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A Two-Period Decision Model for Central Bank Digital Currencies and Households

Guizhou Wang [(a)], *Kjell Hausken* [(a)*]

[(a)] *Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway*

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ABSTRACT

Central bank digital currencies (CBDCs) give rise to many possibilities including those of negative interest rates. A two-period decision model is presented between one central bank and one representative household. The central bank applies the Taylor (1993) rule to choose its interest rate. The household allocates its resources strategically to production, consumption, CBDC holding, and non-CBDC holding. The results are determined analytically and illustrated numerically by varying 19 parameter values. Interesting novelties of the article are that the central bank may choose negative CBDC interest rates when the household holds far more CBDC than non-CBDC, for low inflation rates, low real interest rates, low household's potential production, low weight assigned to inflation in the Taylor (1993) rule, high target inflation rate, and high household's production parameter. That usually causes the household to decrease its CBDC holding and increase its non-CBDC holding, production and consumption. The central bank may increase its CBDC interest rate to compete with an increasing non-CBDC interest rate if the household's transaction efficiencies for CBDC and non-CBDC increase, or the household's transaction efficiency for consumption decreases. Shocks to production, inflation and interest rates are analyzed.

Introduction

Background

Technological developments in cryptography and blockchain have made digital currencies worldwide accessible. Central banks increasingly explore and develop CBDCs (central bank digital currencies). The Bank for International Settlements predicts that central banks for 20% of 7.9 billion people can be expected to issue CBDCs within three years (Boar & Wehrli, 2021). New cryptocurrencies emerge every day. December 30, 2021, 16,211 cryptocurrencies contribute to a market cap of $1.8 trillion.[1]. G. Wang, Zhang, Yu, and Ning (2021) provide a holistic picture of cryptocurrencies and blockchain research. Bhimani, Hausken, and Arif (2022) assess cryptocurrency adoption.

Digital currencies provide new possibilities that include higher transaction efficiencies, universal accessibility, confidentiality and privacy, flexible monetary policy, etc. Theoretically, low or negative interest rates can stimulate production and consumption. Some countries currently choose negative interest rates. For example, Blanke and Krogstrup (2016) cite the negative interest rates -0.75% for Switzerland, -0.5% for Denmark, and -0.1% for Japan. CBDSc make a negative interest policy more widely feasible, which can impact the economy substantially. That suggests a need for thorough analysis.

[1] https://coinmarketcap.com/, retrieved April 28, 2022.

* *Corresponding author. ORCID ID: orcid.org/0000-0001-7319-3876*

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Contribution

This article is the first in a series of two articles. This first article builds the decision model involving the central bank applying the Taylor (1993) rule and a representative household choosing strategically and compares with a benchmark solution assumed to be common in practice. The second article, G. Wang and Hausken (2022), compares with the empirics for the US, China and Russia.

The objective and research question intended to fill the current research gap are to explore the relationship between positive and negative CBDC interest rates and a household's production, consumption, CBDC holding and non-CBDC holding. A CBDC in this article can be interpreted as money supply M2 issued by the central bank. A two-period decision model is developed involving a central bank and a representative household. The central bank applies the Taylor (1993) rule to determine its positive or negative CBDC interest rate. A Cobb Douglas utility with four elasticities for the household accounts for the household allocating its resources strategically to production, consumption, CBDC holding, and non-CBDC holding.

A central bank fully controls its monetary policy and applies a variety of policy instruments, sometimes referred to as discretionary policy. Although no central bank officially uses the Taylor (1993) rule, the rule is frequently used as indicative of what a central bank does or may do. Even for central banks occasionally or more permanently applying a fixed exchange rate strategy, the rule may be indicative if economic conditions are comparable to other countries. The rule was proposed by Taylor (1993) in 1992 to stabilize economic policy by determining an interest rate based on inflation and production.

The four elasticities are adjusted by the CBDC and non-CBDC interest rates, and the household's transaction efficiency which increases with the household's CBDC and non-CBDC holdings and decreases with consumption. Solutions are provided analytically and numerically relative to a benchmark for how 19 parameters impact the central bank's application of the Taylor (1993) rule and the household's strategies. The impacts are analyzed of shocks to production, inflation, and the CBDC, non-CBDC, and real interest rates.

Article organization

Section 2 provides a literature review. Section 3 develops the methodology and the model. Section 4 examines the model. Section 5 shows and exemplifies the solution. Section 6 analyzes shocks to production, inflation, the CBDC interest rate, the non-CBDC interest rate, and the real interest rate. Section 7 discusses the results with economic interpretation. Section 8 concludes.

Literature Review

The literature has four categories. First, CBDCs enable central banks to implement negative interest rates which may become an important policy. Second, the central bank, one of the two actors in the article, provide and design the CBDC, and assess its impact. The third part presents decision theoretic analyses. The fourth part is about CBDCs and policy implications.

Negative interest rates

The article explores how a negative CBDC interest rate is connected to a household's allocations into production, consumption, and holding CBDC and non-CBDC. Grasselli and Lipton (2019) think that CBDCs enable the central bank to overcome any lower interest rate bound. They build a stock-flow macroeconomic model to explore the theoretical effectiveness of negative interest rates. They show that negative interest rates have lower impact on consumption than on investment. In contrast, we show that negative interests greatly and positively impacts both production and consumption.

Czudaj (2020) evaluates the effectiveness of negative interest rates based on expectations data from surveys for 44 economies 2002-2017. He finds reduced expectations for 10-year government bond yields and 3-month money market interest rates, and positive impact on GDP growth and preventing deflation, consistently with the current article.

Jia (2020) presents a model to investigate the macroeconomic impact of negative interest rates on CBDC. He shows that negative interest rates compel agents to save less and consume more, which in turn leads to declining capital investment and output. In the current article agents also save less CBDC and consume more. In contrast, the current article finds that agents save more non-CBDC and produce more.

M. Davoodalhosseini, Rivadeneyra, and Zhu (2020) suggest that an interest-bearing CBDC is a versatile instrument for a central bank. Theoretically, it may boost monetary policy. For instance, it can impose a negative interest rate, carry out non-linear transfers, decrease incentives to use alternative means of payments like cash, etc. That partly relates to the current article's finding that a household's CBDC holding is typically the opposite of its non-CBDC holding.

Assenmacher and Krogstrup (2021) think that digital money removes how monetary policy is constrained by a lower bound. They investigate how a central bank may construct and run a negative interest rate system. They show that without the lower bound constraint, the central bank can stabilize the economy by applying conventional policies. With low to intermediate real interest rates, the central bank can make deflationary spirals and the length of business cycle downturns less likely. That partly relates to the current article's finding of how the CBDC interest greatly impacts each household.

Meaning, Dyson, Barker, and Claytona (2021) point out that central banks may pay positive, zero or negative interest rates, and may impose different rates on different CBDC holders. This flexibility could be an important monetary policy instrument, to stabilize inflation and output, and regulate demand. That issue can be explored in a future extension of the current article by assuming households with different characteristics.

Mooij (2021) explores the legal framework in the Eurozone, including whether CBDCs could be classified as legal tender, and used as a monetary policy instrument. He concludes that the ECB mandate legally permits using CBDCs. He suggests that a CBDC enabling interest can decrease the negative lower bound to near zero, and that negative CBDC interest rates may cause capital flows into cryptocurrencies. That is consistent with the current article's finding that a household's CBDC holding is usually the opposite of its non-CBDC holding.

Some research has focused on pre-commitment rules, dynamic consistency and optimal policy related to negative interest rate. For example, Borio and Zabai (2018) find that various unconventional policies including negative interest rates in varying degrees influence financial conditions. They suggest that the policies are exceptional, for use in specific circumstances, and that the cost-benefit balance is likely to deteriorate over time. They criticize prevailing analyses of helicopter money and explore the risks associated with negative nominal interest rates. Ferrero and Neri (2017) assess reasons for historically low interest rates, including structural factors and cyclical and financial phenomena. They frame their assessment around a so-called natural interest rate and a transmission mechanism for money. They attempt to specify possible policy changes that may follow.

Decision theoretic analyses

The article considers a decision model involving the central bank choosing the CBDC interest rate and a representative household choosing production, consumption, holding CBDC and holding non-CBDC. G. Wang and Hausken (2021) build a model involving a representative household selecting a cryptocurrency or a national currency, analogously to the current article where a household chooses whether to hold CBDC or non-CBDC, and selects the probability of tax evasion for each currency. The government decides how to tax the two currencies, and how to detect and impose penalties for tax evasion. Welburn and Hausken (2015, 2017) investigate economic crises. Six kinds of players are included. These are countries, central banks, firms, banks, households, and financial inter-governmental organizations. Each player has multiple strategies, i.e. choosing interest rate, borrowing, lending, producing, consuming, investing, defaulting, etc.

CBDC design and the economy

The article considers the features of CBDC, i.e., higher transaction efficiencies for consumption, compared to non-CBDCs (including Bitcoin, bonds, stocks, etc.) and flexible monetary policies that include negative interest rates. Kiff et al. (2020) review the literature on central bank experiments, present main considerations on retail CBDCs, and provide a structured framework for CBDC issuance. Allen et al. (2020) argues that CBDCs may achieve a broad range of new capabilities, e.g., frictionless payments, new financial instruments, direct disbursements, broader tax bases, financial inclusion, the overcoming of technological vulnerabilities, etc. But CBDCs also lead to various challenges related to privacy, security, disintermediation of the banking system, etc. They summarize the basic technical design choices of CBDCs, especially as they relate to privacy, security, and performance. Auer and Böhme (2020) focus on retail CBDCs. They depict the CBDC pyramid that maps consumer needs into the CBDC design choice of central banks. They argue that the retail CBDC design needs to make tradeoffs between being secure, accessible, convenient and the safeguarding of privacy. Additional assessments addressed in the current article are how a household compares CBDC against non-CBDC, production and consumption.

Agur, Ari, and Dell'Ariccia (2021) present an optimal CBDC design, where each agent holds cash, CBDCs and bank deposits. The agent chooses based on its preference for anonymity and security. They find that the optimal CBDC design entails a tradeoff between bank intermediation and the maintenance of various payment instruments. H. Wang and Gao (2021) investigate various types of CBDCs and their implications on regulation and global financial networks. They suggest that the optimal CBDC networks will be decentralized, and cause monetary policy diffusion without regulatory convergence. Lee, Yan, and Wang (2021) explore how a CBDC structure can keep a balance between benefits and risks. Advantages of CBDC include inclusiveness, cost-saving, managed anonymity, lower cross-border payments, transaction efficiency, security, and more. The risks of CBDCs include bank disintermediation, blockchain-based technology vulnerabilities, and the regulation of shadow and derivate markets of CBDCs. They conclude that CBDCs will become the primary tools of the digital economy.

Urbinati et al. (2021) present the status quo of CBDC-related work worldwide. They illustrate a potential digital euro solution that will combine an account-based platform and distributed ledger technology. Based on the experiments, they find that this combination may provide a sound solution for regulations and retail demand. Choi, Henry, Lehar, Reardon, and Safavi-Naini (2021) introduce a hypothetical retail CBDC design for the Bank of Canada. They think that the design is sound and feasible because it is scalable, resilient, privacy-centric, and universally accessible. Boar and Wehrli (2021) survey worldwide CBDC developments. They find that central banks for 20% of 7.9 billion people can be expected to issue CBDCs within three years.

CBDCs and policy implications

This article relates to this literature by exploring positive and negative CBDC interest rates, and transaction efficiencies. Böser and Gersbach (2020) examine the impact of an interest-bearing CBDC on bank activities and monetary policy. They point out that setting appropriate collateral requirements will boost aggregate productivity. However, if households hold massive amounts of CBDCs, policy with restrictive collateral requirements is risky for banks related to liquidity. That may induce the central bank to abandon these policies. This illustrates the dilemmas faced by central banks when issuing CBDCs. S. M. Davoodalhosseini (2021) explores the optimal monetary policy when an agent chooses between cash and a CBDC. He finds that only a CBDC may be used if its cost is limited, since more efficient allocations can be achieved.

Beniak (2019) discusses potential challenges of CBDC implementation for monetary policy. He points out that CBDCs will impact the interest rates of the central bank, implementation of policy, and the mechanism for transmission. These impacts depend on the design of, and the demand for, CBDC. Bindseil (2020) summarizes the advantages of CBDCs, which include efficient payments, anti-illegal activities, flexible monetary policy with a negative interest rate, etc. The potential risks of CBDCs are bank disintermediation, systemic runs on banks, possible centralization within the central bank, etc. He introduces a two-tier remuneration of CBDC as a solution. Bindseil and Fabio (2020) think that a two-tier CBDC provides a sound solution to issues like bank disintermediation, negative interest rate policy, financial stability, etc. The CBDC with tiered remuneration has four key objectives, including being an attractive means of payment, being universally accessible, depressing the risks of structural bank disintermediation, and providing negative interest rates.

Methodology: The model

This section specifies how the central bank determines the interest rate through the Taylor (1993) rule. The household's resource constraint for production, consumption, CBDC holding and non-CBDC holding, is specified. The household's utility is built up gradually over four steps. Appendix A shows the nomenclature.

The central bank's Taylor (1993) rule application

The central bank applies in period 1 the Taylor (1993) rule to determine the interest rate

$$I_m = \max\left\{\pi + I_r + a_\pi(\pi - \pi^*) + a_p Log\left(\frac{p^h}{\bar{p}^h}\right), z\right\} \tag{1}$$

where I_r, $I_r \in \mathbb{R}$, is the equilibrium real interest rate, where $\mathbb{R}$ is the set of all real numbers; π, $\pi \in \mathbb{R}$, is the inflation rate (which can be positive or negative); π^*, $\pi^* \in \mathbb{R}$, is the desired inflation rate; p^h, $p \geq 0$, is the representative household's production; h is a production parameter; $\bar{p}^h$, $\bar{p} \geq 0$, is the household's potential production (which can be sustained over the long term); Log is the logarithm with base ten; a_π, $a_\pi \geq 0$, is the weight assigned to inflation; a_p, $a_p \geq 0$, is the weight assigned to production; and z, $z \leq 0$, is the negative lower bound on the interest rate I_m.

The household's strategic choices and utility

The representative household has resources r which comprise labor capacity and convertible assets. The resources r are in period 2 converted at unit cost a into production p, and converted at unit cost 1 into consumption c, CBDC (Central Bank Digital Currency) m, and non-CBDC q, i.e.

$$r = ap + c + m + q \tag{2}$$

where c, m, q are equivalently scaled on a suitable scale, e.g. US$. As CBDCs are not widely available at the time of writing this article, we may interpret CBDC as money supply M2 that the central bank issues, made available to the household. The household's production p follows from applying its labor capacity which may generate a salary or useful products. The non-CBDC q can be a cryptocurrency such as Bitcoin, a CBDC from another central bank, or any asset. The CBDC m and non-CBDC q are money demands which in (2) have the same interpretation as resource allocation into any asset. The household's production p causes productive output p^h, where $h = 1$ means linear production, $h > 1$ means convex production, $0 < h < 1$ means concave production, and $h = 0$ means no production.

The household's Cobb Douglas utility is advanced in four steps. First, the household's Cobb Douglas utility has four output elasticities. The first is $\alpha - MI_m - QI_q$ for production, $0 \leq \alpha - MI_m - QI_q \leq 1$, where α is the basic elasticity from which the CBDC interest rate I_m and the non-CBDC interest rate I_q, with weights M and Q, are subtracted. The reasoning is that when the interest rates I_m and I_q increase, production decreases as is commonly observed, and is thus assigned lower elasticity or weight.

The second elasticity is $\beta - MI_m - QI_q$ for consumption, $0 \leq \beta - MI_m - QI_q \leq 1$, where, analogously, β is the basic elasticity from which the CBDC interest rate I_m and the non-CBDC interest rate I_q, with weights M and Q, are subtracted. The reasoning is that when the interest rates I_m and I_q increase, consumption decreases as is commonly observed, and is thus assigned lower elasticity or weight.

The third elasticity is $\gamma + 2MI_m$ for saving CBDC m, $0 \leq \gamma + 2MI_m \leq 1$, where γ is the basic elasticity to which the CBDC interest rate I_m, with weight $2M$, is added. The reasoning is that when the interest rate I_m increases, the household assigns higher elasticity or weight to saving CBDC m. The weight $2M$ is chosen to ensure that the four elasticities sum to 1.

The fourth elasticity is $1 - \alpha - \beta - \gamma + 2QI_q$ for saving non-CBDC q, $0 \leq 1 - \alpha - \beta - \gamma + 2QI_q \leq 1$, where $1 - \alpha - \beta - \gamma$ is the basic elasticity to which the non-CBDC interest rate I_q, with weight $2Q$, is added. The reasoning is that when the interest rate I_q increases, the household assigns higher elasticity or weight to saving non-CBDC q. The weight $2Q$ is chosen to ensure that the four elasticities sum to 1. The household's utility is thus

$$U_1 = p^{h(\alpha - MI_m - QI_q)} c^{\beta - MI_m - QI_q} m^{\gamma + 2MI_m} q^{1-\alpha-\beta-\gamma+2QI_q} \tag{3}$$

which manifests constant returns to scale. Since the Cobb Douglas elasticities sum to 1, increasing one elasticity means that at least one other elasticity must decrease. For example, if α and β increase, assigning higher weight to production p and consumption c, The four exponents sum to 1. Second, the household earns interest I_m on CBDC m, and earns interest I_q, $I_q \in \mathbb{R}$, on the non-CBDC q. Interest rates are, at least historically, mostly positive. For digital currencies, including CBDC m and non-CBDC q, interest rates can be negative. Hence, we multiply m and q with $1 + I_m$ and $1 + I_q$, respectively, to denote how interest rates are earned. Absorbing these multiplications into (3) gives

$$U_2 = p^{h(\alpha - MI_m - QI_q)} c^{\beta - MI_m - QI_q} \big(m(1 + I_m)\big)^{\gamma + 2MI_m} \Big(q\big(1 + I_q\big)\Big)^{1-\alpha-\beta-\gamma+2QI_q} \tag{4}$$

Third, to transact between consumption c, CBDC m, and non-CBDC q, the household has to pay transaction costs. The household prefers high transaction efficiency, which ceteris paribus expresses lower transaction cost. The household's transaction efficiency E is modeled to increase with holding CBDC m and holding non-CBDC q, and decrease with consumption c, i.e.

$$E = \frac{m^\mu q^\eta}{\theta c^\lambda} \tag{5}$$

where μ, $\mu \geq 0$, is the household's transaction efficiency for CBDC m; η, $\eta \geq 0$, is the household's transaction efficiency for non-CBDC q. The parameter λ is the household's transaction efficiency for consumption c, and $1/\theta$, $\theta \geq 0$, scales the degree or level of the household's transaction efficiency. The requirement $0 \leq \lambda \leq \beta \leq 1$ expresses that the household prefers consumption, shown as c^β in (4), although incurring the transaction cost $1/c^\lambda$ in (5). The assumption $\eta \geq \lambda$ ensures that the household's transaction efficiency η for non-CBDC q is higher than or equal to the household's transaction efficiency λ for consumption c.

In (5) the transaction efficiency E satisfies $\frac{\partial E}{\partial c} \leq 0, \frac{\partial E}{\partial m} \geq 0, \frac{\partial E}{\partial q} \geq 0, \frac{\partial^2 E}{\partial c^2} \geq 0, \frac{\partial^2 E}{\partial m^2} \leq 0 \; when \; \mu \leq 1, \frac{\partial^2 E}{\partial q^2} \leq 0 \; when \; \eta \leq 1, \frac{\partial^2 E}{\partial c \partial m} \leq 0, \frac{\partial^2 E}{\partial c \partial q} \leq 0$, see Appendix B. Hence E decreases convexly in consumption c, and increases in the CBDC m and the non-CBDC q. For other accounts of the transaction efficiency E, often expressed as the transaction cost $1/E$, see Feenstra (1986), Bougheas (1994), and Saygılı (2012).

The inverse $1/E$ of E in (5), interpreted as the transaction cost, is commonly analyzed in the literature, where θ scales the transaction cost. Higher transaction efficiency for CBDC m than for non-CBDC q, to enable negative interest rates $I_m < 0$ on CBDC m, requires $\mu > \eta$. This article does not impose that requirement since we may have even more negative interest rates $I_q < I_m < 0$ for non-CBDC q. Multiplying (5) with (4) gives the household's utility

$$U_3 = p^{h(\alpha - MI_m - QI_q)} c^{\beta - MI_m - QI_q} \big(m(1 + I_m)\big)^{\gamma + 2MI_m} \Big(q\big(1 + I_q\big)\Big)^{1-\alpha-\beta-\gamma+2QI_q} \frac{m^\mu q^\eta}{\theta c^\lambda} \tag{6}$$

Fourth, the household's resource constraint in (2) shows that the household has three free choice variables, i.e. production p, consumption c and CBDC m, where non-CBDC $q = r - ap - c - m$ follows from solving (2) with respect to q. Inserting $q = r - ap - c - m$ into (6) gives the household's utility

$$\begin{aligned} U &= p^{h(\alpha - MI_m - QI_q)} c^{\beta - MI_m - QI_q} \big(m(1 + I_m)\big)^{\gamma + 2MI_m} \\ &\times \Big((r - ap - c - m)\big(1 + I_q\big)\Big)^{1-\alpha-\beta-\gamma+2QI_q} \frac{m^\mu (r - ap - c - m)^\eta}{\theta c^\lambda} \end{aligned} \tag{7}$$

which has three strategic choice variables p, c and m, and which is the household's utility U which we now proceed to analyze.

We analyze a two-period decision model. In period 1 the central bank applies the Taylor (1993) rule to determine its interest rate I_m. In period 2 the household makes three strategic choices, i.e., production p, consumption c, and its holding m of CBDC. Applying (1) gives the household's holding q of non-CBDC.

Analyzing the model

This section determines the household's production p, consumption c, CBDC holding m, non-CBDC holding q, and utility U. An implicit solution is presented for the CBDC interest rate I_m. The signs of the first and second order derivatives of the variables are determined. Further analysis is provided when the CBDC interest rate I_m is a parameter.

Analyzing the household

Assumption 1.

$$\{p \geq 0, c \geq 0, m \geq 0, q \geq 0, U \geq 0\} \Leftrightarrow \left\{\begin{array}{c} 0 \leq \alpha - MI_m - QI_q \leq 1, \\ \beta - \lambda - MI_m - QI_q \geq 0, \\ \gamma + 2MI_m + \mu \geq 0, \\ 1 - \alpha - \beta - \gamma + \eta + 2QI_q \geq 0, \\ 1 - (1-h)\left(\alpha - MI_m - QI_q\right) + \eta - \lambda + \mu \geq 0 \end{array}\right\} \tag{8}$$

Property 1. When Assumption 1 holds, the household's production p, consumption c, CBDC holding m, non-CBDC holding q, and utility U, are

$$\begin{aligned}
p &= \frac{rh\left(\alpha - MI_m - QI_q\right)}{a\left(1 - (1-h)\left(\alpha - MI_m - QI_q\right) + \eta - \lambda + \mu\right)}, \\
c &= \frac{r\left(\beta - \lambda - MI_m - QI_q\right)}{1 - (1-h)\left(\alpha - MI_m - QI_q\right) + \eta - \lambda + \mu}, \\
m &= \frac{r(\gamma + 2MI_m + \mu)}{1 - (1-h)\left(\alpha - MI_m - QI_q\right) + \eta - \lambda + \mu}, \\
q &= \frac{r\left(1 - \alpha - \beta - \gamma + \eta + 2QI_q\right)}{1 - (1-h)\left(\alpha - MI_m - QI_q\right) + \eta - \lambda + \mu}, \\
U &= \frac{\left(1 - \alpha - \beta - \gamma + \eta + 2QI_q\right)}{\theta\left(\beta - \lambda - MI_m - QI_q\right)}(1 + I_m)^{\gamma + 2MI_m}\left(1 + I_q\right)^{1 - \alpha - \beta - \gamma + 2QI_q} \\
&\times \left(\frac{-rh\left(\alpha - MI_m - QI_q\right)}{a\left(\left(1-h\right)\left(\alpha - MI_m - QI_q\right) - \eta + \lambda - \mu - 1\right)}\right)^{h\left(\alpha - MI_m - QI_q\right)} \\
&\times \left(\frac{-r\left(1 - \alpha - \beta - \gamma + \eta + 2QI_q\right)}{(1-h)\left(\alpha - MI_m - QI_q\right) - \eta + \lambda - \mu - 1}\right)^{-\alpha - \beta - \gamma + \eta + 2QI_q} \\
&\times \left(\frac{-r\left(\beta - \lambda - MI_m - QI_q\right)}{(1-h)\left(\alpha - MI_m - QI_q\right) - \eta + \lambda - \mu - 1}\right)^{1 + \beta - \lambda - MI_m - QI_q} \\
&\times \left(\frac{-r(\gamma + 2MI_m + \mu)}{(1-h)\left(\alpha - MI_m - QI_q\right) - \eta + \lambda - \mu - 1}\right)^{\gamma + 2MI_m + \mu}
\end{aligned} \tag{9}$$

Proof. Appendix C. ■

Analyzing the central bank

Property 2. When Assumption 1 holds, the central bank's CBDC interest rate I_m for the household's CBDC holding m is

$$I_m = \max\left\{\pi + I_r + a_\pi(\pi - \pi^*) + a_p hLog\left(\frac{rh\left(\alpha - MI_m - QI_q\right)}{a\left(1 - (1-h)\left(\alpha - MI_m - QI_q\right) + \eta - \lambda + \mu\right)\bar{p}}\right), z\right\} \tag{10}$$

Proof. Follows from inserting p in (9) into (1). ■

Since the CBDC interest rate I_m appears on the left-hand side and twice inside the logarithm Log with base ten in (10), I_m has no analytical solution and is determined numerically.

Analyzing the household and the central bank

The CBDC interest rate I_m in (10) depends on $r,a,\alpha,M,Q,I_q,\mu,\eta,\lambda,I_r,\pi,\pi^*,h,\bar{p},a_\pi,a_p$,z, and hence does not depend on β,γ,θ. Assume that $\bar{p}=\frac{kr}{a}$, which means that the household's potential production $\bar{p}$ is a fraction k, $0\le k\le 1$, where k is a parameter, of the maximum possible production $p=\frac{r}{a}$ obtained when $c=m=q=0$ in (2). Then I_m in (10) also does not depend on r and a. Hence Property 3 determines the derivatives of p,c,m,q,U with respect to β,γ,θ,r,a when $\bar{p}=kr/a$.

Property 3. When Assumption 1 holds, $\frac{\partial p}{\partial\beta}=\frac{\partial p}{\partial\gamma}=\frac{\partial p}{\partial\theta}=0,\frac{\partial^2 p}{\partial\beta^2}=\frac{\partial^2 p}{\partial\gamma^2}=\frac{\partial^2 p}{\partial\theta^2}=0,\frac{\partial p}{\partial r}\ge 0,\frac{\partial^2 p}{\partial r^2}=0,\frac{\partial p}{\partial a}\le 0,\frac{\partial^2 p}{\partial a^2}\ge 0,\frac{\partial c}{\partial\beta}\ge 0,\frac{\partial^2 c}{\partial\beta^2}=0,\frac{\partial c}{\partial\gamma}=\frac{\partial c}{\partial\gamma^2}=0,\frac{\partial c}{\partial\theta}=\frac{\partial c}{\partial\theta^2}=0,\frac{\partial c}{\partial r}\ge 0,\frac{\partial^2 c}{\partial r^2}=0,\frac{\partial c}{\partial a}=\frac{\partial^2 c}{\partial a^2}=0,\frac{\partial m}{\partial\beta}=\frac{\partial^2 m}{\partial\beta^2}=0,\frac{\partial m}{\partial\gamma}\ge 0,\frac{\partial^2 m}{\partial\gamma^2}=0,\frac{\partial m}{\partial\theta}=\frac{\partial^2 m}{\partial\theta^2}=0,\frac{\partial m}{\partial r}\ge 0,\frac{\partial^2 m}{\partial r^2}=0,\frac{\partial m}{\partial a}=\frac{\partial^2 m}{\partial a^2}=0,\frac{\partial q}{\partial\beta}\le 0,\frac{\partial q}{\partial\gamma}\le 0,\frac{\partial^2 q}{\partial\beta^2}=\frac{\partial^2 q}{\partial\gamma^2}=0,\frac{\partial q}{\partial\theta}=\frac{\partial^2 q}{\partial\theta^2}=0,\frac{\partial q}{\partial r}\ge 0,\frac{\partial^2 q}{\partial r^2}=0,\frac{\partial q}{\partial a}=\frac{\partial^2 q}{\partial a^2}=0,\frac{\partial U}{\partial\beta}\lesseqqgtr 0,\frac{\partial^2 U}{\partial\beta^2}>0,\frac{\partial U}{\partial\gamma}\lesseqqgtr 0,\frac{\partial^2 U}{\partial\gamma^2}>0,\frac{\partial U}{\partial\theta}<0,\frac{\partial^2 U}{\partial\theta^2}>0,\frac{\partial U}{\partial r}>0,\frac{\partial^2 U}{\partial r^2}\lesseqqgtr 0,\frac{\partial U}{\partial a}<0,\frac{\partial^2 U}{\partial a^2}\lesseqqgtr 0$.

Proof. Follows from (19), (20), (21), (22), (23) in Appendix D. ■

Property 3 states, first, that the household's consumption c increases linearly, while its non-CBDC holding q decreases linearly, in its output elasticity β for consumption c. As β increases, the household values consumption c more and values non-CBDC q less. The household's production p and CBDC holding m are independent of β. The household's utility U can increase or decrease in β.

Second, the household's CBDC holding m increases linearly, while its non-CBDC holding q decreases linearly, in its output elasticity γ for holding CBDC m. As γ increases, the household values CBDC m more and values non-CBDC q less. The household's production p and CBDC holding m are independent of γ. Also, here the household's utility U can increase or decrease in γ.

Third, the household's production p, consumption c, CBDC holding m, and non-CBDC holding q, are independent of the household's scaling θ of the transaction cost. That's because θ appears only in the denominator in (7), and hence does not impact the household's strategic choices. However, a high θ is costly and impacts the household's utility U which decreases convexly.

Fourth, the household's production p, consumption c, CBDC holding m, and non-CBDC holding q, increase linearly in the household's resources r. That's because more resources are beneficial for the household. Hence the household's utility U also increases in r.

Fifth, the household's production p decreases convexly, causing the household's utility U also to decrease, in the household's unit production cost a. The household's consumption c, CBDC holding m, and non-CBDC holding q are independent of a.

Analyzing the household when I_m is a parameter

Property 4. Assume that Assumption 1 holds, and that I_m is a parameter.

$\frac{\partial p}{\partial I_m}\le 0,\frac{\partial^2 q}{\partial I_m^2}\ge 0,\frac{\partial m}{\partial I_m}\lesseqqgtr 0,\frac{\partial^2 m}{\partial I_m^2}\lesseqqgtr 0.$

If $0\le h\le 1,\frac{\partial^2 p}{\partial I_m^2}\ge 0,\ \frac{\partial c}{\partial I_m}\le 0,\ \frac{\partial^2 c}{\partial I_m^2}\ge 0,\ \frac{\partial q}{\partial I_m}\le 0.$

If $h\ge 1,\frac{\partial^2 p}{\partial I_m^2}\le 0,\ \frac{\partial q}{\partial I_m}\ge 0.$

If $h\gg 1,\frac{\partial c}{\partial I_m}\ge 0,\frac{\partial^2 c}{\partial I_m^2}\ge 0.$

Proof. Follows from (24) in Appendix E. ■

Property 4 states, first, that the household's production p decreases in the CBDC interest rate I_m, since the subtraction of MI_m in the numerator in (9) has higher impact than the role of MI_m in the denominator in (9).[2]

Second, the household's consumption c decreases convexly in I_m if $0\le h<1$, and decreases linearly in I_m if $h=1$. The household's consumption c can increase in I_m if h is sufficiently above 1 as specified in (24).

Third, the household's non-CBDC holding q decreases convexly in I_m if $0\le h\le 1$, due to the competing CBDC m offering more favorable interest rate I_m, and otherwise increases concavely due to the high household's production parameter $h>1$.

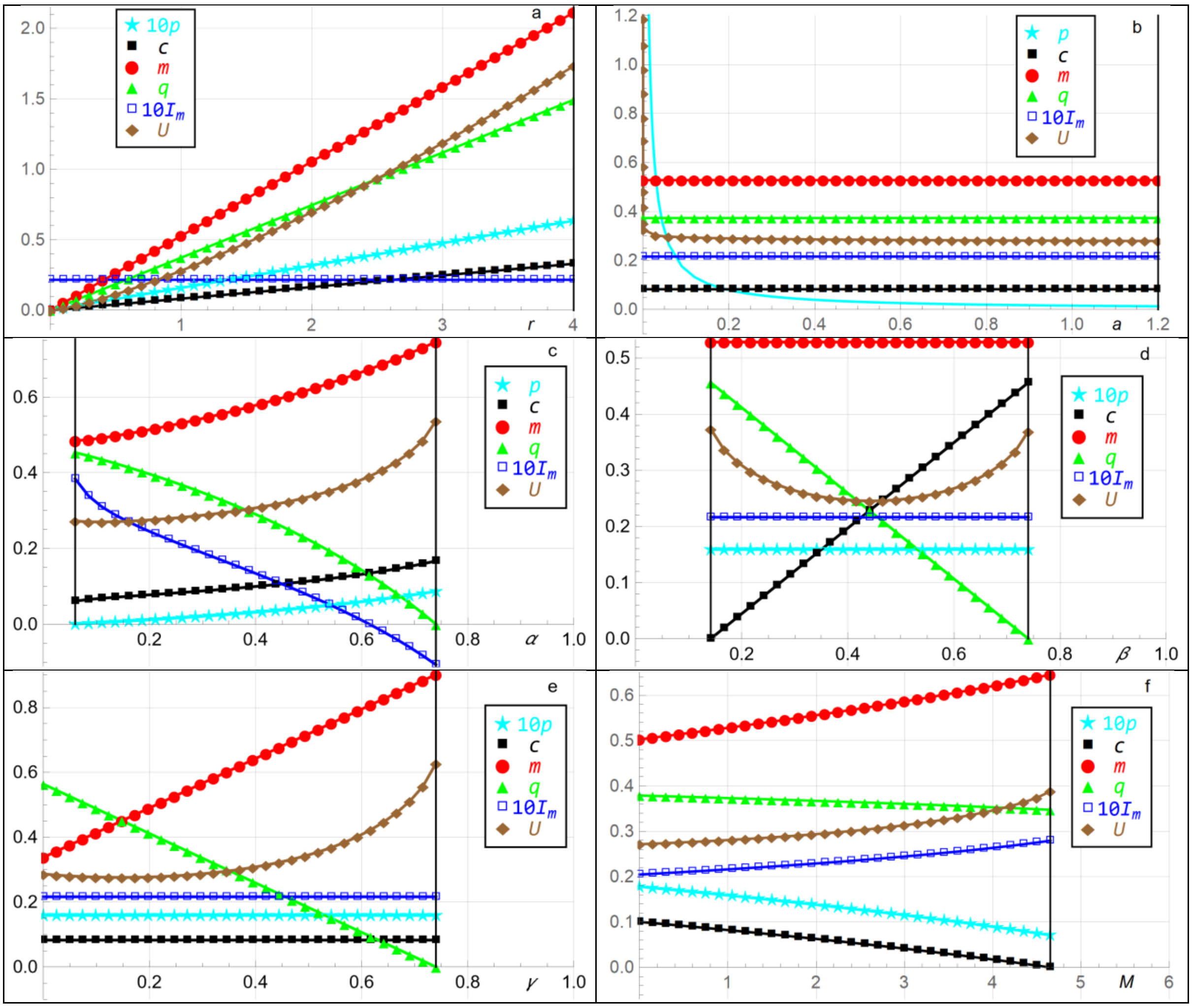
Fourth, the household's CBDC holding m usually increases in I_m, $\frac{\partial m}{\partial I_m}\ge 0$, which tends to make holding CBDC m more attractive.[3]

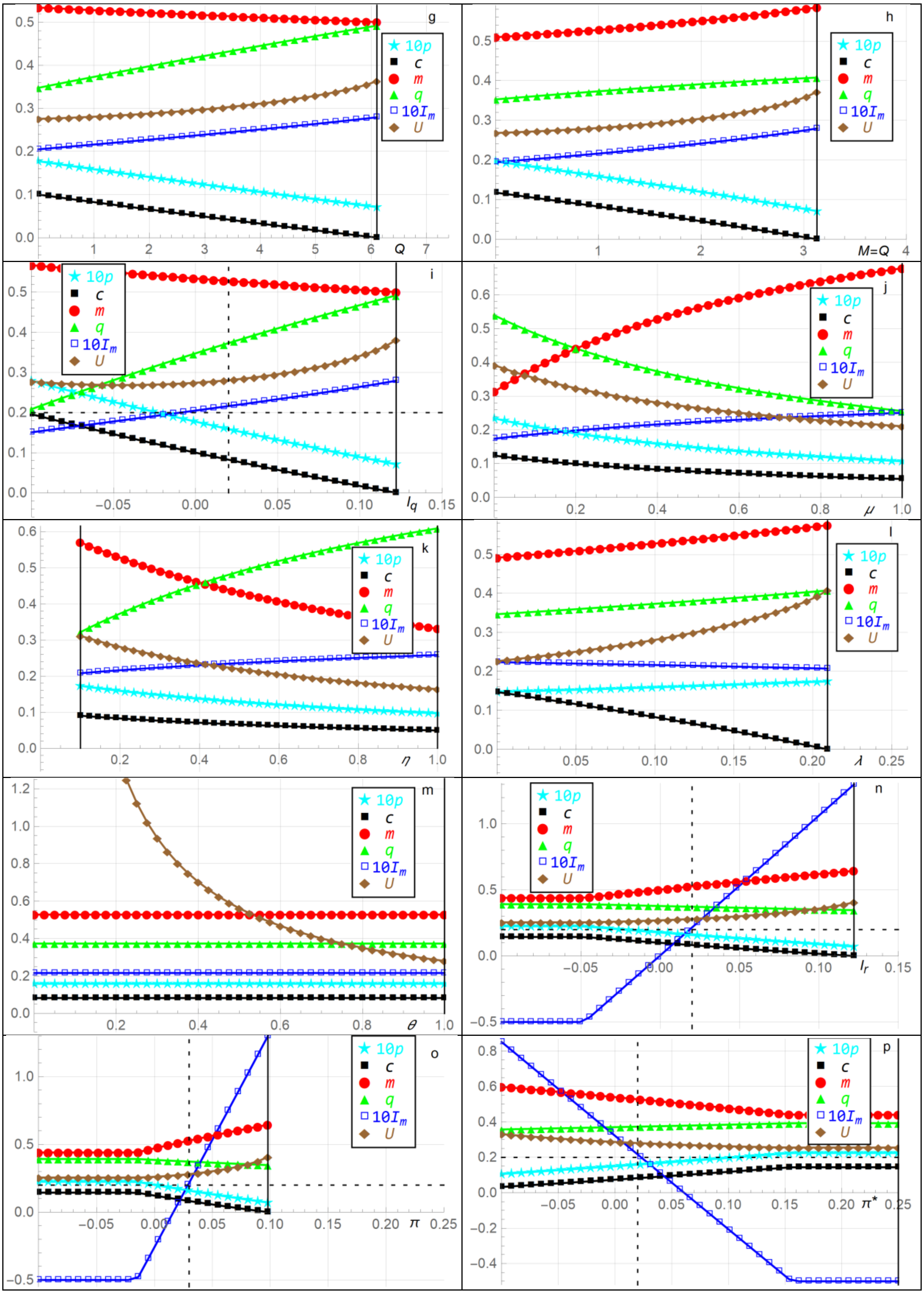
[2] If $h>1$, the production p decreases concavely. If $0\le h<1$, the production p decreases convexly. If $h=1$, the production p decreases linearly.

[3] However, numerical simulation has shown that extreme parameter values, such as negative non-CBDC interest rate I_q, low h, high α, low μ and high λ, may cause the household's CBDC holding m to decrease in I_m.

Illustrating the solution

This section varies the parameter values $r, a, \alpha, \beta, \gamma, M, Q, M = Q, I_q, \mu, \eta, \lambda, \theta, I_r, \pi, \pi^*, h, \bar{p}, a_\pi, a_p$ relative to a benchmark. The benchmark values are $\alpha = \beta = \gamma = \frac{1}{4}, r = a = M = Q = 1, I_q = I_r = 2\%, \eta = \frac{1}{5}, \mu = \frac{2}{5}, \lambda = \frac{1}{10}, \pi = 3\%, \pi^* = 2\%, h = \frac{1}{10}, \bar{p} = \frac{1}{2}, a_\pi = a_p = \frac{1}{2}, z = -5\%$. The benchmark is chosen to be realistic in practice. First, $\alpha = \beta = \frac{1}{4}$ expresses equal weight or elasticity for production p, consumption c, respectively. Second, $\gamma = 1 - \alpha - \beta - \gamma = 1/4$ reflects identical weight or elasticity for CBDC m and non-CBDC q. Third, $\eta = \frac{1}{5}$ depicts a middle-level transaction efficiency for non-CBDC. Fourth, $\mu = \frac{2}{5}$ expresses a higher transaction efficiency or lower cost for CBDC m. Fifth, $\lambda = \frac{1}{10}$ expresses low transaction efficiency for consumption c. Sixth, $I_q = 2\%$ reflects an intermediate interest rate for non-CBDC. $I_r = 2\%$ expresses a desired or equilibrium real interest rate. Seventh, $r = a = M = Q = 1$ are chosen for the sake of simplicity, and value one is also plausible. Eighth, $\pi = 3\%$ presents the inflation rate. Nineth, $\pi^* = 2\%$ reflects a desired inflation rate. Tenth, $\bar{p} = \frac{r}{2a} = \frac{1}{2}$ expresses the potential production, which is 50% of what can be produced if the entire resource r is allocated to production. Eleventh, $h = \frac{1}{10}$ reflects a concave production function for the household. Twelfth, $a_\pi = a_p = \frac{1}{2}$ expresses the common equal weight assigned to inflation and production in the Taylor (1993) rule. Thirteenth, $z = -5\%$ is the negative lower bound on the CBDC interest rate I_m. With these benchmark parameter values, the benchmark solution is $I_m = 2.16\%, p = 0.0159, c = 0.0826, m = 0.5282, q = 0.373, U = 0.281$. In Figure 1 each of the 20 parameters values is changed from its benchmark, as illustrated with labels along the horizontal axis, while the other 19 parameter values remain at their benchmarks. The Wolfram Mathematica 13 software package (wolfram.com) has been used. Multiplication of p and I_m with 10 is for scaling purposes.

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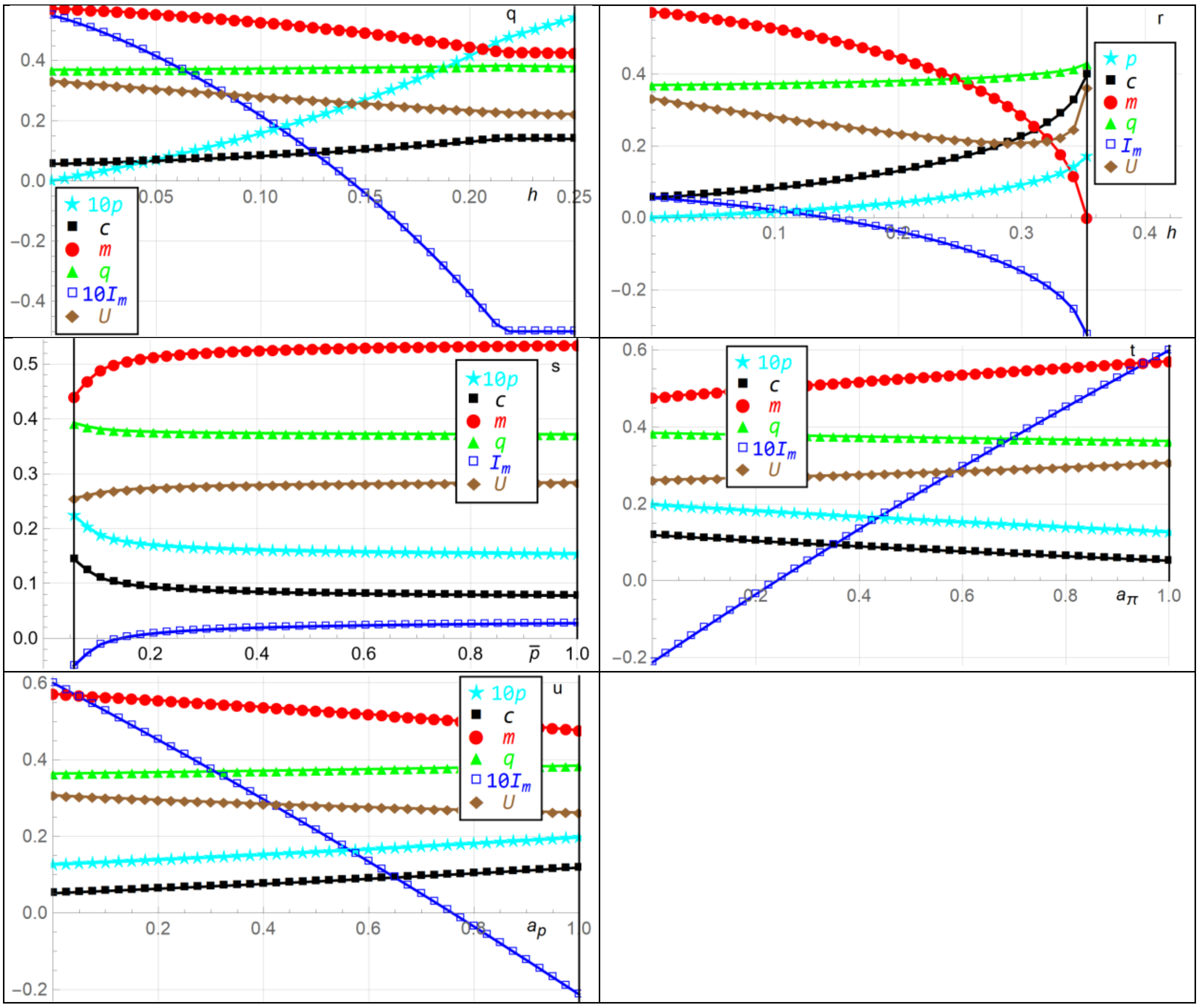

Figure 1: The household's production p, consumption c, CBDC holding m, non-CBDC holding q, and utility U, and the CBDC interest rate I_m, as functions of $r, a, \alpha, \beta, \gamma, M, Q, M = Q, I_q, \mu, \eta, \lambda, \theta, I_r, \pi, \pi^*, h, \bar{p}, a_\pi, a_p$ relative to the benchmark parameter values $\alpha = \beta = \gamma = \frac{1}{4}, r = a = M = Q = 1, I_q = I_r = 2\%, \eta = \frac{1}{5}, \mu = \frac{2}{5}, \lambda = \frac{1}{10}, \pi = 3\%, \pi^* = 2\%, h = \frac{1}{10}, \bar{p} = \frac{1}{2}, a_\pi = a_p = \frac{1}{2}, z = -5\%$. Multiplication of p and I_m with 10 is to ensure proper scaling.

In Figure 1a, if the household's resources r increases, which is intuitively beneficial, its production p, consumption c, CBDC holding m, and non-CBDC holding q, increase linearly according to (9). The central bank's CBDC interest rate I_m remains constant, since resources r are abbreviated in the Taylor (1993) rule in (10) since $\bar{p} = \frac{r}{2a}$. The household's utility U increases convexly according to (9). Specifically, production p increases slowly, while CBDC holding m increases rapidly.

In Figure 1b, if the household's unit production cost a increases, its production p and utility U intuitively decrease convexly. The other variables remain constant, and a is abbreviated in (10). The household intuitively benefits from the unit cost a approaching zero, which causes the production p and expected utility U to approach infinity.

In Figure 1c, if the household's output elasticity α for production p increases from $\alpha = 0.06$, its production p, consumption c, CBDC holding m, and the household's utility U, increase convexly, while non-CBDC holding q decreases convexly reaching $q = 0$ when $\alpha > 0.74$, since the output elasticity $1 - \alpha - \beta - \gamma + 2QI_q$ in (7) decreases. When $\alpha < 0.06$, no production p occurs due to subtraction of $MI_m + QI_q$ in (9), where the CBDC interest rate I_m is high to induce the household to save in CBDC m rather than non-CBDC q. The CBDC interest rate I_m decreases and becomes negative when $\alpha > 0.61$, since the household then saves far more in CBDC m than in non-CBDC q, and the central bank can charge the household for saving in CBDC m. When I_m becomes negative, subtraction of MI_m for production p and consumption c in (9) causes addition of $-MI_m$ which is positive. When $\alpha > 0.61$, the household values consumption c more than non-CBDC q.

In Figure 1d, if the household's elasticity β for consumption c increases from $\beta = 0.14$, its consumption c increases linearly. The CBDC interest rate I_m in (10) is independent of β, and hence production p and CBDC holding m in (9) are also independent of β. The household's non-CBDC holding q decreases linearly reaching $q = 0$ when $\beta > 0.74$, since the output elasticity $1 - \alpha - \beta - \gamma + 2QI_q$ in (7) decreases. Interestingly, the household's utility U is U shaped. That's because holding non-CBDC q causes high utility U when β is low, choosing high consumption c causes high utility U when β is high, and U is intermediate when q and c are intermediate at the intermediate $\beta = 0.44$. When $\beta < 0.14$, no consumption c occurs due to subtraction of $MI_m + QI_q - \lambda$ in (9).

In Figure 1e, if the household's output elasticity γ for holding CBDC m increases, its CBDC holding m intuitively increases, and its non-CBDC holding q decreases reaching $q = 0$ when $\gamma > 0.74$, according to (9). When $\gamma > 0.15$, the household values CBDC m more than non-CBDC q. The CBDC interest rate I_m in (10) is independent of γ, and hence production p and consumption c in (9) are also independent of γ. The household's utility U is U shaped, but less symmetric than in Figure 1d. That's because m crosses q at the low value $\gamma = 0.15$, causing high utility U when m is high, while c crosses q at the intermediate $\beta = 0.44$ in Figure 1d.

In Figure 1f, if the household's weight M of the CBDC interest rate I_m in its output elasticities increases, its CBDC holding m increases, while its non-CBDC holding q decreases. Subtracting MI_m in the household's output elasticity for production p and consumption c in (7), as also shown in the numerator for p and c in (9), causes production p and consumption c to decrease. Consumption c eventually decreases to $c = 0$ when $M > 4.65$. Interestingly, the CBDC interest rate I_m increases, which is not intuitively obvious. It illustrates the multiple tradeoffs that the central bank has to make. The household benefits more from holding CBDC m at an increasing CBDC interest rate I_m, than the costs of decreasing q, p, c. Its utility U thus increases.

In Figure 1g, if the household's weight Q of the non-CBDC interest rate I_q in its output elasticities increases, its non-CBDC holding q increases, while its CBDC holding m decreases. That intuitively stands in contrast to Figure 1f. The other four curves are qualitatively similar to Figure 1f, since the household merely shifts its interest from CBDC m to non-CBDC q. That is, subtracting QI_q in the household's output elasticity for production p and consumption c in (7) causes production p and consumption c to decrease. Consumption c eventually decreases to $c = 0$ when $Q > 6.10$. The CBDC interest rate I_m and the household's utility U increase.

In Figure 1h, if the household's equal weights $M = Q$ of the CBDC and non-CBDC interest rates I_m and I_q in its output elasticities increase, its holding of both CBDC m and non-CBDC q increase. That follows since $2MI_m$ and $2QI_q$ are added to the output elasticities in (7). In contrast, MI_m and QI_q are subtracted from the output elasticities for production p and consumption c in (7), causing these to decrease. Consumption c eventually decreases to $c = 0$ when $M = Q > 3.13$, which is a lower value than in Figure 1f and Figure 1g. As in Figure 1f and Figure 1g, and for the same reason, the CBDC interest rate I_m and the household's utility U increase.

In Figure 1i, if the non-CBDC interest rate I_q increases, the household's non-CBDC holding q increases, while its CBDC holding m decreases. The six variables are qualitatively similar to Figure 1g as functions of Q. That's intuitive since QI_q always appear multiplicatively together in (9) and (10), and never separately alone. Hence production p decreases, and consumption c decreases reaching $c = 0$ when $I_q > 12\%$. That is, the household stops consuming when the non-CBDC interest rate I_q is high, and saves non-CBDC q instead. The CBDC interest rate I_m increases to compete with the increasing I_q, and the household's utility U increases convexly.

In Figure 1j, if the household's transaction efficiency μ for CBDC m increases, its CBDC holding m increases concavely, while its production p, consumption c, non-CBDC holding q, and utility U decrease convexly. When $\mu > 0.20$, the household values CBDC m more than non-CBDC q. The central bank increases its CBDC interest rate I_m.

In Figure 1k, if the household's transaction efficiency η for non-CBDC q increases, its non-CBDC holding q increases concavely, while its production p, consumption c, CBDC holding m, and utility U decrease convexly. Hence the CBDC m and non-CBDC q have switched roles compared to Figure 1j, and p, c, U are qualitatively similar. When $\eta > 0.41$, the household values non-CBDC q more than CBDC m. Interestingly, the central bank increases its CBDC interest rate I_m.

In Figure 1l, if the household's transaction efficiency λ for consumption c increases, its consumption c decreases according to (9), eventually reaching $c = 0$ when $\lambda > 0.21$. In contrast, it saves more. Hence both its CBDC holding m and non-CBDC holding q increase. The household's production p increases marginally, and its utility increases convexly. The central bank decreases its CBDC interest rate I_m.

In Figure 1m, if the household's scaling θ of the transaction cost increases, causing the transaction efficiency E in (5) to decrease, only its utility U is affected and decreases convexly. The other variables, i.e., production p, consumption c, CBDC holding m, non-CBDC holding q, and the CBDC interest rate I_m, remain unchanged.

In Figure 1n, if the equilibrium real interest rate I_r increases, the CBDC interest rate I_m increases according to (10). That induces the household to increase its CBDC holding m, decrease its non-CBDC holding q, and decrease its production p and consumption c which decreases to $c = 0$ when $I_r > 12\%$. The household's utility U increases.

In Figure 1o, if the inflation rate π increases, the impact is qualitatively similar to Figure 1n, except that the CBDC interest rate I_m becomes negative when the inflation rate π decreases below $\pi = 1.6\%$. That's because π appears twice on the right hand side of (10), and $\pi - \pi^*$ is negative when π decreases below the desired inflation rate $\pi^* = 2\%$. The central bank thus combats low and decreasing inflation π below the target inflation π^* by choosing negative CBDC interest rate I_m, thus inducing the household to increase its consumption c, production p, and non-CBDC holding q, and decrease its CBDC holding m, which causes convexly decreasing utility U. The household's consumption c decreases to $c = 0$ when $\pi > 9.80\%$.

In Figure 1p, if the desired inflation rate π^* increases, the impact is opposite that of Figure 1o. All the variables move in the opposite direction. That follows from the term $\pi - \pi^*$ in (10) and the minus sign before π^*. The CBDC interest rate I_m becomes negative when the desired inflation rate π^* increases above $\pi^* = 6.1\%$. When π^* increases above $\pi^* = 15.6\%$, the CBDC interest rate I_m decreases to its negative lower bound $z = -5\%$, causing all the six variables to remain constant when $\pi^* > 15.58\%$. As π^* increases, the household's consumption c, production p, and non-CBDC holding q increase, while its CBDC holding m and its utility U decrease.

In Figure 1q, if the household's production parameter h increases, so that it produces more effectively, its production p, consumption c, and non-CBDC holding q increase, while its CBDC holding m and utility U decrease. The CBDC interest rate I_m becomes negative when h increases above $h = 0.143$. When h increases above $h = 0.215$, the CBDC interest rate I_m decreases to its negative lower bound $z = -5\%$, causing all the six variables to remain constant.

Figure 1r replicates Figure 1q with no lower bound $z = -\infty$ on the interest rate I_m. Then the interest rate I_m decreases to $I_m = -32.5\%$ when the household eliminates its CBDC holding m to $m = 0$ when $h > 0.353$. As h increases to $h = 0.353$, the other four variables increase. That is, the household's production p, consumption c, and non-CBDC holding q increase, and the utility U is U shaped with a minimum at $h = 0.296$ and thereafter increases. The situation when $h > 0.353$ models a world with no central bank where, with these parameter values, the household benefits from high utility U.

In Figure 1s, if the household's potential production $\bar{p}$ increases to its maximum $\bar{p} = r/a = 1$, the central bank increases its CBDC interest rate to $I_m = 2.74\%$. Applying (9), that causes the household to increase its CBDC holding m, which increases its utility U, and decrease its production p, consumption c, and non-CBDC holding q. As $\bar{p}$ decreases, the CBDC interest rate I_m becomes negative when $\bar{p} < 0.144$, and decreases to the negative lower bound $z = -5\%$ when $\bar{p} = 0.067$.

In Figure 1t, if the weight assigned to inflation a_π in the Taylor (1993) rule increases, the impact is qualitatively similar to Figure 1o where the inflation rate π increases. That can be seen mathematically from the term $a_\pi(\pi - \pi^*)$ in (10). The CBDC interest rate I_m becomes negative when a_π decreases below $a_\pi = 0.241$. Since inflation then is assigned low weight a_π, and production is assigned higher weight $a_p = 1 - a_\pi$, the household chooses lower CBDC holding m, and chooses higher production p, consumption c, and non-CBDC holding q, which causes lower utility U.

In Figure 1u, if the weight assigned to production a_p in the Taylor (1993) rule increases, the impact is opposite that of Figure 1s, since $a_p = 1 - a_\pi$. Thus the CBDC interest rate I_m becomes negative when a_p increases above $a_p = 0.759$. Furthermore, the household chooses lower CBDC holding m, and chooses higher production p, consumption c, and non-CBDC holding q, which causes lower utility U.

Shocks to production, inflation, interest rates of CBDC and non-CBDC, and real interest rate

Shocks to production p

The household's production is characterized by its unit cost a of production considered in Figure 1b, and its production parameter h considered in Figure 1q and Figure 1r. Figure 1b shows increased household production p and household utility U, and the other variables are constant, as a decreases. Figure 1q and Figure 1r show increased household production p, consumption c, and non-CBDC holding q, and decreased CBDC holding m and CBDC interest rate I_m, and decreased utility U, up to a certain point, as h increases.

Shocks to inflation π and target inflation π^*

Inflation is characterized by the inflation rate π considered in Figure 1o, and the desired or target inflation rate π^* considered in Figure 1p. Figure 1o shows decreased household production p, consumption c and non-CBDC holding q, and increased CBDC holding m, utility U and CBDC interest rate I_m, as π increases. Figure 1p shows all the variables moving in the opposite direction. Hence the household prefers high inflation rate π and low target inflation rate π^*.

Shocks to the CBDC interest rate I_m

The CBDC interest rate I_m is the central bank's free choice variable. Shocks to I_m may occur if the central bank were to depart from the optimal solution analyzed in the previous sections. Considering I_m as a parameter, Figure 2 plots the household's production p, consumption c, CBDC holding m, non-CBDC holding q, and utility U, as functions of I_m ranging from $I_m = -20\%$ to $I_m = 20\%$.

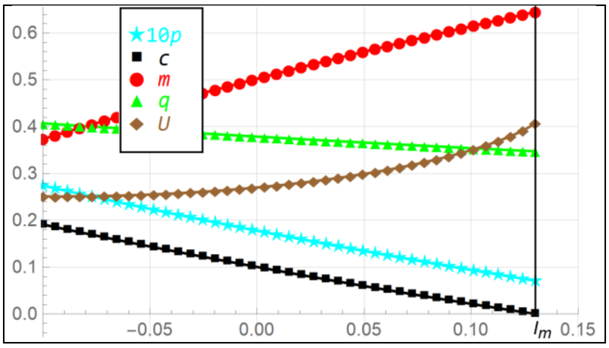

Figure 2: The household's production p, consumption c, CBDC holding m, non-CBDC holding q, and utility U as functions of the CBDC interest rate I_m as a parameter relative to the benchmark $\alpha = \beta = \gamma = \frac{1}{4}, r = a = M = Q = 1, I_q = I_r = 2\%, \eta = \frac{1}{5}, \mu = \frac{2}{5}, \lambda = \frac{1}{10}, \pi = 3\%, \pi^* = h = 2\%, \frac{1}{10}, \bar{p} = \frac{1}{2}, a_\pi = a_p = \frac{1}{2}, z = -5\%$. Multiplication of p with 10 is for scaling purposes.

In Figure 2, if the CBDC interest rate I_m increases, the five variables change as follows: First, and intuitively, the household's CBDC holding m increases. When $I_m > -0.8\%$, the household holds more CBDC m than non-CBDC q. Second, the household's non-CBDC holding q decreases due to the substitution effect between CBDC m and non-CBDC q. Third and fourth, holding more CBDC m induces the household to decrease its production p and consumption c. The household's consumption c eventually decreases to $c = 0$ when $I_m > 13\%$. Fifth, the household's utility U is U shaped, reaching a minimum of approximately 0.25 when the CBDC interest rate $I_m = -0.94\%$. Hence the household prefers I_m to be low or high. When I_m is low, the household derives moderately high utility U due to high production p and consumption c, and substantial non-CBDC holding q. When I_m is high, the household derives high utility U due to substantial CBDC holding m at a high CBDC interest rate I_m.

Shocks to the non-CBDC interest rate I_q

Shocks to the non-CBDC interest rate I_q is considered in Figure 1i which shows decreased household production p, consumption c, and CBDC holding m, and increased non-CBDC holding q, utility U, and CBDC interest rate I_m, as I_q increases. That benefits the household. The central bank needs to increase its interest rate I_m to compete.

Shocks to the real interest rate I_r

Shocks to the real interest rate I_r is considered in Figure 1n which shows decreased household production p, consumption c, and non-CBDC holding q, and increased CBDC holding m, utility U, and CBDC interest rate I_m, as I_r increases. Again, that benefits the household, and the central bank increases its interest rate I_m to compete.

Discussion and economic interpretation

The following results in the previous sections are noteworthy, related to varying 19 parameter values relative to a benchmark.

The household's production, consumption, CBDC holding and non-CBDC holding increase as its resources increase. The intuition is that it is beneficial to have more resources. Thus, as resources increase, the household's utility increases.

If the household's unit production cost increases, it intuitively decreases its production, which decreases the household's utility. The household's consumption, CBDC holding, and non-CBDC holding remain constant if the unit production cost changes.

If the household's output elasticity for production increases, it increases its production, consumption, and CBDC holding. However, the non-CBDC holding decreases since the output elasticities sum to one and higher elasticity assigned to production means lower elasticity assigned to non-CBDC holding. The CBDC interest rate becomes negative when the household holds far more CBDC than non-CBDC. Since the household saves substantially in CBDC, the central bank sees no reason to encourage further saving.

If the household's elasticity for consumption increases, it increases its consumption, but decreases its non-CBDC holding since higher elasticity assigned to consumption means lower elasticity assigned to non-CBDC holding. The CBDC interest rate, production, and CBDC holding remain constant since the CBDC interest rate is independent of the elasticity for consumption. The household's utility is U shaped in the elasticity for consumption. That is, it prefers either high or low elasticity for consumption.

If the household's output elasticity for holding CBDC increases, it increases its CBDC holding, but decreases its non-CBDC holding. The CBDC interest rate is independent of the output elasticity for holding CBDC. Thus, the household's production and consumption remain constant.

If the household's weight of the CBDC interest rate in its elasticities increases, it increases its CBDC holding and decreases its non-CBDC holding. This is similar to the increase of the output elasticity for holding CBDC. The increase of the CBDC interest rate shows the multiple tradeoffs that the central bank strikes. The household's utility increases from holding more CBDC if the CBDC interest rate increases, which offsets the decrease in production, consumption, and non-CBDC holding.

Intuitively and in contrast to the previous point, if the household's weight of the non-CBDC interest rate in its output elasticities increases, it increases its non-CBDC holding and decreases its CBDC holding.

We consider the case when the weight of the CBDC interest rate in the output elasticities equals the weight of the non-CBDC interest rate. If the household's equal weights increase, the household increases both its CBDC holding and its non-CBDC holding, and the central bank increases its CBDC interest rate. Since the weights are subtracted from production and consumption, production and consumption decrease.

If the non-CBDC interest rate increases, the household increases its non-CBDC holding and decreases its CBDC holding. This is because holding non-CBDC becomes more attractive since the household benefits from gaining higher interest from non-CBDC. The central bank chooses to increase the CBDC interest rate to compete with the non-CBDC. Hence a household's CBDC holding is typically the opposite of its non-CBDC holding, as found by Mooij (2021) and partly found by M. Davoodalhosseini et al. (2020).

If the household's transaction efficiency for CBDC increases, it intuitively increases its CBDC holding, and decreases its non-CBDC holding. The central bank increases its CBDC interest rate to support the household's CBDC holding.

In contrast to the previous point, if the household's transaction efficiency for non-CBDC increases, it increases its non-CBDC holding, and decreases its CBDC holding. Interestingly, also here the central bank increases its CBDC interest rate, to compete with the non-CBDC.

If the household's transaction efficiency for consumption increases, its consumption decreases since transactions become more costly. However, the household increases its CBDC holding and non-CBDC holding, and the central bank decreases its CBDC interest rate.

If the household's scaling of the transaction efficiency increases, the transaction efficiency decreases accordingly. Only the household's utility is impacted, and it decreases.

If the real interest rate increases, the CBDC interest rate increases, in accordance with the Taylor (1993) rule. Thus, if the real interest rate increases, the household holds more CBDC and less non-CBDC. Meanwhile, the household's production and consumption decrease.

Similarly to the previous point, if the inflation rate increases, the CBDC interest rate increases, encouraging more CBDC saving and less non-CBDC saving. The central bank combats low inflation via a negative CBDC interest rate.

In contrast to the previous point, if the target inflation rate increases, the CBDC interest rate decreases. The central bank combats a high target inflation rate through a negative CBDC interest rate. That, in turn, induces agents to save less CBDC and consume more, as also found by Jia (2020).

If the household's production parameter increases, it increases its production, consumption, and non-CBDC holding, but decreases its CBDC holding. This is because the household produces more effectively. The central bank decreases its CBDC interest rate, which also enhances the household's production and consumption.

If the household's potential production increases, the central bank increases its CBDC interest rate. Thus, the household increases its CBDC holding, and decreases its production, consumption, and non-CBDC holding.

If the weight assigned to inflation in the Taylor (1993) rule increases, the impact is similar to increasing the inflation rate. The central bank chooses negative interest rate when inflation is assigned low weight. Since a higher weight assigned to inflation means a lower weight assigned to production, the household's production and consumption decrease. In contrast, Grasselli and Lipton (2019) show that negative interest rates have lower impact on consumption than on investment.

The following further results are noteworthy, related to analyzing the impacts of shocks to production, inflation, the CBDC interest rate, the non-CBDC interest rate, and the real interest rate.

Production shocks are captured by the unit cost of production and production parameter. If the unit cost decreases, the household's production increases, and the other variables remain constant. If the production parameter increases, the household's production, consumption, and non-CBDC holding increase, while the household's CBDC holding and utility, and the CBDC interest rate, decrease.

Inflation shocks are characterized by changes to the inflation rate and the target inflation rate. If the inflation rate increases, the household's production, consumption, and non-CBDC holding decrease, and the CBDC interest rate, CBDC holding and utility increase.

If the CBDC interest rate increases, the household increases its CBDC holding, and decreases its non-CBDC holding. The household decreases its production and consumption. Its utility is U shaped. When the CBDC interest rate is low, the household gains utility from production, consumption, and non-CBDC holding. When the CBDC interest rate is high, the household gains utility from holding CBDC with high CBDC interest return.

The non-CBDC interest shock shows that if the non-CBDC interest rate increases, the household decreases its production, consumption, and CBDC holding, increases its non-CBDC holding, and eventually earns higher utility.

The real interest rate shock shows that if the real interest rate increases, the household decreases its production, consumption, and non-CBDC holding, and increases its CBDC holding, while the central bank increases its CBDC interest rate.

Conclusion

The article explores a two-period decision model between a central bank and a representative household. The central bank applies the Taylor (1993) rule to choose a positive or negative interest rate. The representative household owns resources or energy allocated into production, consumption, CBDC (central bank digital currency) holding, and non-CBDC holding. The non-CBDC holding can be various cryptocurrencies like Bitcoin, Ethereum, etc., or stocks, bonds, real estate, etc. A Cobb Douglas utility with elasticities for the household's allocations is presented, and adjusted by the CBDC interest rate, the non-CBDC interest rate, and the transaction efficiency. In period 1, the central bank chooses its interest rate. In period 2, the household determines its production, consumption, CBDC holding and non-CBDC holding.

The article shows that if the household's output elasticities for production, consumption, CBDC holding, and non-CBDC holding change, the household's strategies change as expected. The central bank chooses negative interest rate when the household holds far more CBDC than non-CBDC, to discourage further saving. Increasing the non-CBDC interest rate, which causes the household to hold more non-CBDC and less CBDC, induces the central bank to increase its CBDC interest rate to compete with the threat from the attractive non-CBDC interest rate. Increasing the household's transaction efficiencies for CBDC and non-CBDC cause the central bank to increase its CBDC interest rate, to support the household's holding of CBDC and compete with the non-CBDC, respectively. However, increasing the household's transaction efficiency for consumption has the opposite impact of decreasing the CBDC interest rate. Decreasing the real interest rate or the inflation rate or the household's potential production or the weight assigned to inflation in the Taylor (1993) rule, or increasing the target inflation rate or the household's production parameter, causes lower and eventually negative CBDC interest rate, which induces the household to hold less CBDC, more non-CBDC, produce and consume more, and earn lower utility.

Positive shocks to production cause lower and eventually negative CBDC interest rate. The household holds less CBDC and earns lower utility, but produces and consumes more and holds more non-CBDC. Positive inflation shocks cause the household to hold more CBDC and earn higher utility due to a higher CBDC interest rate, while the production, consumption, and non-CBDC holding decrease. Positive shocks to the CBDC interest rate cause the household to hold more CBDC and less non-CBDC, and conversely for positive shocks to the non-CBDC interest rate. Both these two shocks cause the household to produce and consume less and eventually earn higher utility. Positive shocks to the real interest rate cause higher CBDC interest rate.

Future research, which implicitly illustrates limitations of the article, should consider the interactions of several CBDCs and non-CBDCs. More players can be introduced, e.g. governments, commercial banks, firms, etc. Various negative interest rate bounds, and corner solutions can be analyzed. In addition, the burning and issuance of CBDCs and non-CBDCs should be analyzed. Expansion should be made to heterogeneous households. Each household's Cobb Douglas utility should be expanded to account for additional factors such as safety, convenience, taxes, etc. The analysis can also be generalized to allow each household and one or multiple central banks to choose their strategies simultaneously or sequentially in one-period or repeated games. More extensive empirical research should be conducted.

Appendix A Nomenclature

Parameters

r Household's monetary energy, or resources, $r \geq 0$
a Household's unit cost of production, $a \geq 0$
α Household's output elasticity for production p, $0 \leq \alpha \leq 1$
β Household's output elasticity for consumption c, $0 \leq \lambda \leq \beta \leq 1$
γ Household's output elasticity for CBDC m, $0 \leq \gamma \leq 1$
M Household's weight of the CBDC interest rate I_m in its output elasticities, $M \geq 0$
Q Household's weight of the non-CBDC interest rate I_q in its output elasticities, $Q \geq 0$
$1-\alpha-\beta-\gamma+2QI_q$ Household's output elasticity for non-CBDC q, $0 \leq 1-\alpha-\beta-\gamma+2QI_q \leq 1$
I_q Non-CBDC interest rate, $I_q \in \mathbb{R}$
μ Household's transaction efficiency for CBDC m, $\mu \geq 0$
η Household's transaction efficiency for non-CBDC q, $\eta \geq \lambda$
λ Household's transaction efficiency for consumption c, $0 \leq \lambda \leq \beta \leq 1$
θ Scaling or degree or level of the household's transaction cost, $\theta \geq 0$.
I_r The equilibrium real interest rate, $I_r \in \mathbb{R}$
π The inflation rate, $\pi \in \mathbb{R}$
π^* The desired or target inflation rate, $\pi^* \in \mathbb{R}$
h The household's production parameter, $h \geq 0$

$\bar{p}^h$ The household's potential production, $0 \leq \bar{p} \leq r/a$
a_π The weight assigned to inflation in the Taylor (1993) rule, $0 \leq a_\pi \leq 1$
$a_p = 1 - a_\pi$ The weight assigned to production in the Taylor (1993) rule, $0 \leq a_p \leq 1$
z The negative lower bound on the interest rate I_m, $z \leq 0$

Household's free choice variables
p Household's production, $0 \leq p \leq r/a$
c Household's consumption, $0 \leq c \leq r$
m Household's CBDC holding, $0 \leq m \leq r$

Dependent variables
I_m CBDC interest rate for the household's CBDC holding m, $I_m \in \mathbb{R}$
U Household's utility
$q = r - ap - c - m$ Household's non-CBDC holding, $0 \leq q = r - ap - c - m \leq r$
E Household's transaction efficiency

Appendix B The derivatives for the transaction efficiency *E*

Differentiating the transaction efficiency E in (5) with respect to c, m and q gives

$$\frac{\partial E}{\partial c} = -\frac{c^{-1-\lambda} m^\mu q^\eta \lambda}{\theta} \leq 0, \frac{\partial E}{\partial m} = \frac{c^{-\lambda} m^{-1+\mu} q^\eta \mu}{\theta} \geq 0, \frac{\partial E}{\partial q} = \frac{c^{-\lambda} m^\mu q^{-1+\eta} \eta}{\theta} \geq 0 \tag{11}$$

The second derivatives of the transaction efficiency E in (5) with respect to c, m and q gives

$$\begin{aligned}
&\frac{\partial^2 E}{\partial c^2} = \frac{c^{-2-\lambda} m^\mu q^\eta \lambda(1+\lambda)}{\theta} \geq 0,\\
&\frac{\partial^2 E}{\partial m^2} = \frac{c^{-\lambda} m^{-2+\mu} q^\eta (-1+\mu)\mu}{\theta} \leq 0, when\ \mu \leq 1,\\
&\frac{\partial^2 E}{\partial q^2} = \frac{c^{-\lambda} m^\mu q^{-2+\eta} (-1+\eta)\eta}{\theta} \leq 0, when\ \mu \leq 1,\\
&\frac{\partial^2 E}{\partial c \partial m} = -\frac{c^{-1-\lambda} m^{-1+\mu} q^\eta \lambda\mu}{\theta} \leq 0,\\
&\frac{\partial^2 E}{\partial c \partial q} = -\frac{c^{-1-\lambda} m^\mu q^{-1+\eta} \eta\lambda}{\theta} \leq 0
\end{aligned} \tag{12}$$

Appendix C Proof of Property 1

Calculating the defivative of the household's utility U in (7) with respect to its free choice variables p, c and m, and equating to zero, gives

$$\begin{aligned}
\frac{\partial U}{\partial p} &= \frac{1}{\theta} c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}\\
&\times m^{\gamma+2MI_m+\mu} p^{-1+h(\alpha-MI_m-QI_q)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q}\\
&\times \left(h(r-c-m-ap)\left(\alpha-MI_m-QI_q\right)-ap\left(1-\alpha-\beta-\gamma+\eta+2QI_q\right)\right)=0,
\end{aligned} \tag{13}$$

$$\begin{aligned}
\frac{\partial U}{\partial c} &= \frac{1}{\theta} c^{-1+\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}\\
&\times m^{\gamma+2MI_m+\mu} p^{h(\alpha-MI_m-QI_q)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q}\\
&\times \left((r-m-ap)\left(\beta-\lambda-MI_m-QI_q\right)-c\left(1-MI_m+QI_q-\alpha-\gamma+\eta-\lambda\right)\right)=0,
\end{aligned} \tag{14}$$

$$\begin{aligned}
\frac{\partial U}{\partial m} &= \frac{1}{\theta} c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}\\
&\times m^{-1+\gamma+2MI_m+\mu} p^{h(\alpha-MI_m-QI_q)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q}\\
&\times \left((r-c-ap)(\gamma+2MI_m+\mu)-m(1-\alpha-\beta+\eta+\mu+2MI_m+2IqQ)\right)=0,
\end{aligned} \tag{15}$$

which are solved to yield p, c and m in (9). The dependent variable q follows from solving (2) with respect to q and inserting p, c and m. The second order conditions, inserting (14) to (14), are

$$\begin{aligned}
\frac{\partial^2 U}{\partial p^2} &= -\frac{1}{\theta} c^{\beta-\lambda-MI_m-QI_m}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}\\
&\times m^{\gamma+2MI_m+\mu} p^{h(\alpha-MI_m-QI_q)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q}
\end{aligned} \tag{16}$$

$$\times\left(\frac{2ah(\alpha - MI_m - QI_q)(1-\alpha-\beta-\gamma+\eta+2QI_q)}{p} + \frac{h(r-c-m-ap)\left(1-h(\alpha-MI_m-QI_q)\right)(\alpha-MI_m-QI_q)}{p^2} + \frac{a^2(\alpha+\beta+\gamma-\eta-2QI_q)(1-\alpha-\beta-\gamma+\eta+2QI_q)}{r-c-m-ap}\right),$$

$$\frac{\partial^2 U}{\partial c^2} = -\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_m}(1+I_m)^{\gamma+2MI_m}(1+I_q)^{1-\alpha-\beta-\gamma+2QI_q} \times m^{\gamma+2MI_m+\mu}p^{h(\alpha-MI_m-QI_q)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q} \times\left(\frac{2(1-\alpha-\beta-\gamma+\eta+2QI_q)(\beta-\lambda-MI_m-QI_m)}{c} + \frac{(\alpha+\beta+\gamma-\eta-2QI_q)(1-\alpha-\beta-\gamma+\eta+2QI_q)}{r-c-m-ap} + \frac{(r-c-m-ap)(\beta-\lambda-MI_m-QI_m)(1-\beta+\lambda+MI_m+QI_q)}{c^2}\right), \tag{17}$$

$$\frac{\partial^2 U}{\partial m^2} = -\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_m}(1+I_m)^{\gamma+2MI_m}(1+I_q)^{1-\alpha-\beta-\gamma+2QI_q} \times m^{\gamma+2MI_m+\mu}p^{h(\alpha-MI_m-QI_q)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q} \times\left(\frac{2(1-\alpha-\beta-\gamma+\eta+2I_qQ)(\gamma+2MI_m+\mu)}{m} + \frac{(\alpha+\beta+\gamma-\eta-2QI_q)(1-\alpha-\beta-\gamma+\eta+2QI_q)}{r-c-m-ap} + \frac{(r-c-m-ap)(1-\gamma-\mu-2MI_m)(\gamma+2MI_m+\mu)}{m^2}\right) \tag{18}$$

Appendix D Proof of Property 3

Differentiating (9) gives

$$\begin{aligned}
&\frac{\partial p}{\partial \beta} = \frac{\partial p}{\partial \gamma} = \frac{\partial p}{\partial \theta} = 0, \frac{\partial^2 p}{\partial \beta^2} = \frac{\partial^2 p}{\partial \gamma^2} = \frac{\partial^2 p}{\partial \theta^2} = 0,\\
&\frac{\partial p}{\partial r} = \frac{h(\alpha-MI_m-QI_q)}{a(1-(1-h)(\alpha-MI_m-QI_q)+\eta-\lambda+\mu)}, \frac{\partial^2 p}{\partial r^2} = 0,\\
&\frac{\partial p}{\partial a} = \frac{-hr(\alpha-MI_m-QI_q)}{a^2(1-(1-h)(\alpha-MI_m-QI_q)+\eta-\lambda+\mu)},\\
&\frac{\partial^2 p}{\partial a^2} = \frac{2hr(\alpha-MI_m-QI_q)}{a^3(1-(1-h)(\alpha-MI_m-QI_q)+\eta-\lambda+\mu)}
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\frac{\partial c}{\partial \beta} = \frac{r}{1-(1-h)(\alpha-MI_m-QI_q)+\eta-\lambda+\mu}, \frac{\partial^2 c}{\partial \beta^2} = 0,\\
&\frac{\partial c}{\partial \gamma} = \frac{\partial c}{\partial \gamma^2} = 0, \frac{\partial c}{\partial \theta} = \frac{\partial c}{\partial \theta^2} = 0,\\
&\frac{\partial c}{\partial r} = \frac{\beta-\lambda-MI_m-QI_q}{1-(1-h)(\alpha-MI_m-QI_q)+\eta-\lambda+\mu}, \frac{\partial^2 c}{\partial r^2} = 0,\\
&\frac{\partial c}{\partial a} = \frac{\partial^2 c}{\partial a^2} = 0
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\frac{\partial m}{\partial \beta} = \frac{\partial^2 m}{\partial \beta^2} = 0,\\
&\frac{\partial m}{\partial \gamma} = \frac{r}{1-(1-h)(\alpha-MI_m-QI_q)+\eta-\lambda+\mu}, \frac{\partial^2 m}{\partial \gamma^2} = 0,\\
&\frac{\partial m}{\partial \theta} = \frac{\partial^2 m}{\partial \theta^2} = 0,
\end{aligned} \tag{21}$$

$$\frac{\partial m}{\partial r}=\frac{r}{1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu},\frac{\partial^2 m}{\partial r^2}=0,$$
$$\frac{\partial m}{\partial a}=\frac{\partial^2 m}{\partial a^2}=0$$

$$\begin{aligned}
&\frac{\partial q}{\partial \beta}=\frac{\partial q}{\partial \gamma}=\frac{-r}{1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu},\frac{\partial^2 q}{\partial \beta^2}=\frac{\partial^2 q}{\partial \gamma^2}=0,\\
&\frac{\partial q}{\partial \theta}=\frac{\partial^2 q}{\partial \theta^2}=0,\\
&\frac{\partial q}{\partial r}=\frac{1-\alpha-\beta-\gamma+\eta+2QI_m}{1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu},\frac{\partial^2 q}{\partial r^2}=0,\\
&\frac{\partial q}{\partial a}=\frac{\partial^2 q}{\partial a^2}=0
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\frac{\partial U}{\partial \beta}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}\\
&\qquad\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{1-\alpha-\beta-\gamma+\eta+2QI_q}Log\left(\frac{c}{\left(1+I_q\right)(r-c-m-ap)}\right),\\
&\frac{\partial^2 U}{\partial \beta^2}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}\\
&\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{1-\alpha-\beta-\gamma+\eta+2QI_q}\\
&\times\left(Log\left(\frac{\left(1+I_q\right)(r-c-m-ap)}{c}\right)\right)^2,\\
&\frac{\partial U}{\partial \gamma}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}\\
&\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{1-\alpha-\beta-\gamma+\eta+2QI_q}Log\left(\frac{m(1+I_m)}{\left(1+I_q\right)(r-c-m-ap)}\right),\\
&\frac{\partial^2 U}{\partial \gamma^2}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}\\
&\times m^{\gamma+2MI_m+\mu}p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{1-\alpha-\beta-\gamma+\eta+2QI_q}\\
&\times\left(Log\left(\frac{m(1+I_m)}{\left(1+I_q\right)(r-c-m-ap)}\right)\right)^2,\\
&\frac{\partial U}{\partial \theta}=-\frac{1}{\theta^2}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}\\
&\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{1-\alpha-\beta-\gamma+\eta+2QI_q},\\
&\frac{\partial^2 U}{\partial \theta^2}=\frac{2}{\theta^3}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}\\
&\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{1-\alpha-\beta-\gamma+\eta+2QI_q},\\
&\frac{\partial U}{\partial r}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}\\
&\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q}\left(1-\alpha-\beta-\gamma+\eta+2QI_q\right),\\
&\frac{\partial^2 U}{\partial r^2}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}\\
&\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{-1-\alpha-\beta-\gamma+\eta+2QI_q}\left(1-\alpha-\beta-\gamma+\eta+2QI_q\right)\\
&\times(-\alpha-\beta-\gamma+\eta+2QI_q),\\
&\frac{\partial U}{\partial a}=-\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}\\
&\times p^{1+h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q}\left(1-\alpha-\beta-\gamma+\eta+2QI_q\right),\\
&\frac{\partial^2 U}{\partial a^2}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}\\
&\times p^{2+h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{-1-\alpha-\beta-\gamma+\eta+2QI_q}\left(1-\alpha-\beta-\gamma+\eta+2QI_q\right)\\
&\times(-\alpha-\beta-\gamma+\eta+2QI_q)
\end{aligned} \tag{23}$$

Appendix E Proof of Property 4

Differentiating (9) when I_m is a parameter gives

$$\frac{\partial p}{\partial I_m}=-\frac{rhM(1+\eta-\lambda+\mu)}{a\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^2},$$
$$\frac{\partial^2 p}{\partial I_m^2}=\frac{2rhM^2(1-h)(1+\eta-\lambda+\mu)}{a\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^3},$$
$$\frac{\partial c}{\partial I_m}=-\frac{rM(1+(1-h)(\beta-\alpha-\lambda)+\eta-\lambda+\mu)}{\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^2},$$
$$\frac{\partial^2 c}{\partial I_m^2}=\frac{2rM^2(1-h)(1+(1-h)(\beta-\alpha-\lambda)+\eta-\lambda+\mu)}{\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^3},$$
$$\frac{\partial m}{\partial I_m}=\frac{rM\left(2-2\left((1-h)\left(\alpha-QI_q\right)-\eta+\lambda\right)-(1-h)\gamma+(1+h)\mu\right)}{\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^2}, \tag{24}$$
$$\frac{\partial^2 m}{\partial I_m^2}=-\frac{rM^2(1-h)\left(2-2\left((1-h)QI_q+\alpha-\eta+\lambda\right)-\gamma+\mu+h(2\alpha+\gamma+\mu)\right)}{\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^3},$$
$$\frac{\partial q}{\partial I_m}=-\frac{(1-h)Mr(1-\alpha-\beta-\gamma+\eta+2QI_q)}{\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^2},$$
$$\frac{\partial^2 q}{\partial I_m^2}=\frac{2rM^2(1-h)^2\left(1-\alpha-\beta-\gamma+\eta+2QI_q\right)}{\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^3}$$

Combining $0\le\alpha-MI_m-QI_q\le 1$ and $\beta-\lambda-MI_m-QI_q\ge 0$ gives $\beta-\alpha-\lambda\ge MI_m+QI_q-\alpha\ge -1$. Hence $1+(1-h)(\beta-\alpha-\lambda)\ge 0$ if $0\le h<1$, causing $\frac{\partial c}{\partial I_m}<0$ and $\frac{\partial^2 c}{\partial I_m^2}>0$.

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