobbying operating additively. The article assumes that efforts may differ regarding unit costs, impacts, and contest intensities. Different contest intensities are not commonly analyzed in the literature.

Literature

Axiomatization is made of symmetric contests by Skaperdas (1996), asymmetric contests by Clark and Riis (1998), difference form contest success functions by Cubel and Sanchez-Pages (2016), group contests by Münster (2009), multiplicative Cobb–Douglas type efforts by Arbatskaya and Mialon (2010), contest success functions for networks by Bozbay and Vesperoni (2018), and multiple types of investments by Rai and Sarin (2009).

Blavatskyy (2010) axiomatizes contests when a draw is possible, and the highest effort does not necessarily guarantee winning. Vesperoni and Yildizparlak (2019) axiomatize single-winner contests when a draw is possible, accounting for different rents when a win or a draw occurs. Lu and Wang (2015) axiomatize multi-prize nested lottery contests. Lu and Wang (2016) axiomatize the reverse nested lottery contest, specifying the probabilities of being lowest ranked. Epstein and Hefeker (2003) assume that each player has two efforts; i.e. rent seeking and, if exerted, a second effort which may strengthen the first effort multiplicatively.

Yildirim (2005) assumes that two players first choose one effort each, and thereafter choose whether to add one effort. Melkonyan (2013) assumes that each player ex ante forfeits one resource and commits one resource. Ex post the winning player expends his committed resource. Hausken (2020a, b) analyzes additive efforts for multiple players. Hirshleifer (1995), Skaperdas and Syropoulos (1997) and Hausken (2005) analyze two efforts for each player labelled production and appropriation. Clark and Konrad (2007) analyze many-dimension contests, of which some have to be won to gain the rent. Keskin (2018) considers cumulative prospect theory preferences in rent seeking contests. Dickson et al. (2018) assess how rents may become more strongly contested when they become scarcer. Mercier (2018) develops a revelation mechanism where a contest designer selects a subset of contestants to maximize his profits.

Further literature considers sabotage. Konrad (2000) assumes two efforts which improves a player's contest success, and decreases the contesting players' success, respectively. Chen (2003) assumes two efforts which enhances a player's performance, and sabotages the contesting players' performance, respectively. Amegashie and Runkel (2007) analyze sabotage between four players in a three-stage elimination contest. Minchuk (2021) considers reimbursement as a tool to reduce sabotaging in rent seeking contests. Krakel (2005) considers players which first help, sabotage, or abstain from action, and thereafter exert effort to win the contest. See Chowdhury and Gürtler (2015) for a survey.

Section 2 considers axioms for multiple additive efforts. Section 3 discusses the axioms and considers limitations and future research. Section 4 concludes.

Axioms

Appendix A shows the nomenclature. A static game between two players is analyzed. Player 3 is introduced as a dummy player in the axioms to facilitate the analysis, i.e. to consider subcontests between any two of three players. Player $i \in \{1,2\}$ exerts K_i rent seeking efforts x_{ik}, $k \in \{1,\dots,K_i\}$ at unit cost $c_{ik} > 0$. The rent has value $S \geq 0$. Define $\boldsymbol{x_i} = \left(x_{i1},\dots,x_{iK_i}\right)$ as the vector of player $i's$ K_i efforts. Player $i's$ expected utility is

$$\mathrm{u}_i\left(\boldsymbol{x_1},\boldsymbol{x_2}\right) = p_i\left(\boldsymbol{x_1},\boldsymbol{x_2}\right)S - \sum_{k=1}^{K_i} c_{ik}x_{ik}, \tag{1}$$

where $p_i^{(2)}\left(\boldsymbol{x_1},\boldsymbol{x_2}\right) = p_i\left(\boldsymbol{x_1},\boldsymbol{x_2}\right)$ and $p_i^{(3)}\left(\boldsymbol{x_1},\boldsymbol{x_2},\boldsymbol{x_3}\right) = p_i\left(\boldsymbol{x_1},\boldsymbol{x_2},\boldsymbol{x_3}\right)$ are player i's contest success functions, i.e. winning probabilities, in the two-player and three-player contests, respectively. For simplicity we suppress the superscripts (2) and (3) and let the number of arguments in p_i determine the domain, i.e. $\left(\boldsymbol{x_1},\boldsymbol{x_2}\right) \in \mathbb{R}_+^{\sum_{i=1}^2 K_i}$ for the two-player contest and $\left(\boldsymbol{x_1},\boldsymbol{x_2},\boldsymbol{x_3}\right) \in \mathbb{R}_+^{\sum_{i=1}^3 K_i}$ for the three-player contest. For risk neutral players, p_i can be conceived as player i's proportion of the rent won. In accordance with Arbatskaya and Mialon (2010), we consider subcontests between any two of three players, $i \in I \equiv \{1, 2, 3\}$. We define $\boldsymbol{x} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x}_3)$ as the vector of the three players' overall number $\sum_{i=1}^{3} K_i$ of efforts, $\boldsymbol{x_{-i}}$ as the vector of two of the players' $\sum_{j=1, j \neq i}^{3} K_j$ efforts except player $i's$ K_i efforts, $\boldsymbol{x}_{-ik}$ as the vector of all the three players' $\left(\sum_{i=1}^{3} K_i\right) - 1$ efforts except player $i's$ effort $k \in \{1, \ldots, K_i\}$, and $p_i(\boldsymbol{x})$ as player $i's$ contest success function.

Axiom 1 Probability. (i) For all $i \in I$ and $x_{ik} \in \mathbb{R}_{+}^{1}$ (i.e. $x_{ik} \geq 0$), $p_i(\boldsymbol{x}) \geq 0$ and $\sum_{i \in I} p_i(\boldsymbol{x}) = 1$.

(ii) For all $i \in I$, if $x_{ik} \in \mathbb{R}_{++}^{1}$ (i.e. $x_{ik} > 0$) for at least one $k \in \{1, \ldots, K_i\}$, $\boldsymbol{x}_{-ik} \in \mathbb{R}_{+}^{\sum_{i=1}^{3} K_i - 1}$, then $p_i(\boldsymbol{x}) > 0$. If $\boldsymbol{x_{-i}} = (\boldsymbol{0}, \boldsymbol{0})$ and $x_{ik} \in \mathbb{R}_{++}^{1}$ for at least one $k \in \{1, \ldots, K_i\}$, then $p_i(\boldsymbol{x}) = 1$.

(iii) For all $i \in I$, if $\boldsymbol{x_i} = \boldsymbol{0}$, $\boldsymbol{x_{-i}} \in \mathbb{R}_{+}^{\sum_{i=1}^{2} K_i}$, and $\boldsymbol{x_{-i}} \neq (\boldsymbol{0}, \boldsymbol{0})$, then $p_i(\boldsymbol{x}) = 0$.

(iv) If $\boldsymbol{x} = (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$, then $p_1(\boldsymbol{x}) = p_2(\boldsymbol{x})$ and $p_3(\boldsymbol{x}) = 0$.

Axiom 1(i) assumes that a player's winning probability is non-negative regardless of all the players' efforts, and that the sum of all the players' winning probabilities is one since only one rent is allocated and the rent is inevitably allocated. Axiom 1(ii) expresses that if player i exerts at least one effort which is strictly positive, then he wins with strictly positive probability, for all choices of efforts by the other players, and for all other choices of efforts by player i. Throughout Axiom 1 we assume that the sum of the probabilities is one. Then the whole rent S always gets distributed. Arbatskaya and Mialon (2010) do not necessarily assume that all winning probabilities sum to one, in their Axiom 1(i). More specifically, they require in their Axiom 1(iv) that all the K_i efforts by minimum one player must be strictly positive so that the winning probabilities sum to one. Axiom 1(ii) contrasts with Arbatskaya and Mialon (2010) who require that all player i's K_i efforts must be strictly positive so that his probability of winning the rent is strictly positive for all possible combinations of efforts by the other players. Axiom 1(ii) also expresses that if player i chooses minimally one effort which is strictly positive, while all the other players choose no efforts (i.e. efforts equal to zero), then player i wins the rent with probability one. Since all the winning probabilities sum to one, all other players' winning probabilities are zero. This contrasts with Arbatskaya and Mialon (2010). Axiom 1(iii) is the same as Arbatskaya and Mialon's (2010) Axiom 1(iii), except that they do not require $\boldsymbol{x_{-i}} \neq (\boldsymbol{0}, \boldsymbol{0})$ which we require to enable the new Axiom 1(iv) where all players exert no efforts. Axiom 1(iii) expresses that if player i exerts no efforts, and minimally one other player exerts minimally one effort which is strictly positive, then player i is guaranteed not to win the rent (i.e. wins the rent with probability zero). Axiom 1(iv) assumes that if all players withdraw from exerting effort, then the winning probabilities of players 1 and 2 are equal, while the dummy player 3's winning probably is zero.

The four parts of Axiom 1 are independent and do not imply each other for the following reasons. First, Axiom 1(i) is a simple assumption. Second, Axiom 1(ii) requires that at least one effort by player i is strictly positive, i.e., at a minimum, $x_{ik} \in \mathbb{R}_{++}^{1}$. Axiom 1(ii) also assumes complementarily no effort by all the other players when $\boldsymbol{x_{-i}} = (\boldsymbol{0}, \boldsymbol{0})$. Third, Axiom 1(iii) assumes no effort by player i, and at least one positive effort by at least one other player, i.e. $\boldsymbol{x}_{-i} \neq (\boldsymbol{0}, \boldsymbol{0})$. Fourth, Axiom 1(iv) assumes no effort by any player, i.e. $\boldsymbol{x} = (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$.

Axiom 2 Independence from irrelevant alternatives. For all $i \neq r \neq s \in I$, the odds ratio $\frac{p_i(\boldsymbol{x})}{p_r(\boldsymbol{x})}$ does not depend on $\boldsymbol{x_s}$ for $\boldsymbol{x_i} \in \mathbb{R}_{+}^{K_i}$, $\boldsymbol{x_r} \in \mathbb{R}_{++}^{K_r}$, and $\boldsymbol{x_s} \in \mathbb{R}_{+}^{K_s}$.

Axiom 2 is a version of the independence from irrelevant alternatives, and is as in Arbatskaya and Mialon (2010). It states that the ratio of any two players' winning probabilities does not depend on the third player's efforts.

Axiom 3 Inactive third player. For all $i \in \{1, 2\}$, $(\boldsymbol{x_1}, \boldsymbol{x_2}) \in \mathbb{R}_{+}^{\sum_{i=1}^{2} K_i}$, $p_i(\boldsymbol{x_1}, \boldsymbol{x_2}) = p_i(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x}_3 = \boldsymbol{0})$.

Axiom 3 expresses that if the third player is inactive, then the contest reduces to a two-player contest. The number of arguments in p_i determines the domain, as explained above. Axiom 3 is as in Arbatskaya and Mialon (2010).

Lemma 1 *If Axioms 1, 2, 3 hold, then player i's contest success function, $i \in \{1, 2\}$, in a two-player contest where each player i exerts K_i efforts is*

$$p_i(\boldsymbol{x_1}, \boldsymbol{x_2}) = \begin{cases} \frac{f_i(\boldsymbol{x}_i)}{f_1(\boldsymbol{x}_1) + f_2(\boldsymbol{x}_2)} \text{ if } x_{1k} \in \mathbb{R}_{++}^1 \text{ for at least one } k \in \{1, \dots, K_1\} \\ \qquad \text{or } x_{2k} \in \mathbb{R}_{++}^1 \text{ for at least one } k \in \{1, \dots, K_2\} \\ 1/2 \text{ if } \boldsymbol{x} = (\boldsymbol{0}, \boldsymbol{0}) \end{cases}, \quad (2)$$

where player i's production (impact) function $f_i(\boldsymbol{x_i})$ satisfies

$$f_i(\boldsymbol{x_i}) \begin{cases} > 0 \text{ if } x_{ik} \in \mathbb{R}_{++}^1 \text{ for at least one } k \in \{1, \dots, K_i\} \\ \geq 0 \text{ if } x_{ik} \in \mathbb{R}_{+}^1 \text{ for at least one } k \in \{1, \dots, K_i\} \\ = 0 \text{ if } \boldsymbol{x_i} = \boldsymbol{0} \end{cases}. \quad (3)$$

Proof Appendix B.

Lemma 1 differs from Arbatskaya and Mialon (2010) e.g. in that it is sufficient that one of player $i's$ K_i efforts is strictly positive or positive to cause $f_i(\boldsymbol{x_i})$ and $p_i(\boldsymbol{x_1}, \boldsymbol{x_2})$ to be strictly positive or positive. That implies $x_{ik} \in \mathbb{R}_{+}^1 \backslash \mathbb{R}_{++}^1 = \{0\}$ which is a singleton. In contrast, they demand strict positivity for all K_i efforts by player i. That implies $\boldsymbol{x_i} \in \mathbb{R}_{+}^{K_i} \backslash \mathbb{R}_{++}^{K_i}$ expressing a positive orthant compilation of coordinate hyperplanes. The probability $p_i(\boldsymbol{x_1}, \boldsymbol{x_2}) = 0$ follows when $f_i(\boldsymbol{x_i}) = 0$ and $f_{-i}(\boldsymbol{x}_{-i}) > 0$, and $p_i(\boldsymbol{x_1}, \boldsymbol{x_2}) = 1$ follows when $f_i(\boldsymbol{x_i}) > 0$ and $f_{-i}(\boldsymbol{x}_{-i}) = 0$.

Axiom 4 Conventional monotonicity. For all $i \neq r$, $i \in \{1, 2\}$, $r \in \{1, 2\}$, $k \in \{1, \dots, K_i\}$, and $n \in \{1, \dots, K_r\}$, $p_i(\boldsymbol{x_1}, \boldsymbol{x_2})$ is nondecreasing in x_{ik} if $x_{ik} \in \mathbb{R}_{+}^1$ and $x_{rn} \in \mathbb{R}_{+}^1$ (i.e. $x_{ik} \geq 0$ and $x_{rn} \geq 0$) for at least one $n \in \{1, \dots, K_r\}$, and continuous and strictly increasing in x_{ik} if $x_{ik} \in \mathbb{R}_{++}^1$ and $x_{rn} \in \mathbb{R}_{++}^1$ for at least one $n \in \{1, \dots, K_r\}$.

Axiom 4 states that each player's success probability is nondecreasing in any given effort by that player if that given effort is positive and at least one effort by the other player is positive. Furthermore, each player's success probability is continuous and strictly increasing in any given effort by that player if that given effort is strictly positive and at least one effort by the other player is strictly positive. This contrasts with Arbatskaya and Mialon's (2010) Axiom 4 which requires that each player's success probability is continuous and strictly increasing with that player's efforts in each activity if the efforts by both players in every activity are positive; and otherwise, each player's success probability is nondecreasing with that player's efforts in each activity.

Axiom 5 Homogeneity. For all $i \in \{1,2\}$ and $\lambda > 0$, $p_i(\lambda \boldsymbol{x_1}, \lambda \boldsymbol{x_2}) = p_i(\boldsymbol{x_1}, \boldsymbol{x_2})$.

Axiom 5 equals Münster's (2009, p. 351) Axiom 6. It states that an equiproportionate change in both players' effort vectors $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ does not impact the players' success probabilities. This requirement differs from Arbatskaya and Mialon's (2010) Axiom 5 which requires that an equiproportionate change in corresponding single efforts x_{1k} and x_{2k}, $k \in \{1, \dots, K_i\}$, for both players does not impact the players' success probabilities. Their Axiom 5 implies Cobb–Douglas production functions. To illustrate the difference, inserting the special case $f_i(\boldsymbol{x_i}) = x_{i1} + x_{i2}$ into (2) gives $p_i(\boldsymbol{x_1}, \boldsymbol{x_2}) = \frac{x_{i1}+x_{i2}}{x_{11}+x_{12}+x_{21}+x_{22}}$ (when $\boldsymbol{x} \neq (\boldsymbol{0}, \boldsymbol{0})$). Axiom 5 is then satisfied since λ gets abbreviated so that $p_i(\boldsymbol{x_1}, \boldsymbol{x_2}) = \frac{\lambda(x_{i1}+x_{i2})}{\lambda(x_{11}+x_{12})+\lambda(x_{21}+x_{22})} = \frac{x_{i1}+x_{i2}}{x_{11}+x_{12}+x_{21}+x_{22}}$. However, if λ is applied on only one x_{i1}, and not on the whole vector $\boldsymbol{x_i}$, inserting $f_i(\boldsymbol{x_i}) = \lambda x_{i1} + x_{i2}$ into (2) gives $p_i(\boldsymbol{x_1}, \boldsymbol{x_2}) = \frac{\lambda x_{i1}+x_{i2}}{\lambda x_{11}+x_{12}+\lambda x_{21}+x_{22}}$ (when $\boldsymbol{x} \neq (\boldsymbol{0}, \boldsymbol{0})$). Axiom 5 is then not satisfied since λ cannot be abbreviated.

An alternative to homogeneity in Axiom 5 is the following where $\mathbf{1} = (1, \dots, 1)$ is a vector with K_i elements equal to 1.

Axiom 6 Alternative to homogeneity. For all $i \in \{1,2\}$ and $\lambda > 0$, $p_i(\boldsymbol{x_1} + \lambda \mathbf{1}, \boldsymbol{x_2} + \lambda \mathbf{1}) = p_i(\boldsymbol{x_1}, \boldsymbol{x_2})$ if players 1 and 2 have equivalent effort technologies $f_1(\boldsymbol{x_1}) = f_2(\boldsymbol{x_2})$.

Axiom 6, which departs from Arbatskaya and Mialon (2010), states that adding a constant λ to all efforts by both players does not impact the players' success probabilities.

Axiom 7 Multiplicative and additive monotonicity. For all $i \neq s$, $i \in \{1,2\}$, $s \in \{1,2\}$, if $\boldsymbol{x_i} b + a\mathbf{1} \geq \boldsymbol{x_i}$, then $p_i(\boldsymbol{x_i} b + a\mathbf{1}, \boldsymbol{x_s}) \geq p_i(\boldsymbol{x_i}, \boldsymbol{x_s})$.

A sufficient condition for $\boldsymbol{x_i} b + a\mathbf{1} \geq \boldsymbol{x_i}$ is $a \in \mathbb{R}^1_+$ and $b \geq 1$. Then Axiom 7 states that adding a positive constant a to a player's effort vector $\boldsymbol{x_i}$, and multiplying a player's effort vector $\boldsymbol{x_i}$ with a constant b larger than one, causes higher success probability.

Axiom 8 Anonymity. For any $\left(\boldsymbol{x_1},\boldsymbol{x_2}\right) \in \mathbb{R}_{+}^{\sum_{i=1}^{2} K_i}$, $p_1\left(\boldsymbol{x_1},\boldsymbol{x_2}\right) = p_2\left(\boldsymbol{x_2},\boldsymbol{x_1}\right)$ if $\boldsymbol{x_1} = \boldsymbol{x_2}$ and players 1 and 2 have equivalent effort technologies $f_1\left(\boldsymbol{x_1}\right) = f_2\left(\boldsymbol{x_2}\right)$.

Axiom 8 assumes that interchanging players to enable anonymity is possible only when effort technologies are equivalent.

Axiom 9 Summation. For all $i \in \{1,2\}$ and $k \in \left\{1,\dots,K_i\right\}$, fixate any $\Delta_i > 0$ such that $\Delta_i \le x_{ik}$. Define

$$\hat{\boldsymbol{x}}_{\boldsymbol{ikn}} = \left(x_{i1},\dots,x_{i,k-1},x_{ik} - \Delta_i, x_{i,k+1},\dots,x_{i,n-1},x_{in} + \Delta_i, x_{i,n+1},\dots,x_{iK}\right). \tag{4}$$

Then

$$p_i\left(\hat{\boldsymbol{x}}_{\boldsymbol{ikn}},\boldsymbol{x_{-i}}\right) = p_i\left(\boldsymbol{x_i},\boldsymbol{x_{-i}}\right) \tag{5}$$

assuming that efforts x_{ik} and x_{in} in the contest success function $p_i\left(\boldsymbol{x_1},\boldsymbol{x_2}\right)$ are substitutable for all $k \in \left\{1,\dots,K_i\right\}$ and $n \in \left\{1,\dots,K_i\right\}$ while preserving $\Delta_i > 0$.

Axiom 9 states that if two efforts x_{ik} and x_{in} are substitutable in the contest success function $p_i\left(\boldsymbol{x_1},\boldsymbol{x_2}\right)$, and player i exerts less effort, by the amount Δ_i, into effort k yielding effort $x_{ik} - \Delta_i$, and exerts more effort, by the same amount Δ_i, into effort n yielding effort $x_{in} + \Delta_i$, then the winning probabilities of both players are unaffected. Hence only the sum $x_{ik} - \Delta_i + x_{in} + \Delta_i = x_{ik} + x_{in}$ matters, independently of Δ_i. Axiom 9 equals Münster's (2009, p. 354) Axiom 8 for groups where, if one group member exerts more effort while another group member exerts equally less effort, the group's winning probability remains unchanged.

Lemma 2 *Consider any contest success function $p_i\left(\boldsymbol{x_1},\boldsymbol{x_2}\right)$ from the class characterized in Lemma 1. A success function $p_i\left(\boldsymbol{x_1},\boldsymbol{x_2}\right)$ satisfying Axioms 1, 2, 3, 4, 7 is compatible with a CES production (impact) function $f_i\left(\boldsymbol{x_i}\right)$ taking the form*[1]

$$f_i\left(\boldsymbol{x_i}\right) = \left(\sum_{k=1}^{K_i} d_{ik} x_{ik}^{m_{ik}}\right)^r, \tag{6}$$

where $d_{ik} \ge 0$ scales impact proportionally, $m_{ik} \ge 0$ is player i's contest intensity, and $r \ge 0$. Axiom 5 is compatible with (6) when $m_{ik} = m$. Axiom 8 is compatible with (6) when $\boldsymbol{x_1} = \boldsymbol{x_2}$, $d_{1k} = d_{2k}$, $m_{1k} = m_{2k}$, $K_1 = K_2$. Axiom 9 is compatible with (6) when $d_{ik} = d_{in}$ and $m_{ik} = m_{in} = 1$, $k,n \in \left\{1,\dots,K_i\right\}$, required for substitutability between x_{ik} and x_{in} in Axiom 9.

Proof Münster (2009, p. 355) proves Lemma 2 in his Proposition 2 assuming his axioms A1, A2, A4, A5, A6, A8 for the special case that $d_{ik} = d_i$ and $m_{ik} = 1$. Appendix C generalizes Münster's (2009) proof from x_{ik} to $d_{ik}x_{ik}^{m_{ik}}$ inside the

[1] Equation (6) is a generalization of the CES (Constant elasticity of substitution) function common in production economics.

summation applying Axioms 1, 2, 3, 4, 7, some of which correspond to Münster's (2009) axioms.

Lemma 3 *Consider any contest success function $p_i(\boldsymbol{x_1}, \boldsymbol{x_2})$ from the class characterized in Lemma 1. A success function $p_i(\boldsymbol{x_1}, \boldsymbol{x_2})$ satisfying Axioms 1, 2, 3, 4, 7 is compatible with a production (impact) function $f_i(\boldsymbol{x_i})$ taking the logistic form*

$$f_i(\boldsymbol{x_i}) = a_i \mathrm{Exp}\left(r \sum_{k=1}^{K_i} d_{ik} x_{ik}^{m_{ik}} \right), \tag{7}$$

where $d_{ik} \geq 0$, $m_{ik} \geq 0$, $a_i \geq 0$, and $r \geq 0$. Axiom 6 is compatible with (7) when $d_{1k} = d_{2k}, m_{ik} = 1, K_1 = K_2$.

Proof Münster (2009, pp. 355–356) proves Lemma 3 in his Proposition 3 by contradiction assuming his axioms A1, A2, A4, A5, A7, A8 for the special case that $d_{ik} = d_i$ and $m_{ik} = 1$. Appendix D generalizes Münster's (2009) proof from x_{ik} to $d_{ik} x_{ik}^{m_{ik}}$ inside the summation applying Axioms 1, 2, 3, 4, 7, some of which correspond to Münster's (2009) axioms.

Whereas $p_i(\boldsymbol{x_1}, \boldsymbol{x_2})$ interpreted as the ratio form contest success function (Skaperdas, 1996; Tullock, 1980) in Lemma 2 is compatible with Axiom 5 when $m_{ik} = m$, $p_i(\boldsymbol{x_1}, \boldsymbol{x_2})$ interpreted as the logistic contest success function (Hirshleifer, 1989) in Lemma 3 is compatible with Axiom 6 when $d_{1k} = d_{2k}, m_{ik} = 1, K_1 = K_2$. Arbatskaya and Mialon (2010) also assume contest intensities $m_{ik} = 1$. A contest success function $p_i(\boldsymbol{x_1}, \boldsymbol{x_2})$ and production (impact) function $f_i(\boldsymbol{x_i})$ from the classes characterized in Lemma 2 is compatible with Axiom 9 only for the special case that $d_{ik} = d_{in}$ and $m_{ik} = m_{in} = 1$, $k, n \in \{1, \dots, K_i\}$. That special case, usually restricted to $d_{ik} = d_{in} = 1$, is the case commonly studied in the literature, and thus constitutes a comparison benchmark.

Lemma 4 *Consider any contest success function $p_i(\boldsymbol{x_1}, \boldsymbol{x_2})$ from the class characterized in Lemma 1. If this success function $p_i(\boldsymbol{x_1}, \boldsymbol{x_2})$ satisfies Axiom 1(ii), or Axiom 9, its production (impact) function $f_i(\boldsymbol{x_i})$ cannot take the form of the Cobb–Douglas production function*

$$f_i(\boldsymbol{x_i}) = d_i \prod_{k=1}^{K_i} x_{ik}^{m_{ik}}. \tag{8}$$

In contrast, if $p_i(\boldsymbol{x_1}, \boldsymbol{x_2})$ in Lemma 1 satisfies Axiom 5 and $m_{ik} = m$, it is possible for $f_i(\boldsymbol{x_i})$ to take the form in (8).

Proof Appendix E.

Axiom 9 drives a wedge between the additive production function in (6) and multiplicative production functions. Axiom 9 is compatible with (6) when $d_{ik} = d_{in}$

and $m_{ik} = m_{in} = 1$, $k, n \in \{1, \dots, K_i\}$. Axiom 9 excludes multiplicative production functions including Cobb–Douglas production functions since it is mathematically impossible to find one special case where a fixed amount Δ_i is added to one effort and subtracted from another effort while, simultaneously, the product of the efforts, where each effort is raised to an exponent as in the Cobb–Douglas production function, remains unchanged. The model has $3K_1 + 3K_2 + 2$ parameters, i.e. $K_1 + K_2$ unit efforts costs c_{ik}, $K_1 + K_2$ scaling parameters d_{ik}, $K_1 + K_2$ contest intensities m_{ik}, and $K_1 + K_2$ for the numbers of efforts x_{ik} which are the players' strategic choice variables.

Discussion, limitations and future research

This article's departs from Arbatskaya and Mialon (2010) is Axiom 6 which states that winning probabilities remain unchanged when a fixed positive amount is added to all players' efforts. One advantage of Axiom 6 is that it allows for additive efforts and therefore efforts can be perfect substitutes, in contrast to Arbatskaya and Mialon (2010). That efforts might be perfect substitutes affects the distribution of efforts among different activities which has important consequences when analyzing possible equilibria, rent dissipation, contest design, etc., as illustrated by Hausken (2020a, b).

Axiom 6 is reminiscent of Translation Invariance but imposes directly a restriction on player i's production (impact) function $f_i(\boldsymbol{x_i})$ rather than player i's contest success function $p_i(\boldsymbol{x})$. Future research may develop this further to obtain a restriction on $p_i(\boldsymbol{x})$ which then could lead to a functional representation via the existence of some $f_i(\boldsymbol{x_i})$.

The functional forms in this article are restricted to a class that coincides with the one characterized in Lemma 1 of Arbatskaya and Mialon (2010) except for cases with zero efforts. These cases are covered in Axiom 1 which specifies whether a winning probability should be zero when certain efforts are zero. Axioms 1, 2 and 3 are otherwise similar to the corresponding axioms in Arbatskaya and Mialon (2010) and lead to an analogous characterization. This limits the value of Lemma 1.

Axioms 4–9 state variations of conventional properties of success functions. That is, Axiom 4 expresses standard monotonicity, and Axiom 5 expresses standard momogeneity (or Scale Invariance). Axiom 7 is a variation of Monotonicity. Axiom 8 is a weaker form of the standard Anonymity axiom. It requires symmetry only if effort vectors are symmetric across players, whereas the standard anonymity axiom is typically based on permutation of identities. Axiom 9 is a form of Independence influenced by Münster's (2009) Axiom 8. It serves the analogous purpose of identifying a linear form for $f_i(\boldsymbol{x_i})$. Axiom 9 crucially excludes multiplicative production functions including Cobb–Douglas production functions, and furnishes a foundation for additive production functions.

One limitation of the article is that Axioms 4–9 do not lead to any characterization, that is, a statement claiming that a contest success function satisfies a set of axioms if and only if it takes a certain functional form. Instead, Axioms 4–9 are employed in Lemmas 2, 3 and 4 which express compatibility between axioms and functional forms. Future research should extend these expressions on compatibility to "if and only" characterizations.

Future research should furthermore axiomatize more general contest success functions and production functions (e.g. which depend less separably on the multiple efforts), consider more than two players, and analyze empirically which forms are descriptive, including when multiple efforts are additive or multiplicative.

Conclusion

Axioms and lemmas are developed for rent seeking where two players exert multiple additive efforts. The implication for theory is that the satisfied axioms assume the sufficiency of exerting one effort under certain conditions, because efforts can substitute for each other. On the other hand, the satisfied axioms for the multiplicative Cobb–Douglas production function analyzed in the earlier literature require that all available efforts are exerted by a player, because efforts complement each other. The additive production function furthermore assumes a so-called summation axiom where adding an amount to one effort and subtracting the same amount from a second equivalent substitutable effort does not change the players' probabilities of winning the contest. In contrast, the Cobb–Douglas production function analyzed in the earlier literature assumes a homogeneity axiom where an equiproportionate change in two players' matched efforts does not affect the probabilities of winning.

The implication for practice is that an axiomatic foundation provides rent seeking practitioners, regulators, analysts and others with a more rigorous basis for understanding the role, logic, mechanism, and impact of rent seeking with multiple efforts. This article informs candidates for elected office, entrepreneurs seeking government privileges, firms seeking various rents, and policy makers, how to understand when multiple efforts by their nature are additive or multiplicative. The article specifies when multiple efforts are substitutes, complements, or neither substitutes nor complements, and how multiple efforts impact when they have different relevance, different unit costs, different impacts, and different contest intensities.

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Declarations

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