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Alexander Olevskii and Alexander Ulanovskii

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Fourier Quasicrystals with Unit Masses

Alexander Olevskii[a] **and Alexander Ulanovskii**[*, b]

[a] School of Mathematics, Tel Aviv University, 69978 Ramat Aviv, Israel.
[b] Stavanger University, 4036 Stavanger, Norway.
E-mails: olevskii@post.tau.ac.il (A. Olevskii), Alexander.Ulanovskii@uis.no
(A. Ulanovskii)

Abstract. The sum of δ-measures sitting at the points of a discrete set $\Lambda \subset \mathbb{R}$ forms a Fourier quasicrystal if and only if Λ is the zero set of an exponential polynomial with imaginary frequencies.

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1. Introduction

By a Fourier quasicrystal one usually means a complex measure with discrete support and spectrum. This concept goes back to works of Yves Meyer in the 1970-ies and it reappeared later in connection with an unexpected phenomenon in crystallography discovered by Dan Shechtman in the 1980-ies, see [5].

More precisely, following [6] we call a measure μ on $\mathbb{R}$ a *crystalline measure*, if it is an atomic measure which is a tempered distribution, its distributional Fourier transform $\widehat{\mu}$ is an atomic measure and both the support Λ and the spectrum S of μ are locally finite sets. If in addition the measures $|\mu|$ and $|\widehat{\mu}|$ are also tempered, then μ is called a *Fourier quasicrystal* (FQ).

The classical example of an FQ is the Dirac comb (the crystal)

$$\mu = \sum_{k \in \mathbb{Z}} \delta_k,$$

where δ_x is the unit mass at point x. Then the Poisson summation formula reads $\widehat{\mu} = \mu$.

Examples of aperiodic quasicrystals were presented in [3] and then in [1, 6, 7]. Recently a new progress was achieved by P. Kurasov and P. Sarnak [2] who discovered examples of FQs with unit masses

$$\mu = \sum_{\lambda \in \Lambda} \delta_\lambda, \tag{1}$$

where $\Lambda \subset \mathbb{R}$ is a uniformly discrete aperiodic set. An alternative construction of such measures was suggested by Y.Meyer [8].

* Corresponding author.

Below we present one more construction and prove that it characterizes all FQs of form (1). A preliminary publication of our results was given in arXiv [9, 10].

The Theorem 1 below reveals a fundamental connection between FQs with unit masses and the zero sets $Z(p) := \{z \in \mathbb{C} : p(z) = 0\}$ of exponential polynomials p with imaginary frequencies.

Theorem 1.

(i) *Let p be an exponential polynomial*

$$p(t) = \sum_{1 \le j \le N} c_j e^{2\pi i \gamma_j t}, \quad N \in \mathbb{N}, c_j \in \mathbb{C}, \gamma_j \in \mathbb{R}, \tag{2}$$

which has only simple real zeros. Then the measure μ defined in (1) *with $\Lambda = Z(p)$ is an FQ.*

(ii) *Conversely, let μ be an FQ of form* (1). *Then there is an exponential polynomial p of form* (2) *with real simple zeros such that $\Lambda = Z(p)$.*

We will sketch the proof of part (ii), see [10] for the proof of part (i).

Using Theorem 1 (i) one may construct simple examples of aperiodic FQs.

Lemma 2. *Fix a real number ϵ satisfying $0 < |\epsilon| \le 1/2$ and set*

$$p_\epsilon(t) := \sin(\pi t) + \epsilon \sin t. \tag{3}$$

Then p_ϵ has only simple real zeros and

$$Z(p_\epsilon) = \{k + \epsilon_k : k \in \mathbb{Z}\}, \quad \epsilon_k \in [-1/6, 1/6].$$

For a proof see [10].

Theorem 1 and Lemma 2 show that the sum of δ-measures sitting at the points of $Z(p_\epsilon)$ is an FQ.

Let p_ϵ be given in (3). One may check that the numbers ϵ_k in Lemma 2 satisfy $\max_k |\epsilon_k| \to 0$ as $\epsilon \to 0$. Therefore, the set $Z(p_\epsilon)$ “approaches” the set of integers $\mathbb{Z}$:

Corollary 3. *For every $\epsilon > 0$ there is an aperiodic set*

$$\Lambda = \{k + \epsilon_k : k \in \mathbb{Z}\}, \quad 0 < |\epsilon_k| < \epsilon, k \in \mathbb{Z},$$

such that the corresponding measure in (1) *is an FQ.*

2. Proof of Part (ii) of Theorem 1

In what follows we consider the standard form of the Fourier transform

$$\widehat{h}(u) := \int_{\mathbb{R}} e^{-2\pi i u t} h(t)\, dt, \quad h \in L^1(\mathbb{R}).$$

Let us start with a result which may have intrinsic interest:

Proposition 4. *Let μ be a positive measure which is a tempered distribution, such that its distributional Fourier transform $\widehat{\mu}$ is a measure satisfying*

$$\left|\widehat{\mu}\right|(-R, R) = O\left(R^m\right),\ R \to \infty, \quad \textit{for some } m > 0, \tag{4}$$

which means that $|\widehat{\mu}|$ is a tempered distribution. Then there exists C such that

$$\mu(a, b) \le C(1 + b - a), \quad -\infty < a < b < \infty. \tag{5}$$

Proof. It suffices to prove (5) for every interval (a,b) satisfying $b-a\geq 2$.

Fix any non-negative Schwartz function $g(x)$ supported by $[-1/2,1/2]$ and such that
$$\int_{\mathbb{R}} g(x)\,dx = 1.$$
Set
$$f(x) := \left(g * 1_{(a-1/2,\,b+1/2)}\right)(x) \in S(\mathbb{R}).$$
Clearly,
$$\left|\widehat{f}(t)\right| = \left|\widehat{g}(t)\widehat{1}_{(a-1/2,\,b+1/2)}(t)\right| \leq (1+b-a)\left|\widehat{g}(t)\right|.$$
Using this inequality and (4), we get
$$\int_{\mathbb{R}} f(x)\mu(dx) = \int_{\mathbb{R}} \widehat{f}(t)\widehat{\mu}(dt) \leq (1+b-a)\int_{\mathbb{R}} \left|\widehat{g}(t)\right|\,\left|\widehat{\mu}\right|(dt) = C(1+b-a).$$
On the other hand, clearly,
$$f(x) = g(x) * 1_{(a-1/2,\,b+1/2)}(x) = 1, \quad x \in (a,b).$$
Hence,
$$\int_{\mathbb{R}} f(t)\mu(dt) \geq \mu(a,b),$$
which proves the Proposition 4.

Recall that a set $\Lambda\subset\mathbb{R}$ is called uniformly discrete, if
$$\inf_{\lambda',\lambda\in\Lambda,\lambda\neq\lambda'} \left|\lambda-\lambda'\right| > 0.$$
A set Λ is called relatively uniformly discrete if it is a union of finite number of uniformly discrete sets.

Proposition 4 implies

Corollary 5. *Let μ be a measure of form* (1) *whose distributional Fourier transform is a measure satisfying* (4)*. Then its support Λ is a relatively uniformly discrete set.*

Assume μ is an FQ of form (1). This means that a Poisson-type formula
$$\sum_{\lambda\in\Lambda} f(\lambda) = \sum_{s\in S} a_s\widehat{f}(s), \quad f\in S(\mathbb{R}), \tag{6}$$
is true where $S(\mathbb{R})$ denotes the Schwartz space, S is locally finite set and the coefficients a_s satisfy
$$\sum_{s\in S,|s|<R} |a_s| \leq CR^m,\ R>1, \quad \text{for some } C,m>0. \tag{7}$$
To prove part (ii) of Theorem 1 we have to show that $\Lambda = Z(p)$ for some exponential polynomial p of form (2). We will prove this under the additional restrictions that Λ is a symmetric set, $-\Lambda=\Lambda$ and $0\notin\Lambda$. For the general case see [9].

Set
$$\psi(z) := \prod_{\lambda\in\Lambda}\left(1-\frac{z}{\lambda}\right) = \prod_{\lambda\in\Lambda,\lambda>0}\left(1-\frac{z^2}{\lambda^2}\right), \quad z\in\mathbb{C}. \tag{8}$$
The product converges (uniformly on compacts) due to Corollary 5.

Lemma 6. *ψ is an entire function of order one and finite type, i.e. there exist $C,\sigma>0$ such that*
$$\left|\psi(z)\right| \leq Ce^{\sigma|z|}, \quad z\in\mathbb{C}.$$

This lemma follows from Corollary 5 and the symmetry of Λ by standard estimates.

Lemma 7. *The following representation is true:*
$$\frac{\psi'(z)}{\psi(z)} = -2\pi i\left(a_0/2 + \sum_{s\in S\cap(-\infty,0)} a_s e^{-2\pi i s z}\right), \quad \operatorname{Im} z>0, \tag{9}$$
where a_s are the coefficients in (6)*.*

By (7), the series in (9) converges absolutely for every z, $\operatorname{Im} z > 0$.

Let us sketch a proof of Lemma 7. It follows from (8) that

$$\frac{\psi'(z)}{\psi(z)} = \sum_{\lambda\in\Lambda} \frac{1}{z-\lambda}, \quad z \in \mathbb{C}. \tag{10}$$

The next step is to check that

$$\sum_{\lambda\in\Lambda} \frac{1}{z-\lambda} = -2\pi i \left(a_0/2 + \sum_{s\in S\cap(-\infty,0)} a_s e^{-2\pi i s z} \right), \quad \operatorname{Im} z > 0. \tag{11}$$

This can be done as follows: For every fixed z, $\operatorname{Im} z > 0$, set

$$e_z(u) = \begin{cases} 2\pi e^{-2\pi i z u} & u < 0 \\ 0 & u \ge 0 \end{cases}$$

Then the inverse Fourier transform of e_z is the function $i/(z-t)$. Fix any function $h \in S(\mathbb{R})$ such that $h(0) = 1$ and the Fourier transform $H := \widehat{h}$ is even, non-negative and vanishes outside $(-1,1)$. Then use (6) with $f(t) = h(\epsilon t)/(z-t)$:

$$\sum_{\lambda\in\Lambda} \frac{h(\epsilon\lambda)}{z-\lambda} = -i \sum_{s\in S} a_s \left(e_z(u) * \frac{1}{\epsilon} H(u/\epsilon) \right)(s).$$

Finally, to prove Lemma 7 one lets $\epsilon \to 0$ and checks that the right and left hand-sides above converge to the corresponding sides of (11).

Now, it follows from (9) that there exists $K \in \mathbb{C}$ such that

$$\psi(z) = K \exp\left(-\pi i a_0 z + \sum_{s\in S\cap(-\infty,0)} (a_s/s) e^{-2\pi i s z} \right), \qquad \operatorname{Im} z > 0.$$

Set

$$p(z) := e^{\pi i a_0 z} \psi(z)/K = \exp\left(\sum_{s\in S\cap(-\infty,0)} (a_s/s) e^{-2\pi i s z} \right), \quad \operatorname{Im} z > 0. \tag{12}$$

Recall that S is a locally finite set. Therefore, by (7) the series above converges absolutely for every z, $\operatorname{Im} z > 0$.

Denote by S_k the sets

$$S_1 := S \cap (-\infty, 0), \quad S_2 := S_1 + S_1, \quad S_3 := S_1 + S_1 + S_1, \ldots$$

Denote by $a_{s,k}$ the coefficients of the series

$$\frac{1}{k!} \left(\sum_{s\in S\cap(-\infty,0)} (a_s/s) e^{-2\pi i s z} \right)^k = \sum_{s\in S_k} a_{s,k} e^{-2\pi i s z}, \quad k \in \mathbb{N}, \operatorname{Im} z > 0.$$

Then by (12) we get a representation

$$p(z) = 1 + \sum_{k=1}^{\infty} \sum_{s\in S_k} a_{s,k} e^{-2\pi i s z},$$

where the double series converges absolutely for every z, $\operatorname{Im} z > 0$. Set

$$U := \{0\} \bigcup_{j=1}^{\infty} S_j \subset (-\infty, 0].$$

One may check that U is a locally finite set and that p admits a representation

$$p(z) = \sum_{u\in U} d_u e^{-2\pi i u z}, \quad \operatorname{Im} z > 0, \tag{13}$$

where the series converges absolutely.

To prove part (ii) of Theorem 1 it remains to check that the series in the right hand-side of (13) contains only a finite number of terms. This can be done as follows: Since ψ is an entire function of order one and finite type, the same is true for p. By (13), p is bounded on every line $\operatorname{Im} z = const > 0$. It follows that (see [4, Lecture 6, Theorem 2]) p is an entire function of exponential type, i.e. it satisfies

$$\left|p(x+iy)\right| \le Ce^{\sigma|y|}, \quad x, y \in \mathbb{R},$$

with some $C, \sigma > 0$. Now, to check that in (13) we have $d_u = 0$ for every $u \in U$, $|u| > \sigma$, one simply integrates both sides against $e^{2\pi iuz}(\sin\epsilon z/\epsilon z)^2$, where $\epsilon > 0$ is so small that $|u| - \epsilon > \sigma$ and $U \cap (u-\epsilon, u+\epsilon) = \{u\}$.

We note that one can extend Theorem 1 to measures with integer masses,

$$\mu = \sum_{\lambda \in \Lambda} c_\lambda \delta_\lambda, \quad c_\lambda \in \mathbb{N}, \lambda \in \Lambda. \tag{14}$$

Theorem 8.

(i) *If a measure μ of form* (14) *is an FQ, then there is an exponential polynomial p of form* (2) *with real zeros such that $\Lambda = Z(p)$ and $c(\lambda)$ is the multiplicity of zero λ.*

(ii) *Conversely, let p be an exponential polynomial of form* (2) *with real zeros and let $c(\lambda)$ be the multiplicity of zero λ. Then the measure μ of form* (14) *where $\Lambda = Z(p)$ is an FQ.*

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