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An introduction to differential forms and topological field theory

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Abstract

In this thesis, we will use topological field theory to do an explicit computation of a one-loop partition function over a 6-dimensional manifold. This is a topological invariant, which can be used to distinguish geometries. To set us up for this task, we will introduce various topics in mathematics and physics. The topics covered are; manifolds, differential forms, cohomology, Hodge theory, Lagrangian formalism, and quantum field theory.

Acknowledgements

Thanks to my supervisor Eirik Eik Svanes for teaching me about the topics covered in this thesis, his valuable feedback, and his enthusiastic teaching. He has also shown me a bit of how research within mathematics and physics works. Building on the computation of this thesis, he will, together with some colleagues, publish an article later this year, where I will also contribute. In like manner, I want to thank him for having a great interest in showing me the possibilities for Ph.D. positions. He also made it possible for me to attend a winter school on general relativity and black holes earlier this year.

Thanks to my girlfriend and family for supporting me throughout my five years of study.

Contents

Chapter 1

Introduction

This thesis is in the realm of mathematical physics, the cross-section between mathematics and physics. Our goal for the thesis will be to compute a so-called one-loop partition function over a 6-dimensional manifold. The reason for looking at six dimensions stems from String Theory [1]. String theory is an alternative description of how the universe works and one of the theories that tries to unify the Standard Model together with general relativity. Since the dawn of physics, one has tried to find one grand theory that describes everything about how the universe works, a "theory of everything". The Standard Model has managed to do so to quite some extent. However, the Standard Model fails at a rather crucial point. Namely, it does not include gravity. Simply speaking, this theory fails to unify quantum field theory and general relativity, which is where String Theory has tried to come up with the solution. The driving force behind String Theory is to be a so-called theory of everything and resolve the problem of a quantum description of gravity. For String Theory to work out, we need more dimensions than our four-dimensional spacetime. In fact, the theory requires ten dimensions to work. Therefore, what one does in String Theory is to split up these ten dimensions into our 4-dimensional spacetime and six dimensions which are compactified and often modeled as a so-called Calabi-Yau manifold. This is a complex 3-dimensional manifold, or one can view it as a 6-dimensional real manifold.

To this date, String Theory is still governed as very theoretical. One of the problems in String Theory is that one quickly runs into extremely complicated mathematics. Therefore, one has tried to look for results that might simplify the theory. Examples of such simplifying results are reductions of the theory to topological sectors, where one can compute topological invariants. This thesis will then focus on developing the tools needed in order to look at topological invariants in the form of using topological field theories to compute partition functions. These will be examples of topological quantum invariants, which are topological invariants calculated using the partition function defined in quantum field theory (QFT). These invariants will then be related to local QFT.

This thesis is meant as a brief introduction to concepts in topological field theory and computations of the one-loop partition function for topological field theories. We will embark on the exciting topics of manifolds, differential forms, cohomology, Hodge theory, Lagrangian formalism, and some quantum field theory. Then we will use all of this to look at a topological field theory. The goal of the thesis is to perform a one-loop computation of geometric invariants using the one-loop partition function of certain topological field theories inspired by String Theory. Hence most of the thesis will focus on building up the required knowledge to be able to perform the computation. For the sake of brevity, many of the proofs will be dropped, but the interested reader can find the proofs in the references. The first chapter will cover manifolds and what these mathematical objects are. They will be the spaces for which we do our computations over. In the following chapter, we cover differential forms that will act as our fields in the final theory. Then in the third chapter, we take a look at something called de Rham cohomology, and we will do some explicit computations of cohomology groups. These groups will be our first encounter with topological invariants in this thesis. They are so-called classical invariants and are related to classical field theory; from cohomology, it is possible to count the solutions of equations of motion and other types of differential equations. The next chapter covers Hodge theory which is our final mathematical tool. We then make a visit to analytical mechanics and quantum field theory for our final tools needed to compute the partition function for a topological field theory. The last chapter will solely be devoted to topological field theory, where we start with a discussion about regularising before we go over to do some explicit computations of partition functions.

The chapters regarding the mathematics, chapter 2-5, will closely follow the books “An Introduction to Manifolds” by Loring W. Tu [2] and “Geometry, topology and physics” by M. Nakahara [3]. Chapter 6, regarding analytical mechanics and quantum field theory, will closely follow “Geometry, topology and physics” by M. Nakahara [3] and “Quantum field theory in a nutshell” by A. Zee [4].

1.1 Prerequisites

This thesis will require the reader to be familiar with concepts introduced in courses within mathematics and physics, such as courses on calculus, linear algebra, abstract algebra, and quantum mechanics. We will also assume the reader to be familiar with basic point-set topology. See appendix A of [2] if you are not familiar with basic topology. Here we mention some of the most important definitions from topology and linear algebra that will be used in the upcoming chapters. Feel free to skip this chapter if you are familiar with topology and tensors.

1.1.1 Topology

Definition 1 (Topological space, [2] definition A.2)**.** A *topological space* consist of a set X, and a set T consisting of open subsets $U \subset X$. Where

- Any $\cup_i U_i \in \mathrm{T}$ if all $U_i \in \mathrm{T}$.
- Any finite $\cap_i U_i \in \mathrm{T}$ if all $U_i \in \mathrm{T}$
- X and $\emptyset \in \mathrm{T}$, where $\emptyset$ is the empty set.

Definition 2 (Subspace Topology, [2] appendix A.2)**.** If $A \subset X$ and (X, T) is a topological space,

$$T_A := \{U \cap A | U \in T\}$$

we call the *subspace topology* of A and any element of T_A is said to be open in A. (A, T_A) is a topological space.

Definition 3 (Compact, [2] definition A.32)**.** Any subset $\{U_i\} \subset T$ is called a *cover* of X if $X \subset \cup_i U_i$. A subset of cover which is itself a cover is called a *subcover*. Then a topological space (X, T) is called *compact* if every cover of X has a finite subcover.

Definition 4 (Hausdorff, [2] definition 4.20)**.** A topological space (X, T) is called *Hausdorff* if $\forall x, y \in X$ with $x \neq y$, $\exists U_x, U_y \in \mathrm{T}$ such that $U_x \cap U_y = \emptyset$, where $x \in U_x$ and $y \in U_y$.

If (X, T) is Hausdorff, then for any $A \subset X$, (A, T_A) is Hausdorff. Example of a Hausdorff space is $\mathbb{R}^n$ w.r.t the standard topology. It then follows that any subset of $\mathbb{R}^n$ is Hausdorff.

Definition 5 (Basis, [2] definition A.13)**.** Any $B \subset \mathrm{T}$ in a topological space (X, T) is called a *basis* (for T) if every $U \in \mathrm{T}$ is a union of elements in B.

Definition 6 (Second countable, [2] definition A.17)**.** A topological space is called *second countable* if it admits a countable basis.

$\mathbb{R}^n$ with the standard topology is second countable with the countable basis $B_{\mathbb{Q}} = \{B_\epsilon(x) | \epsilon \in \mathbb{Q}^+, x \in \mathbb{Q}^n\}$. And if (X, T) is second countable, then for any $A \subset X$, (A, T_A) is second countable. So any subset of $\mathbb{R}^n$ is second countable w.r.t. the standard topology.

Definition 7 (Homeomorphism, [2] definition A.39)**.** A map $f : X \to Y$ that is bijective is a *homemorphism* if both f and f^{-1} is continuous.

Definition 8 (Diffeomorphism, [2] definition 6.4)**.** A bijective map $f : U \to V$ where $U, V \in \mathbb{R}^n$ is called a *diffeomorphism* if it is smooth and has smooth inverse $f^{-1} : V \to U$.

1.1.2 Linear Algebra

Definition 9 (Dual Space, [2] chap. 3.1)**.** Let V and W be two real vector spaces, then the vector space of all linear maps $f : V \to W$, is denoted $\mathrm{Hom}(V, W)$. Then the vector space of all real-valued linear functions on V is called the *dual space*, denoted V^*, i.e. $V^* = \mathrm{Hom}(V, \mathbb{R})$. We call the elements of V^* covectors.

Definition 10 (Tensor, [2] chap. 3.3)**.** For a vector space V a *k-tensor* on V is a k-linear function

$$f : V^k \to \mathbb{R}, \tag{1.1}$$

where $V^k = V \times \cdots \times V$ k-times.

Definition 11 (Alternating vector space, [2] chap. 18)**.** Given a vector space V we define $A_k(V) = \bigwedge^k(V) = V \wedge V \wedge \cdots \wedge V$ k-times, to be the vector space of alternating k-linear functions or alternating k-tensors. A k-tensor f is said to be alternating if for any permutation $\sigma \in \mathrm{S}_k$

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (\mathrm{sgn}(\sigma)) f(v_1, \ldots, v_k). \tag{1.2}$$

I.e. doing an odd number of permutations of the elements leaves a minus sign and even number of permutations leaves a plus sign. An alternating k-tensor is also called a k-covector. This wedge product $\wedge$ is an alternating/antisymmetric product that will be discussed in more detail in chapter 3. $A_1(V) = V$ and $A_0(V) = \mathbb{R}$.

Chapter 2

Manifolds

(This chapter is based on [2] chap. 5 and 21 and [5] chap. 2)

Manifolds are one of the most important and fundamental concepts in mathematics and physics. By studying mathematics or physics you are destined to run into manifolds in some way or another. Throughout the studies of maths and physics one gets familiar with the Euclidean space $\mathbb{R}^n$, often represented by the set of n-tuples $(x_1, ..., x_n)$. The idea behind manifolds is to have a space that can have complicated structure, with curves and non-trivial topology globally, but that will look just like $\mathbb{R}^n$ locally. In other words, a manifold is a mathematical structure that will look like $\mathbb{R}^n$ locally, but not necessarily globally. Some examples of manifolds are:

- Euclidean space $\mathbb{R}^n$.
- The n-sphere S^n.
- The n-torus T^n.
- Lie groups like $Gl(n, \mathbb{R})$, $SO(n)$, $U(n)$, etc.

After giving the definitions of a *topological manifold*, and a *smooth manifold* in the upcoming sections we will show explicitly why some of these examples are manifolds.

2.1 Topological manifold

(Based on [2] chap. 5.)

Before we define a manifold we have to give some definitions. We start of by defining what locally Euclidean means. A topological space M is called *locally Euclidean* if every pt. $p \in M$ is contained in an open set $U \in M$ that is homeomorphic (with homeomorphism ϕ) to an open set $\phi(U) \subset \mathbb{R}^m$, for some $m \in \mathbb{N}$. The pair (U, ϕ) is called a (coordinate) *chart* about p, with coordinate neighbourhood U and coordinate map ϕ. See figure 2.1.

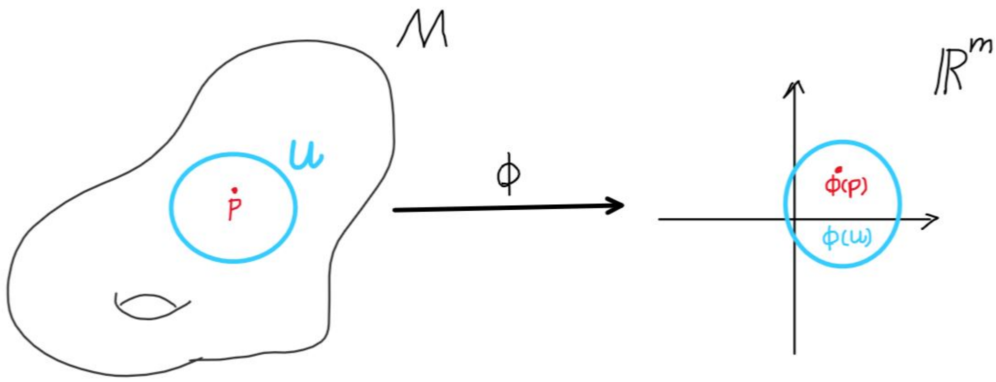

Figure 2.1: Visualising a manifold M being locally Euclidean

If $\phi(p) = 0$, then the chart (U, ϕ) is said to be *centered* at p. If m is the same at each chart on M, then m is called the *dimension* of M, and we write $\dim(M) = m$.

Example 1. An intuitive example of something being locally Euclidean is our earth. Since for us down on the surface of the earth it looks rather flat. But when we go far enough out into space we see that it has the shape of a sphere.

Example 2 ([2] example 5.3)**.** $\mathbb{R}^m$ is locally Euclidean of dimension m w.r.t. a single chart $(\mathbb{R}^m, id_{\mathbb{R}^m})$ (id is the identity map), and so is any open $U \subset \mathbb{R}^m$ w.r.t. (U, id_U).

Now we have what we need to define what a topological manifold is.

Definition 12 (Topological manifold, [2] definition 5.2)**.** A topological space M which is locally Euclidean, is called a topological manifold if it is Hausdorff and second countable.

Let's look at an example of a topological manifold.

Example 3 ([2] example 5.3)**.** One of the examples we gave above of a manifold was $\mathbb{R}^n$. We have actually already showed all the steps necessary to show that is a topological manifold. $\mathbb{R}^n$ is a topological space w.r.t. the standard topology, and it is of course locally Euclidean as showed above. We also saw in the previous chapter that $\mathbb{R}^n$ is both Hausdorff and second countable. Hence $\mathbb{R}^n$ is our first example of a topological manifold. It also follows that any open set $U \subset \mathbb{R}^n$ is a topological manifold w.r.t. the chart (U, id_U).

2.2 Smooth manifolds

(Based on [2] chap. 5)

Before we can go on to define what a smooth manifold is, we need to discuss what happens when we have charts overlapping. In this case the points in the intersection will have more than one coordinate system.

Definition 13 (Compatible, [2] Definition 5.6)**.** Two charts (U, ϕ) and (V, ψ) on a topological manifold M is said to be compatible if

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is a diffeomorphism.

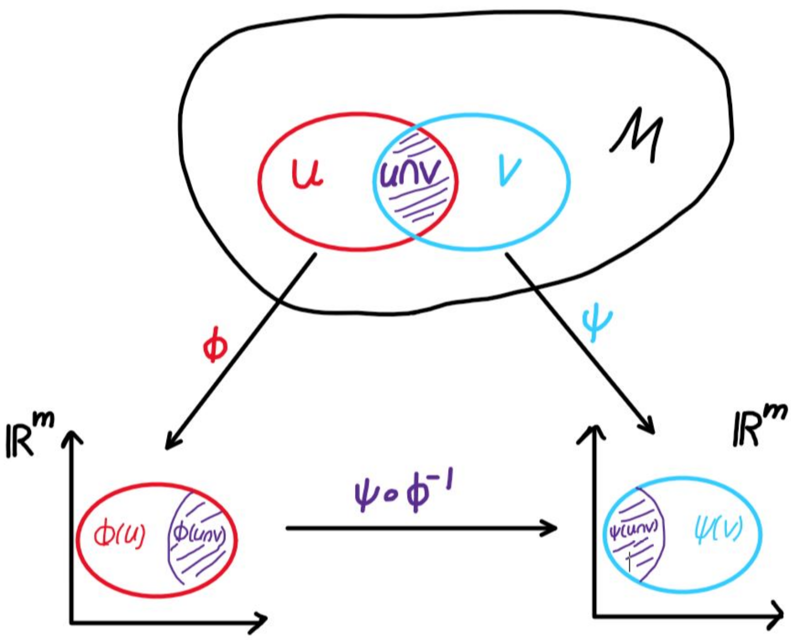

Figure 2.2: Visualization of compatible charts.

Remark. *$U \cap V$ is open in M so $\phi(U \cap V)$ and $\psi(U \cap V)$ are open in $\mathbb{R}^m$.*

The smooth maps $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are called the *transition functions* between the two charts (U, ϕ) and (V, ψ). A collection $A = \{(U_\alpha, \phi_\alpha)\}$ of pairwise compatible charts on a topological manifold M is called an *atlas* if $M \subset \cup_\alpha U_\alpha$, i.e. A covers M. A given chart (V, ψ) is compatible with an atlas $A = \{(U_\alpha, \phi_\alpha)\}$ if it is compatible with every chart (U_α, ϕ_α) of the atlas.

Lemma 1. *([2] Lemma 5.8) Let $A = \{(U_\alpha, \phi_\alpha)\}$ be an atlas on a locally Euclidean space. If two charts (V, ψ) and (W, χ) are both compatible with the atlas A, then they are compatible with each other.*

We say that two atlases $A = \{(U_\alpha, \phi_\alpha)\}$ and $B = \{(V_\beta, \psi_\beta)\}$ are compatible if every (U_α, ϕ_α) and (V_β, ψ_β) are compatible. Note that A and B are compatible if and only if $A \cup B$ is an atlas. An atlas $\mathcal{M}$ is called *maximal* if it is not contained in any larger atlas, i.e. if A is some other atlas containing $\mathcal{M}$, then $A = \mathcal{M}$.

Definition 14 (Smooth manifold [2] Definition 5.9)**.** A C^∞ or *smooth manifold* is a topological manifold M equipped with a maximal atlas $\mathcal{M}$, and $\mathcal{M}$ is called a *differentiable structure* on M.

A smooth manifold M of $\dim M = m$ is a denoted as a m-manifold. Describing manifolds of different dimensions, we then see that a 1-manifold is just a curve, and a 2-manifold is a surface. In practice when we want to check if a topological manifold M is a smooth manifold, it is not necessary to have a maximal atlas. This follows from the following lemma.

Lemma 2. *([2], Proposition 5.10) Any atlas $A = \{(U_\alpha, \phi_\alpha)\}$ on a topological manifold M is contained in a unique maximal atlas.*

One can therefore check if a topological space M is a smooth manifold if it satisfies;

1. M is Hausdorff and second countable.

2. M has a C^∞ atlas (which does not have to be maximal).

For convenience we will from now on write just *manifold* when we mean a smooth manifold. Now that we have established what a manifold is let us look at some examples.

2.3 Examples of Smooth Manifolds

In this section we will see some examples of manifolds. Hopefully, you will recognize a lot of these objects from math or physics courses.

Example 4 ($\mathbb{R}^n$, [2] example 5.11)**.** The simplest and most well known manifold is the Euclidean space $\mathbb{R}^n$. $\mathbb{R}^n$ is a smooth manifold with a single chart $(\mathbb{R}^n, id_{\mathbb{R}^n}) = (\mathbb{R}^n, x^1, ..., x^n)$ defining the atlas of $\mathbb{R}^n$, w.r.t the standard coordinates $(x^1, ..., x^n)$.

Example 5 (Sub-manifold, [2] example 5.12)**.** If M is a manifold and $N \subset M$ is open, then N is also a manifold. Start by looking at the atlas $A = \{(U_\alpha, \phi_\alpha)\}$ for M, then we can define an atlas for N as

$$A_N = \{(U_\alpha \cap N, \phi_\alpha|_{U_\alpha \cap N})\}.$$

Example 6 (Generalized linear group, [2] example 5.14)**.** One of the Lie groups that you probably have encountered in an abstract algebra course or some similar course, is the real generalized linear group $GL(n, \mathbb{R})$. The group consisting of real $n \times n$ matrices with an inverse. This is in fact a manifold, and we will take advantage of the previous example to show this.

We start of by identifying the set of real $n \times n$ matrices as $\mathbb{R}^{n\times n}$ which is isomorphic to the Euclidean space $\mathbb{R}^{n^2}$. Let us equip this with the standard topology. The determinant map

$$\det : \mathbb{R}^{n\times n} \to \mathbb{R}$$

is continuous, since it is a polynomial. Another way of writing $GL(n, \mathbb{R})$ is:

$$GL(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} | \det A \neq 0\} = \det^{-1}(\mathbb{R} \setminus \{0\}).$$

The set $\mathbb{R} \setminus \{0\}$ is open as a subset of $\mathbb{R}$. Hence $Gl(n\mathbb{R})$ is open, so it is a manifold of dimension n^2.

Example 7 (S^1 and S^n, [2] example 5.15)**.** The circle is another example of a manifold. We can denote the unit circle as $x^2 + y^2 = 1$ in $\mathbb{R}^2$. To make an atlas for the circle we will cover it by four open sets. This will be the open upper and lower semicircles, U_1 and U_2 respectively, and the open left and right semicircles, U_3 and U_4 respectively.

We then define the homemorphisms to be:

$$\phi_i : U_i \to (-1, 1), (x, y) \mapsto x, \text{ for } i = 1, 2,$$

and

$$\phi_j : U_j \to (-1, 1), (x, y) \mapsto y, \text{ for } j = 3, 4.$$

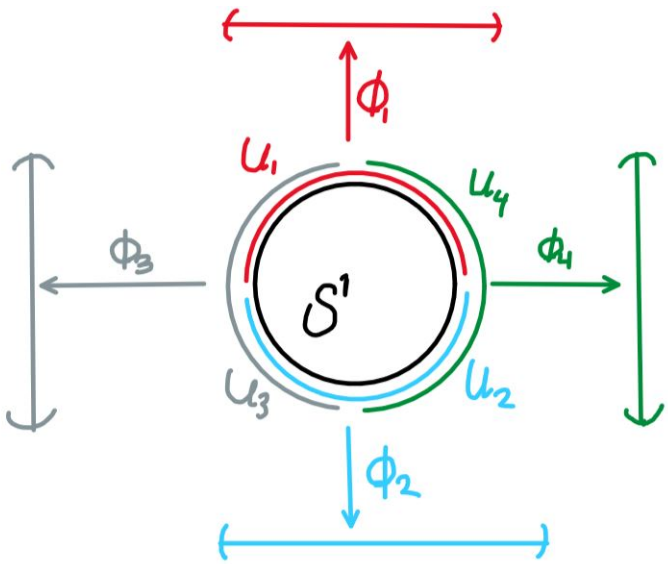

Figure 2.3: S^1 atlas consisting of four charts.

It is then easy to check that the transition functions are smooth on all the non-empty intersections $U_\alpha \cap U_\beta$. For example on $U_1 \cap U_4$, the transition function is

$$\phi_4 \circ \phi_1^{-1}(x) = \phi_4(x, \sqrt{1-x^2}) = \sqrt{1-x^2}.$$

On $U_2 \cap U_3$ it becomes,

$$\phi_3 \circ \phi_2^{-1}(x) = \phi_3(x, -\sqrt{1-x^2}) = -\sqrt{1-x^2}.$$

Both of these transition functions and their inverses are C^∞ where they are defined. One can check that the rest of the transition functions are smooth. Hence $\{(U_i, \phi_i)\}_{i=1}^4$ forms an atlas for S^1. So by using Lemma 2 we know that this atlas is contained in a unique maximal atlas. So S^1 is a manifold of dimension 1.

In a similar way one can show that the n-sphere

$$S^n = \{x^1, \ldots, x^{n+1} \in \mathbb{R}^{n+1} | (x^1)^2 + \cdots + (x^{n+1})^2 = 1\}$$

is a manifold of dimension n. To make an atlas for S^2, the unit sphere, one would need at least six charts to cover the sphere using the above method. This could be the open upper and lower hemisphere (not containing the equator). The open left and right hemispheres. Now we have covered almost the whole sphere except for the two points $(1, 0, 0)$ and $(-1, 0, 0)$. Hence we need at least two more charts which then can be the front and back open hemispheres to cover these two remaining points.

It is possible to use only two charts to cover S^2 using different open sets and homeomorphims. If one uses projection, then one could make a homeomorphism between the open unit sphere not containing the north pole and an open subset of $\mathbb{R}^2$, and a homemorphism using projection between the open sphere not containing the south pole and an open subset of $\mathbb{R}^2$. However, this is the minimum amount of charts one would need to cover S^2.

The last example of a manifold we mentioned in the beginning of the chapter was T^2, the torus. Before we can show that this is a manifold, we have to look at what a product manifold is.

Lemma 3 (Product manifold, [2] proposition 5.17)**.** *If M and N are manifolds with atlas $A_M = \{(U_\alpha, \phi_\alpha)\}$ and $A_N = \{(V_\beta, \psi_\beta)\}$. Then*

$$A_{M\times N} = \{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)\}$$

defines an atlas for $M \times N$. Hence $M \times N$ is manifold.

Example 8 (T^2 and T^n, [2] example 5.18)**.** Now we can easily show that T^2 is manifold. Since by definition the torus is simply the product of two circles S^1,

$$T^2 = S^1 \times S^1.$$

So by the above lemma T^2 is a manifold of dimension 2.

And this can of course be generalized to the n-torus;

$$T^n = S^1 \times ... \times S^1 \text{ (n-times)}.$$

T^n is a manifold of dimension n.

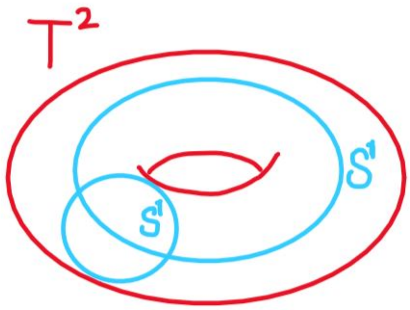

Figure 2.4: Torus as product of two circles.

2.4 Other types of manifolds

Here we briefly mention some definitions of other types of manifolds that will be useful for later.

Definition 15 (Compact manifold [6])**.** If M is a manifold, then M is a compact manifold if it is compact as a topological space.

Definition 16 (Paracompact manifold, [3] definition 5.7)**.** If you have a manifold M for which you can take an open covering $\{U_i\}$ such that all points of M can be covered by a finite number of U_i. Then M is called *paracompact*, if this is always possible.

Definition 17 (Contractible manifold, [2] definition 26.6)**.** A manifold is called contractible if it can be continuously deformed to a point, i.e. the manifold has the homotopy type of a point.

An example of such a manifold is for instance an open hemisphere in $\mathbb{R}^3$. Start of by S^2 and cut it in two along the equator and remove the boundary. Then you are left with two open hemispheres which both can be continuously deformed to a point. Another example of a contractible manifold is an open disk in $\mathbb{R}^2$. A manifold which would not be contractible is for example the torus.

2.5 Manifolds with boundary

(Based on [2] chap. 21)

Just as you have seen in a course on standard topology, one can talk about sets with boundaries. In much the same manner can manifolds also have boundaries. If we have a manifold M with boundary, we denote the boundary as ∂M. A *closed manifold* is a compact manifold without boundary. Some examples of closed manifolds are for example the n-sphere and the n-torus. If M is a compact m-manifold then ∂M is $(m-1)$-manifold.

Let's see how an atlas for ∂M can be defined, letting the manifold M be of dimension m. We take an arbitrary chart (U, ψ) of M and by using this we can denote $\tilde{\psi} = \psi|_{U\cap\partial M}$ to be the coordinate map ψ restricted to the boundary. Then it can be assumed that $\tilde{\psi}$ will map boundary points to boundary points,

$$\tilde{\psi} : U \cap \partial M \to \partial\mathbb{H}^m = \mathbb{R}^{m-1}. \tag{2.1}$$

Here $\mathbb{H}^m$ denotes the closed upper half-space

$$\mathbb{H}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m | x^m \geq 0\}, \tag{2.2}$$

together with the subspace topology of $\mathbb{R}^m$, this is one of the prime examples of a manifold with boundary.

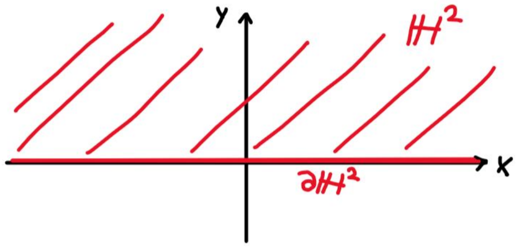

Figure 2.5: Visualization of the upper half plane $\mathbb{H}^2$ in $\mathbb{R}^2$.

Continuing with defining an atlas for ∂M. If M has two charts (U, ψ) and (V, χ) then one can show that the following map is smooth

$$\tilde{\chi} \circ (\tilde{\psi})^{-1} : \tilde{\psi}(U \cap V \cap \partial M) \to \tilde{\chi}(U \cap V \cap \partial M). \tag{2.3}$$

Hence if M has an atlas $\{(U_\alpha, \psi_\alpha)\}$, then this will induce the following atlas for ∂M; $\{(U_\alpha \cap \partial M, \psi_\alpha|_{U_\alpha \cap \partial M})\}$. This also shows that ∂M is a manifold of dimension $(m-1)$ without a boundary. In later chapters unless otherwise stated we will deal with manifolds with $\partial M = 0$.

This concludes our discussion of manifolds. Manifolds are the objects for which we will construct our final topological theory over. This theory will be used to compute invariants for manifolds, which in turn can be used to tell them apart.

Chapter 3

Differential forms

(This chapter is based on [2] chap. 4, 8, 17, 18 and 22, [3] chap. 5, and [7])

Now that we have defined what a manifold is, we wish to get a way to do calculus on them, and this is where differential forms come into play. Differential forms are interesting mathematical objects that will be defined and discussed in this chapter. The critical feature of differential forms, in terms of extending the machinery of multi-variable calculus to manifolds, is that they are independent of coordinates. Hence differential forms allow us a way to do calculus on curves, surfaces, and higher-dimensional manifolds. They will play an important role in defining the final theory.

In fact, you have been working with differential forms throughout your whole "calculus career" without knowing it. Roughly speaking, one can say that differential forms are the objects that appear under an integral sign. The theorems one sees in a course on multi-variable calculus, such as; Green's theorem, divergence theorem, etc., are all special cases of one grand theorem called *the generalized Stokes theorem* or just *Stokes theorem.* This will be the final punchline of this chapter. However, before we start rambling on about differential forms, we take a look at a key ingredient in the definition of a differential form, namely the tangent space.

3.1 Tangent Space

(Based on [2] chap. 8, and [7])

As the name suggests, a *tangent space* is a vector space that consists of the tangent vectors at any given point. Take for instance a smooth curve, $C \in C^\infty(\mathbb{R}^2)$, and take a point $p \in C$. Then the tangent space of C at p is simply the tangent at p, i.e., a line that just touches the curve C at p. The slope of this tangent line, as you might guess, is just the derivative.

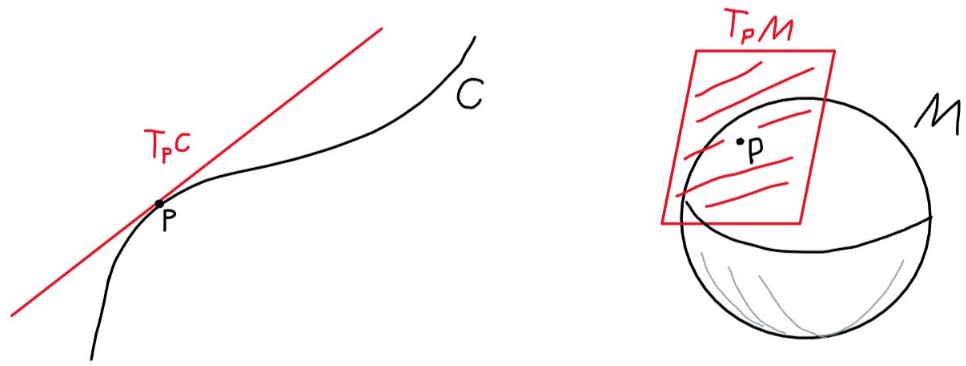

Figure 3.1: Visualisation of the tangent space of a curve C, $T_p(C)$, and the tangent space of a surface M, $T_p(M)$, represented as a plane.

Choosing an open set $U \in \mathbb{R}^n$, we can assign to any point $p \in U$, a vector $v \in \mathbb{R}^n$, called a tangent vector at p. The tangent space, $T_p(\mathbb{R}^n)$, is the vector space of all tangent vectors at p.

Let's present a more formal definition in terms of derivations. Start of with a smooth manifold M, and some smooth function $f \in C^\infty(M)$. We then take an arbitrary point $p \in M$. The definition of a derivation at p is a linear map $D : C^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule, i.e. for any smooth functions f and g, D satisfies

$$D(fg) = D(f)g + fD(g).$$

We can then obtain a vector space of the set of derivations at p by the following addition and scalar multiplication rules;

- $(D_1 + D_2)(f) := D_1(f) + D_2(f)$ and
- $(\alpha \cdot D)(f) := \alpha \cdot D(f)$.

This vector space is denoted as $T_p(M)$, the tangent space of M at p. A basis for the tangent space $T_p(M)$ can be expressed via the coordinates of a chart (U, ϕ), $\phi : U \to \mathbb{R}^n$, where $p \in U$. Using the charts coordinates $(x^1 \dots x^n)$ we may define the following ordered basis $\left\{\left(\frac{\partial}{\partial x^1}\right), \dots, \left(\frac{\partial}{\partial x^n}\right)\right\}$ for $T_p(M)$. Where each of the basis elements of the tangent is defined such that for any $i \in \{1, \dots, n\}$, and any $f \in C^\infty(M)$ it satisfies;

$$\left(\frac{\partial}{\partial x^i}\right)\bigg|_p (f) := \left(\frac{\partial}{\partial x^i}(f \circ \phi^{-1})\right)(\phi(p)).$$

This gives us a way to express the tangent vectors $v \in T_p(M)$ in a simple way as

$$v = \sum_{i=1}^{n} v^i \cdot \left(\frac{\partial}{\partial x^i} \right).$$

Where v^i are scalars. Resembling a familiar way to express the tangent vectors as linear combinations of the basis vectors $\left(\frac{\partial}{\partial x^i}\right)_p \in T_p(M)$.

3.2 Differential k-forms on Euclidean Space

(Based on [2] chap. 4)

There are many ways to define or introduce differential forms. Here we will state the definition in terms of looking at differential forms as objects on $\mathbb{R}^n$, and then later generalize it to manifolds. Above we claimed that you have already been exposed to and worked with differential forms in calculus courses. You have for instance seen things like, dx, dy, dz, which occur in derivation $\frac{dy}{dx}$ and integration $\int_S f(x,y)\,dxdy$. These are examples of differential forms. Let's start off by defining a differential form ω of degree k, also called a k-form.

Definition 18 (Differential k-form [2] chap. 4.2)**.** A general k-form is a function on an open subset $U \subset \mathbb{R}^n$ that to every point $p \in U$ assigns an alternating k-linear function on the tangent space $T_p(\mathbb{R}^n)$, so $\omega_p \in A_k(T_p\mathbb{R}^n)$. I.e. a differential k-form ω is as a k-linear function $\omega^k : \bigwedge^k T_p(\mathbb{R}^n) \to \mathbb{R}$, hence $\omega^k \in \bigwedge^k(T_p^*(\mathbb{R}^n))$.

You may also think of a k-form as a k-tensor. A slightly different version of definition 10 is that a tensor is a (j,k) multi-linear map, mapping j elements from $T_p^*(M)$ and k elements from $T_p(M)$ to real numbers. Hence a differential forms is a $(0,k)$ type tensor.

In the case when $k = 1$ we have that $A_1(T_p\mathbb{R}^n) = T_p^*(\mathbb{R}^n)$, which is the cotangent space or the dual of the tangent space. Hence an element of the cotangent space $T_p^*(\mathbb{R}^n)$ is a covector or a linear functional on the tangent space $T_p(\mathbb{R}^n)$. So in this case when $k = 1$, a 1-form is a function on an open subset $U \in \mathbb{R}^n$ that assigns to each point p in U a covector $\omega_p \in T_p^*(\mathbb{R}^n)$. For example, if we take any smooth function $f : U \to \mathbb{R}$ we can construct a 1-form, $\mathrm{d}f = \partial_i f dx^i$, called the differential of f. We denote a general differential k-form on $\mathbb{R}^n$ by:

$$\omega^k = \frac{1}{k!} a_{i_1,\ldots,i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

Where $a_{i_1,\ldots,i_k}$ are function coefficients: $a_I : U \to \mathbb{R}$. The dx^i's are called differentials, and they generate a basis for differential forms. The exterior product is $dx^i \wedge dx^j = -dx^j \wedge dx^i$, and will be elaborated on below.

Remark. *We will occasionally use a shorthand notation to denote a k-form:*

$$\omega = a_I dx^I,$$

where I is a multi-index set: $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$. And using Einsteins summation convention over repeated indices.

A k-form is called smooth if all the coefficient functions are smooth on the subset U. And $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ forms a basis for the vector space of k-forms, $A_k(T_p\mathbb{R}^n)$. The $\frac{1}{k!}$ is for normalisation. We use $\Omega^k(U)$ to denote vector space of smooth k-forms on U. This is a vector space since differential forms satisfy the axioms of a vector space with addition and scalar multiplication.

Example 9 ([2] Example 4.5)**.** Let's take a look at some examples of differential forms for some specific k:

- A 0-form is just a function since $A_0(T_p(\mathbb{R}^n)) = \mathbb{R}$, hence a 0-form only assigns scalars to each point p.
- A 1-form on $\mathbb{R}^n$ we can write as

$$\omega^1 = f_1 dx^1 + f_2 dx^2 + \ldots + f_n dx^n.$$

- A 2-form on $\mathbb{R}^3$ we can write as

$$\omega^2 = f_1 dx^1 \wedge dx^2 + f_2 dx^2 \wedge dx^3 + f_3 dx^3 \wedge dx^1.$$

- A 3-form on $\mathbb{R}^3$

$$\omega^3 = f_1 dx^1 \wedge dx^2 \wedge dx^3.$$

It seems natural that 1-forms can be integrated over curves. 2-forms we can integrate over surfaces. And in general we can integrate k-forms over k-manifolds. We will come back to this later when we discuss integration of differential forms.

3.3 Exterior product

(Based on [3] chap. 5.4.1)

Differential forms form an algebra w.r.t. $\wedge$ called the *exterior product* or wedge product. It is the same product as the one seen above in the previous section (e.g. $dx^1 \wedge dx^2$). The exterior product is an anti-symmetric product, that follows the following properties:

Let $\omega \in \Omega^p(\mathbb{R}^n)$ and $\chi \in \Omega^q(\mathbb{R}^n)$, then the product is $\omega \wedge \chi \in \Omega^{p+q}(\mathbb{R}^n)$

1. Anti-commutative: $\omega \wedge \chi = (-1)^{pq}\chi \wedge \omega$.
2. Associative: $(dx^1 \wedge dx^2) \wedge dx^3 = dx^1 \wedge (dx^2 \wedge dx^3)$.
3. Distributive: $(dx^1 + dx^2) \wedge dx^3 = dx^1 \wedge dx^3 + dx^2 \wedge dx^3$
   $dx^3 \wedge (dx^1 + dx^2) = dx^3 \wedge dx^1 + dx^3 \wedge dx^2$.

Remark. *If ω is an odd-order form then:*

$$\omega \wedge \omega = -\omega \wedge \omega \implies 2\omega \wedge \omega = 0 \implies \omega \wedge \omega = 0$$

3.4 Exterior derivative

(Based on [2] chap. 4)

Now that we have defined the exterior product, let's define the *exterior derivative* d also known as the *de Rham operator*. The exterior derivative is a generalization of the ordinary derivative known from calculus. Together with differential forms, the exterior derivative will make it possible to unify many theorems of calculus in $\mathbb{R}^3$. It will allow us to differentiate differential forms on manifolds. One of the most important features of d is that we can express the derivatives in a coordinate-free manner. This will play an important role in Stokes theorem at the end of this chapter.

We start by defining the exterior derivative for a function (0-form).

Definition 19 ([2] chap. 4.4)**.** Let f be a smooth function on an open subset $U \in \mathbb{R}^n$, $f \in C^\infty(U)$. Then its differential is;

$$\mathrm{d}f = \sum \frac{\partial f}{\partial x^i} dx^i \in \Omega^1(U).$$

For a general k-form the differential is defined to be:

Definition 20 ([3] Definition 5.5)**.** If $\omega = \frac{1}{k!} a_{i_1,\ldots,i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \Omega^k(U)$, then

$$\mathrm{d}\omega = \frac{1}{k!} \frac{\partial}{\partial x^\mu}(a_{i_1,\ldots,i_k}) dx^\mu \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \Omega^{k+1}(U).$$

We see that the exterior derivative is essentially a map from k-forms to (k+1)-forms; $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$. The exterior derivative obeys the following properties.

Proposition 1 ([2] Proposition 4.13)**.** Let $\omega \in \Omega^k(U)$ and $\chi \in \Omega^l(U)$.

(i) $\mathrm{d}(\omega + \chi) = \mathrm{d}\omega + \mathrm{d}\chi$.

(ii) $\mathrm{d}(\omega \wedge \chi) = \mathrm{d}\omega \wedge \chi + (-1)^k(\omega \wedge \mathrm{d}\chi)$.

(iii) $\mathrm{d}^2 = 0$.

We follow the proof of [2] Proposition 4.13.

Proof. (i) Follows from linearity of the exterior derivative.

(ii) This equation is linear on both sides, hence it will be sufficient to check for two 1-forms, $\omega = f dx^I$ and $\chi = g dx^J$. We then have;

$$\begin{aligned}\mathrm{d}(\omega \wedge \chi) &= \mathrm{d}(f g dx^I \wedge dx^J) \\ &= \sum \frac{\partial(fg)}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\ &= \sum \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \wedge g dx^J + \sum f \frac{\partial g}{\partial x^i} dx^i \wedge dx^I \wedge dx^J.\end{aligned}$$

From the second sum we obtain a $(-1)^k$ by anti-commutativity from moving the 1-form $(\frac{\partial g}{\partial x^i})dx^i$ across the k-form dx^I. Hence

$$\begin{aligned}\mathrm{d}(\omega \wedge \chi) &= \mathrm{d}\omega \wedge \chi + (-1)^k \sum f dx^I \wedge \frac{\partial g}{\partial x^i} dx^i \wedge dx^J \\ &= \mathrm{d}\omega \wedge \chi + (-1)^k(\omega \wedge \mathrm{d}\chi)\end{aligned}$$

(iii) Also to prove the third property we can use the linearity of d. Therefore it will again be sufficient to check for $\omega = f dx^I$ that $d^2\omega = 0$. Hence

$$\begin{aligned}\mathrm{d}^2(f dx^I) &= \mathrm{d}(\sum \frac{\partial f}{dx^i} dx^i \wedge dx^I) \\ &= \sum \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^I\end{aligned}$$

In the sum we see that when $i = j$, then $dx^j \wedge dx^i = 0$. And if $i \neq j$ $\frac{\partial^2 f}{\partial x^j \partial x^i}$ is symmetric in i and j, but $dx^j \wedge dx^i$ is alternating in i and j, so the terms that arise from $i \neq j$ pairs up and perfectly cancels out. For example;

$$\frac{\partial^2 f}{\partial x^1 \partial x^2} dx^1 \wedge dx^2 + \frac{\partial^2 f}{\partial x^2 \partial x^1} dx^2 \wedge dx^1$$
$$= \frac{\partial^2 f}{\partial x^1 \partial x^2} dx^1 \wedge dx^2 + \frac{\partial^2 f}{\partial x^1 \partial x^2}(-dx^1 \wedge dx^2) = 0$$

□

We claimed at the beginning of this section that the exterior derivative is a generalisation of the derivative and can make it possible to unify theorems from vector calculus. So let us have a glimpse at some examples of this.

Example 10 ([2] chap. 4.6)**.** Recall that if you have a vector field on $\mathbb{R}^3$ it is just a vector-valued function. In $\mathbb{R}^3$ we have three operators that can act on scalar functions and vector-valued functions. These are the gradient, the curl and the divergence. Scalar and vector-valued functions then forms the following sequence together with the three operators:

$$\{scalar func.\} \xrightarrow{\text{grad.}} \{vector func.\} \xrightarrow{\text{curl}} \{vector func.\} \xrightarrow{\text{div.}} \{scalar func.\}$$

We use small letters to denote scalar functions and capital letters to denote vector valued functions. Subscript denotes differentiation, $f_x = \frac{\partial f}{\partial x}$.

$$grad\ f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix},$$

$$curl \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} W_y - V_z \\ -(W_x - U_z) \\ V_x - U_y \end{bmatrix},$$

$$div \begin{bmatrix} U \\ V \\ W \end{bmatrix} = U_x + V_y + W_z.$$

From vector calculus we are familiar with the following propositions.

Proposition 2 ([2] Proposition A, B and C, chap. 4)**.** Let f be a scalar function and U, V, W be vector fields.

1. $curl(grad\ f) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

2. $div \left(curl \begin{bmatrix} U \\ V \\ W \end{bmatrix} \right) = 0$.

3. If the curl of a vector field F on $\mathbb{R}^3$ is equal to zero, $curl F = 0$, then F is the gradient of a scalar function f. If we let F denote a force, then if this requirement, $curl F = 0$, is met then F is a conservative force.

All 1-forms on $\mathbb{R}^3$ are linear combinations of function coefficients attached to dx, dy and dz. So we can identify 1-forms with vector fields on $\mathbb{R}^3$ in the following way:

$$U dx + V dy + W dz \longleftrightarrow \begin{bmatrix} U \\ V \\ W \end{bmatrix}.$$

In the same way we can identify 2-forms too on $\mathbb{R}^3$ with vector fields.

$$U dy \wedge dz + V dz \wedge dx + W dx \wedge dy \longleftrightarrow \begin{bmatrix} U \\ V \\ W \end{bmatrix}.$$

Using the same identifications we can see what the exterior derivative acting on a 0-form f corresponds to

$$\mathrm{d}f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \longleftrightarrow \begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \\ \partial f / \partial z \end{bmatrix} = grad\ f.$$

Acting with the exterior derivative on a 1-form in $\mathbb{R}^3$ gives;

$$\begin{aligned} &\mathrm{d}(U dx + V dy + W dz) \\ &\quad = (W_y - V_z) dy \wedge dz - (W_x - U_z) dz \wedge dx - (V_x - U_y) dx \wedge dy, \end{aligned}$$

which we see can be expressed as

$$curl \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} W_y - V_z \\ -(W_x - U_z) \\ V_x - U_y \end{bmatrix}$$

Finally the exterior derivative acting on a 2-form is

$$\begin{aligned}\mathrm{d}(Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy)\\ = (P_x + Q_y + R_z)dx \wedge dy \wedge dz,\end{aligned}$$

corresponding to

$$div \begin{bmatrix} U \\ V \\ W \end{bmatrix} = U_x + V_y + W_z.$$

Hence we have just identified that the exterior derivative acting on 0-forms, 1-forms and 2-forms in $\mathbb{R}^3$, are the operators gradient, curl and divergence respectively.

3.5 Differential forms on manifolds

(Based on [2] chap. 17 and 18, [3] chap. 5)

Now that we are familiar with the concept of differential forms on $\mathbb{R}^n$, let us generalize this to looking at differential forms on general manifolds. This process of defining differential forms on a manifold M will require little effort. As we saw in the chapter about manifolds, $\mathbb{R}^n$ is a special case of a manifold. Hence most of what we have already seen about differential forms will follow. One reason we want to look at differential forms on manifolds is that forms make it possible to integrate on a manifold. In the final chapter, we will express our theories in terms of an integral of differential forms over a manifold, for which we will compute the partition function. However, let us start by looking at the definition of a 1-form on a manifold.

Definition 21 (1-form on M, [2] chap. 17)**.** Let p be an arbitrary point $p \in M$, where M is smooth manifold. The cotangent space of M at p is denoted in the usual way by $T_p^*(M)$, also known as the dual space of the tangent space $T_p(M) \cong T_{\phi(p)}(\mathbb{R}^n)$ using the chart ϕ. It is the elements of the cotangent space $T_p^*(M)$ that is known as covectors at p. As we saw when looking at forms on $\mathbb{R}^n$ we can express a covector ω_p at p as a linear function

$$\omega_p : T_pM \to \mathbb{R}.$$

A *differential 1-form* on M is then what one can call a *covector field*. So a 1-form on M is a function that assigns every p in M a covector at p. You can therefore view 1-forms on a manifold as the dual to vector fields. Since a 1-form assigns to every point a covector, and a vector field assigns to every point a tangent vector.

Then let's have a look at the local expression of a 1-form. We let $(U, \phi) = (U, x^1, ..., x^n)$ be a coordinate chart on M. Then the differentials $dx^1, ..., dx^n$ are 1-forms on U, just as when we looked at forms on $\mathbb{R}^n$. From this follows the following proposition:

Proposition 3 ([2] Proposition 17.3)**.** We form a basis for the cotangent space $T_p^*(M)$ at each $p \in U$ by the covectors $(dx^1)_p, ..., (dx^n)_p$ which is dual to the basis for the tangent space T_pM, $(\partial/\partial x^1)_p, ..., (\partial/\partial x^n)_p$.

Now that we have defined a general 1-form on a manifold, let us define a k-form on M. It will follow in much the same way as with k-forms on $\mathbb{R}^n$.

Definition 22 (k-forms on M, [2] chap. 18)**.** The vector space $A_k(T_pM)$, often denoted $\bigwedge^k(T_p^*M)$ is the vector space consisting of all alternating k-tensor or k-covectors on the tangent space $T_p(M)$. A k-covector field or *differential k-form* on M is a function ω that assigns to each point $p \in M$ a k-covector $\omega \in \bigwedge^k(T_p^*M)$. Hence a k-form at a point p can be written as a function

$$\omega_p : \bigwedge^k(T_p(M)) \to \mathbb{R}.$$

Suppose again that we look at the coordinate chart $(U, x^1, ..., x^n)$ on a manifold M. Above we defined the 1-forms $dx^1, ..., dx^n$ on U. We saw that at each point $p \in U$ we could form a basis for T_p^*M by $(dx^1)_p, ..., (dx^n)_p$. We can then construct a basis for $\bigwedge^k(T_p^*M)$ by the set

$$(dx^{i_1})_p \wedge ... \wedge (dx^{i_k})_p,\ 1 \leq i_1 < ... < i_k \leq n.$$

Hence we can express a k-form locally on U by a linear combination $\omega = \sum a_I dx^I$. Where a_I are smooth functions on U. This way we can look at the differential forms locally as forms on $\mathbb{R}^n$.

A manifold can have overlapping charts. So a differential form lying in the intersection will have multiple coordinates. In this case we can do a change of coordinates, since differential forms are defined without any reference to any specific coordinate system. For a point $p \in U_k \cap U_l$ of two charts (U_k, ϕ_k) and (U_l, ϕ_l), we have for 1-forms;

$$\omega = \omega_a dx^a = \tilde{\omega}_b dy^b,$$

where $x = \phi_k(p)$ and $y = \phi_l(p)$. Using the exterior derivative $dy^b = \frac{\partial y^b}{\partial x^a} dx^a$, we see that $\omega_a = \tilde{\omega}_b \frac{\partial y^b}{\partial x^a}$, defining the transformation rule for the coefficient function between charts. A similar rule holds for higher forms. When we look closer at how to integrate differential forms on manifolds this coordinate independence will be discussed further, and it is an important feature for us to be able to integrate forms over manifolds. We will denote the vector space of smooth k-forms on M by $\Omega^k(M)$. If M is an n-dimensional manifold, then an n-form is the highest differential form on M. Differential forms can be added, multiplied, differentiated, and, as we will see later, integrated on manifolds. The exterior product and exterior derivative can be extended to a manifold. As mentioned, $\mathbb{R}^n$ is a special case of a manifold. Hence the definition of the exterior product is defined in the same way as above for a general manifold. The same holds for the exterior derivative. The exterior derivative also induces the following sequence for an m-manifold;

$$0 \xrightarrow{i} \Omega^0(M) \xrightarrow{\mathrm{d}_0} \Omega^1(M) \xrightarrow{\mathrm{d}_1} ... \xrightarrow{\mathrm{d}_{m-2}} \Omega^{m-1}(M) \xrightarrow{\mathrm{d}_{m-1}} \Omega^m(M) \xrightarrow{\mathrm{d}_m} 0.$$

Where i is the inclusion map $0 \hookrightarrow \Omega^0(M)$. This sequence is known as the *de Rham complex*, and will be more closely investigated in the following chapter. We already encountered a special case of this complex for $\mathbb{R}^3$ in example 10.

$$\{scalar func.\} \xrightarrow{\text{grad.}} \{vector func.\} \xrightarrow{\text{curl}} \{vector func.\} \xrightarrow{\text{div.}} \{scalar func.\}$$

An equivalent way of writing this sequence for an open subset $U \in \mathbb{R}^3$ is in the following way;

$$\Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U) \xrightarrow{\mathrm{d}} \Omega^2(U) \xrightarrow{\mathrm{d}} \Omega^3(U).$$

We will use the de Rham complex later to compute cohomology groups, which are important classical topological invariants of manifolds. There we will also run into the notion of closed and exact forms. In short, a k-form $\omega \in \Omega^k(M)$, is called *closed* if $\mathrm{d}\omega = 0$, and ω is called *exact* if there exists an $(k-1)$-form τ such that $\omega = \mathrm{d}\tau$. This will turn out to be closely related to the de Rham complex. We easily see that the exact forms have to be a subset of the closed forms since $\mathrm{d}^2 = 0$. Then if we go back to the 3. point mentioned in proposition 2, it can be rephrased to; A 1-form on $\mathbb{R}^3$ is closed if and only if it is exact. Nevertheless, this proposition need not be valid for any sub-region of $\mathbb{R}^3$. Let us have a look at the following famous example.

Example 11 ([2] Example 4.16)**.** Let $U = \mathbb{R}^3 \setminus \{z - axis\}$, and F is the following vector field

$$F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$$

on $\mathbb{R}^3$. We then have that $curl F = 0$, but F is not the gradient of any smooth function on U. So expressed in terms of differential forms, we can say that the 1-form

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

is closed but not exact on U. We see that U is has non-trivial topology. Hence it is not a contractible manifold. If F was the gradient of some smooth function f on U, then the line integral

$$\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$$

over a closed curve C would evaluate to zero. However, if one chooses the curve C to be the unit circle, with $x = \cos\theta$ and $y = \sin\theta$, with $0 \leq \theta \leq 2\pi$. The integral becomes;

$$\int_0^{2\pi} -(\sin\theta)d(\cos\theta) + (\cos\theta)d(\sin\theta) = 2\pi \neq 0,$$

reflecting that U has non-trivial topology.

In general, one way to measure the failure of closed k-forms to be exact is the quotient vector space

$$H^k(U) := \frac{\{\text{closed k-forms on U}\}}{\{\text{exact k-forms on U}\}}$$

which is called the *k-th de Rham cohomology group* of U. These cohomology groups are topological invariants hence they depend only on the topology of U. We will study cohomology in greater detail in the following chapter. The generalization of the 3. point in proposition 2 is the following lemma.

Lemma 4 (Poincaré lemma, [3] theorem 6.3)**.** *Let M be a contractible manifold, then every closed form is also exact.*

Before we start talking abut integration of differential forms in the upcoming section, we want to mention one last feature of differential forms. Differential forms has the property that they can be pulled back.

Definition 23 (Pullback between manifolds [2] chap. 23.3)**.** Let M and N be manifolds. Then any smooth map $\phi : M \to N$ will induce a pullback map $\phi^* : \Omega^k(N) \to \Omega^k(M)$ of differential forms. Written out explicitly in local coordinates: $\omega \in \Omega^k(N)$;

$$\phi^*(\omega) = \frac{1}{k!}\omega_{i_1,\ldots,i_k}(\phi(x))d\phi^{i_1} \wedge \ldots \wedge d\phi^{i_k}$$

where the basis $d\phi^{i_l} = \frac{\partial \phi^{i_l}}{\partial x^{j_l}}dx^{j_l}$ is expressed in terms of the local coordinates $\{x^a\}$ on M. The pullback ϕ^* also commutes with d:

$$\phi^* \circ \mathrm{d} = \mathrm{d} \circ \phi^*.$$

It follows that the pullback induces a map $\phi^* : H^k(N) \to H^k(M)$. Pullbacks are hence useful for comparing cohomology groups of different spaces. E.g. if $\phi : M \to N$ is a diffeomorphism, then the pullback induces an isomorphism $\phi^* : H^k(N) \to H^k(M)$.

3.6 Integration of differential forms

(Based on [3] chap. 5.5)

Before we start of defining integration of differential forms, we have to take a look into orientation. The reason for this is that integration of differential forms over a manifold M is only possible in the case when M is orientable. We therefore start of by discussing and defining what it means for M to have an orientation and being orientable.

Orientation

On a manifold M we will often encounter points laying in the intersection of two or more charts. Let's say we have a point $p \in U_\alpha \cap U_\beta \subset M$, for which $U_\alpha \cap U_\beta \neq \emptyset$. U_α being a chart with basis $\{e_a\} = \{\partial/\partial x^a\}$ with x^a as the local coordinates. And U_β being another chart with basis $\{\tilde{e}_b\} = \{\partial/\partial y^b\}$ with y^b being the local coordinates. Then the tangent space $T_p(M)$ can have its basis spanned by either $\{e_a\}$ or $\{\tilde{e}_b\}$. The basis will then change according to;

$$\tilde{e}_b = \Big(\frac{\partial x^a}{\partial y^b}\Big) e_a.$$

This leads us to how orientation is defined on a manifold.

Definition 24 (Orientation [3] chap. 5.5.1)**.** If the Jacobian is positive definite, $J = \det(\partial x^a/\partial y^b) > 0$, on $U_\alpha \cap U_\beta$, then $\{\partial/\partial x^a\}$ and $\{\partial/\partial y^b\}$ admits the same orientation. If the Jacobian is negative, $J < 0$, then they admit opposite orientation.

Definition 25 (Orientable manifold ([3] Definition 5.6))**.** Let M be a connected manifold covered by $\{U_\alpha\}$. M is called *orientable* if there exists local coordinates $\{x^a\}$ for U_α and $\{y^b\}$ for U_β , such that $J = \det(\partial x^a/\partial y^b) > 0$, for any overlapping charts $U_\alpha \cap U_\beta$.

There of course also exist manifolds that are so-called non-orientable, and probably the most famous example is the Möbius strip. A manifold is called non-orientable if J is not positive in all intersections.

Given that M is m-manifold which is orientable, then there exists a m-form ω such that ω is nowhere vanishing on M. Such a m-form ω is called a *volume form* or a *top-form*, and it is these forms that we use to integrate over manifolds. It is only possible to integrate forms of the same degree as the dimension of the manifold. I.e., integration over a 3-dimensional manifold can only be done using a 3-form. The top-forms play the role of a measure for when we integrate a smooth function $f \in C^\infty(M)$ over M, i.e. we can use the top-forms to integrate 0-forms over our manifold. Two top-forms ω and $\tilde{\omega}$ are said to be equivalent if there exists a strictly positive function $h \in C^\infty(M)$ such that $\omega = h\tilde{\omega}$.

Remark. *We can also have a function $\tilde{h} \in C^\infty(M)$ that is negative definite. Hence it will result in an inequivalent orientation on M. Therefore, on any manifold that is orientable, it admits two inequivalent orientations, which are called right handed and left handed respectively.*

Let us have a look at an example of a top-form. Taking a m-form ω on an orientable manifold M of dimension m. We can then define ω in terms of a positive definite smooth function $h(p)$;

$$\omega = h(p)dx^1 \wedge ... \wedge dx^m. \tag{3.1}$$

Since M is an orientable manifold, we can extend ω throughout M such that the component h is positive-definite on any chart U_α. Thus our m-form ω is a top-form. To properly show that this is a top-form, we need to examine that h is positive definite, independent of the coordinates. For instance, let $p \in U_\alpha \cap U_\beta \neq \emptyset$ and we let x^a and y^b be the local coordinates of U_α and U_β respectively. Then our top-form above becomes:

$$\omega = h(p)\frac{\partial x^1}{\partial y^{a_1}}dy^{a_1} \wedge ... \wedge \frac{\partial x^m}{\partial y^{a_m}}dy^{a_m} = h(p)\det(\frac{\partial x^a}{\partial y^c})dy^1 \wedge ... \wedge dy^m.$$

Where the determinant is just the Jacobian and since M is orientable, the Jacobian has to be positive by definition. When M is not orientable it is not possible to define a ω with positive-definite component on M. This is the case for the Möbius strip. If we walk around the strip with our given set of oriented coordinates, after having walked along the strip once, we come back with opposite orientation. I.e. $\omega = dx \wedge dy$ changes signature, $dx \wedge dy \to -dx \wedge dy$, when we arrive back at the starting point. Hence a positive-definite measure cannot be defined.

Integration of forms

After that discussion of orientation and orientable manifolds we are set to define integration of a smooth function $f : M \to \mathbb{R}$, over an orientable m-manifold M. We start of by taking a top-form ω. Then the integration of our

m-form $f\omega$ in a coordinate chart (U_α, ϕ) with local coordinates $(x^1, \ldots, x^m)$ is defined as;

$$\int_{U_\alpha} f\omega := \int_{\phi(U_\alpha)} f(\phi_i^{-1}(x))h(\phi_i^{-1}(x))dx^1...dx^m. \tag{3.2}$$

The right hand side here is simply a multiple integral of a function consisting of m variables that you probably recognize from some vector calculus course. Now that we have defined the integral over a single chart U_α we want to extend it to the whole manifold. The integral of f over the whole manifold M can be done using something called the *partition of unity* that will be defined now.

Definition 26 ([3] Definition 5.7)**.** Let M be a paracompact manifold, then we can take an open covering $\{U_i\}$ of M s.t. every point of M is covered with a finite number of U_i. Next if a collection $\{\epsilon_i(p)\}$ of functions $\epsilon_i(p)$ are differentiable satisfying the following conditions:

1. $0 \leq \epsilon_i(p) \leq 1$,
2. $\epsilon_i(p) = 0$ if $p \notin U_i$ and,
3. $\epsilon_1 + \epsilon_2 + ... = 1$ for any point $p \in M$.

$\{\epsilon_i(p)\}$ is then called a *partition of unity* subordinate to the covering $\{U_i\}$.

Remark ([2] chap 13.2)**.** *An equivalent way of defining the two first conditions is that we require the functions $\epsilon_i(p)$ to have compact support, supp$(\epsilon_i) \subset U_i$.*

It then follows from the 3. point that

$$f(p) = \sum_i f(p)\epsilon_i(p) = \sum_i f_i(p)$$

where $f_i(p) := f(p)\epsilon_i(p)$ vanish outside U_i by the 2. condition. Hence if we assume M to be paracompact, so that for any $p \in M$ the summation over i only contains finite terms, we can define the integral for each $f_i(p)$ according to 3.2. We then finally arrive at the integral of f on M given as

$$\int_M f\omega := \sum_i \int_{U_i} f_i\omega. \tag{3.3}$$

This integral is independent of the choice of atlas, i.e. it would remain the same when using a different atlas $\{(W_i, \chi_i)\}$ with a different partition of unity. To get a feeling of the procedure let us take a look at a familiar example of a manifold, and use the partition of unity to integrate a function.

Example 12 ([3] Example 5.13)**.** We revisit S^1, but this time using the atlas illustrated on the picture.

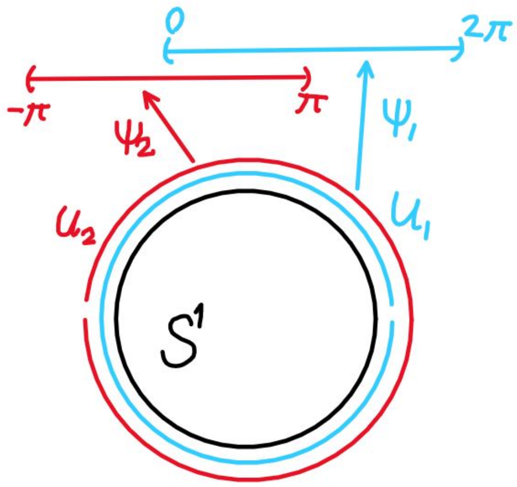

Figure 3.2: S^1 with atlas composed of two charts (U_1, ψ_1) and (U_2, ψ_2).

We let the blue chart be (U_1, ψ_1) where $U_1 = S^1 \setminus \{(1,0)\}$, and $\psi_1 : U_1 \to (0, 2\pi)$. And the red chart be (U_2, ψ_2) where $U_2 = S^1 \setminus \{(-1,0)\}$, and $\psi_2 : U_2 \to (-\pi, \pi)$. We then use the following partition of unity $\epsilon_1(\theta) = \sin^2 \frac{\theta}{2}$ and $\epsilon_2(\theta) = \cos^2 \frac{\theta}{2}$. Then we can for instance integrate $f = \sin^2 \theta$ using the partition of unity:

$$\int_{S^1} d\theta \sin^2\theta = \int_0^{2\pi} d\theta \quad \sin^2\theta \sin^2\theta/2 + \int_{-\pi}^{\pi} d\theta \sin^2\theta \cos^2\theta/2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi,$$

which is the correct answer to the integral of $\sin^2\theta$ from 0 to 2π.

3.7 Stokes theorem

(Based on [2] chap. 22, and [8])

Finally, we arrive at the general Stokes Theorem. This theorem is one of the most beautiful and powerful theorems in calculus. Especially since all the theorems you know from multi-variable calculus, such as Green's theorem, divergence theorem, and so on, are all special cases of the Stokes theorem. Before we present the theorem, we want to give an intuitive explanation of what the theorem is saying. Stokes theorem says that: "the sum of the little changes on the inside, equals the total change on the outside"[8]. This might seem a little vague so let us have a look at a quick example:

Example 13. Say you have a function f and you want to see how much the function increase from a point a to a point b. Then if we substitute the words "little change" with the derivative. We get that the sum of the derivative of f from a to b, is equal to the total change on the outside, i.e. $f(b) - f(a)$. Or written more mathematically:

$$\int_a^b \frac{df(x)}{dx} dx = f(b) - f(a).$$

Which is just the fundamental theorem of calculus or Stokes theorem in 1 dimension, and we can play the same game in higher dimensions.

Theorem 1 (Stokes Theorem ([2] Theorem 22.8))**.** *If M is a compact orientable manifold of dimension k, and $\omega \in \Omega^{k-1}(M)$ then:*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{3.4}$$

We follow the proof to theorem 22.8 in [2].

Proof. We will omit some more general technical aspects for the sake of clarity. We start off by choosing an atlas $\{(U_a, \phi_a)\}$ for our manifold M in such a way that each of the open subsets U_a is diffeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n$, ensuring that the diffeomorphism preserves the orientation. This is easily done since there is a result stating that every open disk is diffeomorphic to $\mathbb{R}^n$. We then choose a smooth partition of unity $\{\epsilon_a\}$ that is subordinate to $\{U_a\}$. Then let us assume that the Stokes theorem holds for $\mathbb{R}^n$ and $\mathbb{H}^n$. This then means that it holds for all the charts U_a in the atlas since these are diffeomorphic either to $\mathbb{R}^n$ or $\mathbb{H}^n$. A nice result that will be used is that the boundary of the manifold intersected with a chart gives the boundary of the chart, $(\partial M) \cap U_a = \partial U_a$.

We then have that

$$\int_{\partial M} \omega = \int_{\partial M} \sum_a \epsilon_a \omega$$

Rewriting, remembering that $\sum_a \epsilon_a = 1$,

$$= \sum_a \int_{\partial M} \epsilon_a \omega,$$

since $\sum_a \epsilon_a \omega$ is finite we can interchange the sum and the integral sign.

This equals

$$\sum_a \int_{\partial U_a} \epsilon_a \omega$$

which comes from the fact that $\epsilon_a = 0$ outside U_a. Using our assumption, we get

$$\sum_a \int_{U_a} \mathrm{d}(\epsilon_a \omega),$$

invoking Stokes theorem for U_a. This is further equal to

$$\sum_a \int_M \mathrm{d}(\epsilon_a \omega),$$

since the space where ϵ_a is nonzero is a subset of U_a.
We can rewrite this as

$$\int_{\partial M} \mathrm{d}(\sum_a \epsilon_a \omega),$$

interchanging since $\epsilon_a \omega$ is only non-zero for finitely many a. We finally get

$$= \int_M \mathrm{d}\omega.$$

Therefore in order to prove Stokes theorem we have to prove the theorem both for $\mathbb{R}^n$ and $\mathbb{H}^n$. We only show the part for the upper half plane $\mathbb{H}^2$, the general case of $\mathbb{H}^n$ follows in a similar fashion. The proof regarding $\mathbb{R}^n$ will not be shown here. The interested reader may visit [2] problem 22.3 for a proof of $\mathbb{R}^n$.

We let x, y denote coordinates of $\mathbb{H}^2$. The orientation of $\mathbb{H}^2$ is then given by $dx \wedge dy$, and the boundary is $\partial \mathbb{H}^2$ is oriented by dx. We then choose a general 1-form

$$\omega = f(x,y)dx + g(x,y)dy$$

where f, g are smooth functions with compact support. Compact support is an equivalent way of stating the two first axioms of a partition of unity 26. Because f and g have compact support it means that they are finite and zero outside some domain. We therefore choose some number $a > 0$ that is large enough to capture the supports of f and g in the interior of the square made up of the line segments $[-a, a] \times [-a, a]$. To make the calculation simpler we denote the partial derivatives of f w.r.t. x and y as f_x and f_y. Now acting with the exterior derivative on ω gives

$$\mathrm{d}\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy = (g_x - f_y) dx \wedge dy$$

We then get when integrating over $\mathbb{H}^2$

$$
\begin{aligned}
\int_{\mathbb{H}^2} \mathrm{d}\omega &= \int_{\mathbb{H}^2} g_x|dxdy| - \int_{\mathbb{H}^2} f_y|dxdy| \\
&= \int_0^\infty \int_{-\infty}^\infty g_x|dxdy| - \int_{-\infty}^\infty \int_0^\infty f_y|dxdy| \\
&= \int_0^a \int_{-a}^a g_x|dxdy| - \int_{-a}^a \int_0^a f_y|dxdy|.
\end{aligned} \tag{3.5}
$$

Now since $\mathrm{supp}(g)$ lies in the interior of the square $[-a, a] \times [-a, a]$, the integral over g gives

$$\int_{-a}^a g_x(x, y)dx = g(x, y)|_{-a}^a = 0$$

Since both $g(a, y) = 0$ and $g(-a, y) = 0$. Likewise the integral over f gives

$$\int_0^a f_y(x, y)dy = f(x, y)|_0^a = -f(x, 0) \tag{3.6}$$

since like in the case of g, $f(x, a) = 0$. The equation 3.5 therefore just becomes

$$\int_{\mathbb{H}^2} \mathrm{d}\omega = \int_{-a}^a f(x, 0)dx$$

For the boundary $\partial\mathbb{H}^2$ it lays along the x-axis, where $dy = 0$ on the boundary. Therefore our 1-form takes the form $\omega = f(x, 0)dx$ when restricting to $\partial\mathbb{H}^2$, hence we get

$$\int_{\partial\mathbb{H}^2} \omega = \int_{-a}^a f(x, 0)dx.$$

This proves Stokes Theorem for the upper half-plane $\mathbb{H}^2$. □

Example 14 (Divergence theorem)**.** If we have a finite volume V in $\mathbb{R}^3$ with boundary ∂V being the surface of the volume. Our boundary ∂V has orientation denoted by an outward pointing normal vector $\vec{n}$. If we then want to integrate a vector field $\mathbf{F}$ over the surface the divergence theorem states that:

$$\int_V \nabla \cdot \mathbf{F} dV = \int_{\partial V} \mathbf{F}\vec{n} dS,$$

which we recognize as just Stokes theorem in 3D.

Example 15 (Integrating a 2-form over 3-manifold)**.** Let's have a look at a rather simple example using Stokes theorem. We let M_3 be compact orientable 3-manifold with $\partial M_3 = 0$, and we let $\psi \in \Omega^2(M_3)$. We then have from Stokes theorem

$$\int_{M_3} \mathrm{d}\psi = \int_{\partial M_3} \psi = \int_0 \psi = 0.$$

More generally, integrating exact forms over compact manifolds without boundary gives zero. This is useful later when we need to "integrate by parts".

This ends our discussion of differential forms in the sense of defining them, and showing how to do calculus with them. Throughout the rest of the thesis, differential forms will show up everywhere. In the upcoming chapter, we discuss something slightly off-topic when it comes to doing the final computation of a partition function. Namely, Cohomology, which we saw some glimpse of earlier, and this will give us examples of topological invariants, just as our final computation of a quantum partition function. Cohomology groups are relevant for classical field theory, as they often count solutions to equations of motion.

Chapter 4

Cohomology

(This chapter is based on [2] chap. 23-27, [9] chap.7, and [3] chap. 6)

In multi-variable calculus, one gets introduced to the divergence, the gradient, and the curl. One also sees theorems like Green's theorem, divergence theorem, and Stokes theorem. For example, Stokes theorem one can use over a surface if your vector field is the curl of another vector field. Therefore, one is often interested to know whether a vector field is the gradient of a function or if it is the curl of another vector field. This can then determine which theorem to use. Vector fields can be expressed in terms of differential forms, and with differential forms, the question above translates to whether the form is exact. Henri Poincaré started to look for which conditions had to be satisfied for a differential form ω to be exact on $\mathbb{R}^n$. One necessary condition is, of course, that the forms are closed. In 1887 Poincaré published a proof that k-forms are exact iff it is closed for $k = 1, 2, 3$ on $\mathbb{R}^n$. We showed the Poincaré Lemma in the last chapter, which implies this. Two years later, Vito Volterra published a proof of the Poincaré lemma for any k. This question of whether a closed form on a manifold is exact comes down to the topology of the manifold. An example is as we saw that on $\mathbb{R}^2$ every closed form is exact, but if we look at the punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$ there will be closed 1-forms that are not exact. As mentioned, the way we measure to which extent closed forms fails to be exact is done using the de Rham cohomology. These de Rham cohomology groups are some of the most important topological invariants of manifolds. Using Hodge-theory, which we will see in the next chapter, they can be used to count solutions of differential equations, such as Maxwell's equations.

In this chapter, we will start by defining de Rham cohomology. We will then devote some sections to looking at some mathematical tools in the form of cochain complexes. Together with the Mayer-Vietoris theorem, this will give us a quite simple way to perform some exact computations of cohomology groups for some manifolds.

4.1 de Rham Cohomology

(Based on [3] chap.6.2)

We briefly touched upon cohomolgy in chapter 3.5. If we let M be a smooth m-manifold, then the set of closed p-forms is denoted $Z^p(M)$. And the set of exact p-forms is denoted $B^p(M)$. Due to that closed forms are defined as $\{\alpha \in \Omega^p(M) | \mathrm{d}\alpha = 0\}$ and the exact forms is defined as $\{\alpha = d\beta | \beta \in \Omega^{p-1}(M)\}$ it implies that every exact form is also closed since $\mathrm{d}^2 = 0$. Hence the space of exact forms is a subspace of the closed forms, $B^p(M) \subset Z^p(M)$. These spaces have a group structure, so we may construct a factor group, which lead us to the definition of the p'th cohomology group.

Definition 27 ([3] definition 6.2)**.** The *p'th de Rham cohomology group* is the factor group:

$$H^p(M) = \frac{Z^p(M)}{B^p(M)} = \frac{\{\alpha \in \Omega^p(M) | \mathrm{d}\alpha = 0\}}{\{\alpha = \mathrm{d}\beta | \beta \in \Omega^{p-1}(M)\}}.$$

The cohomology groups works as a tool that "measures" to which extent the closed forms fails to be exact, just as we saw in the example 11. These groups are as we will see later on, another example of topological invariants, meaning that they are invariant under any smooth geometric change.

Within the set of closed forms $Z^p(M)$, are two p-forms defined to be equivalent if they only differ by an exact form;

$$\omega = \tilde{\omega} + d\tau.$$

In this case ω and $\tilde{\omega}$ are called *cohomologous*. They also give rise to the same cohomology class, $[\omega] = [\tilde{\omega}]$. Below is a definition and a few propositions stated that will be helpful when we later want to explicitly calculate the cohomolgy groups for a few simple manifolds.

Definition 28 ([3], page. 235)**.** The dimension of the p'th cohomology group of M is denoted $b^p(M)$ and is called *p'th betti number*, i.e $b^p(M) := \dim(H^p(M))$.

Proposition 4 ([3])**.** Let M be a smooth compact orientable manifold, then $H^p(M) \cong \mathbb{R}^k$, for $0 \leq k < \infty$. Note that this is a topological property. In fact, cohomology groups are invariant under continuous deformation (homotopy). It follows from various results in Nakahara [3].

Proposition 5 ([2] Proposition 23.1)**.** If M is a manifold consisting of k connected components, then the zeroth de Rham cohomology is $H^0 \cong \mathbb{R}^k$.

Proposition 6 ([2] Proposition 23.2)**.** Let M be an m-dimensional manifold, then the de Rham cohomology $H^p(M) = 0$, for $p > m$.

4.2 Long exact sequences in Cohomology

(Based on [2] chap. 24)

As mentioned, one of the powerful features of cohomology is that we can use it to determine the solution space of differential equations. The betti numbers often count the number of independent solutions, and they are usually easier to compute than constructing the solution explicitly. So through cohomology, we can identify if there exists a solution for a particular differential equation without having to solve it! Nevertheless, before showing this, we have to develop some mathematical tools; so-called cochain complexes, short exact sequences (SES), and long exact sequences (LES).

A cochain complex is essentially a sequence of vector spaces, $\{V^k\}_{k\in\mathbb{Z}}$, connected by homomorphisms, $d_k : V^k \to V^{k+1}$. These homomorphisms have to satisfy $d_{k+1} \circ d_k = 0$, or dropping indices $d^2 = 0$. These homomorphisms are often called *differentials* or *boundary operators*. A cochain complex can then be written in the following way:

$$... \xrightarrow{d_{-1}} V^0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} V^2 \xrightarrow{d_2} V^3 \xrightarrow{d_3} ... \tag{4.1}$$

Remark. *A cochain complex is the dual of a chain complex, for which can be written in much the same way but with homomorphisms going the other way;*

$$... \xleftarrow{d_{-1}} V_{-1} \xleftarrow{d_0} V_0 \xleftarrow{d_1} V_1 \xleftarrow{d_2} V_2 \xleftarrow{d_3} V_3 \xleftarrow{d_4} ...$$

We have already seen a special case of this notion of a cochain complex. Because if we look at the vector space of differential forms on a manifold M, $\Omega^*(M)$, together with the exterior derivative d as our differential, we have the de Rham complex of M:

$$0 \to \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \Omega^3(M) \xrightarrow{\mathrm{d}} ...$$

Back in chapter 3, we saw an example where M was $\mathbb{R}^3$. Let us go over now to define a short exact sequence, but first, we need to define what an exact sequence is.

Definition 29 (Exact Sequence [2] Definition 24.1)**.** Vector spaces connected by homomorphisms in a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact at B if im(f) =ker(g). It then follows that a sequence of vector spaces and homomorphisms

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-1}} A_n$$

is an *exact sequence* if it is exact at each A_i, except for the first an last.

Definition 30 (Short exact sequence [2] Definition 24.1)**.** If we have the sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

where A, B and C are vector spaces. Cochain complexes are infinite dimensional in general. But the following proposition will be useful. f and g are linear maps, and if we require f to be injective, g to be surjective and $\mathrm{Im}(f) = \ker(g)$ we get what is known as a *short exact sequence* (SES).

Proposition 7 ([2] appendix D.1)**.** From linear algebra we have that for finite dimensional vector spaces A and B with linear map $f : A \to B$;

$$\dim(A) = \dim(\ker(f)) + \dim(\mathrm{Im}(f)).$$

When dealing with exact sequences the following proposition can also be useful.

Proposition 8 ([2] Proposition 24.4)**.** The sequence of vector spaces

$$0 \to A \xrightarrow{f} B \to 0$$

is exact iff f is an isomorphism.

4.2.1 Cohomology of cochain complexes

(Based on chapter [2] chapter 24 and [9] section 25)

Now that we have gained some knowledge of what a cochain complex and an exact sequence are, we can start discussing the cohomology of cochain complexes. If we let $\mathcal{V}$ be a cochain complex then by the fact that $\mathrm{d}_{k+1} \circ \mathrm{d}_k = 0$ we have that,

$$\mathrm{im}(\mathrm{d}_k) \subset \ker(\mathrm{d}_{k+1}).$$

Thus we can construct the *k'th cohomology vector space* for a cochain complex $\mathcal{V}$, as the following quotient vector space

$$H^k(\mathcal{V}) := \frac{\ker(d_k)}{\operatorname{im}(d_{k-1})}.$$

Just as we saw earlier with differential forms, the cohomology group measures to which extent closed forms fails to be exact. Likewise, the cohomology of a cochain complex measures to which extent the cochain complex $\mathcal{V}$ fails to be exact at V^k. The elements of $\ker(\mathrm{d}_k)$ and $\operatorname{im}(\mathrm{d}_{k-1})$ are called k-cocycle and k-coboundary respectively. The equivalence class $[v] \in H^k(\mathcal{V})$ of a k-cocycle $v \subset \ker \mathrm{d}_k$ we call the *cohomology class* of v. The subspaces of k-cocycles and k-coboundaries of $\mathcal{V}$ we denote $Z^k(\mathcal{V})$ and $B^k(\mathcal{V})$ respectively. Again we simplify the notation and drop the subscript of d_k if it does not cause confusion.

Example 16 ([2] Example 24.5)**.** As you probably noticed, we are using a similar notation for the k-cocycles and k-coboundaries as we did for the closed and exact forms when we discussed the cohomology group of a manifold earlier on. This is because when we are talking about the de Rham complex, a cocycle is simply a closed form, and a coboundary is an exact form.

If we have two cochain complexes $\mathcal{A}$ and $\mathcal{B}$ with their respective differentials d and $\tilde{d}$. Then we define the following map, $\phi : \mathcal{A} \to \mathcal{B}$ as a collection of linear maps $\phi_k : A^k \to B^k \ \forall k$, such that they commute with d and $\tilde{d}$:

$$\tilde{d} \circ \phi_k = \phi_{k+1} \circ d,$$

then ϕ is called a *cochain map*. I.e. if ϕ is cochain map it implies that the diagram below is commutative:

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{d} & A^{k-1} & \xrightarrow{d} & A^k & \xrightarrow{d} & A^{k+1} & \xrightarrow{d} & \dots \\
 & & \Big\downarrow{\scriptstyle \phi_{k-1}} & & \Big\downarrow{\scriptstyle \phi_k} & & \Big\downarrow{\scriptstyle \phi_{k+1}} & & \\
\dots & \xrightarrow{\tilde{d}} & B^{k-1} & \xrightarrow{\tilde{d}} & B^k & \xrightarrow{\tilde{d}} & B^{k+1} & \xrightarrow{\tilde{d}} & \dots
\end{array}$$

.

Also in this case one usually omits the subscript in ϕ_k. One useful feature of a cochain map $\phi : \mathcal{A} \to \mathcal{B}$ is that it induces a linear map in cohomology;

$$\phi^* : H^k(\mathcal{A}) \to H^k(\mathcal{B})$$

by $\phi^*[a] = [\phi(a)]$.

Let's have a look at an example of a cochain map that we have briefly mentioned before.

Example 17 ([9] example 25.4)**.** If we have two manifolds M and N, with a smooth map $f : N \to M$, then the pullback map $f^* : \Omega^*(M) \to \Omega^*(N)$ is a cochain map since f^* commutes with d due to the following proposition

Proposition 9 ([9] Proposition 19.5)**.** If $f : N \to M$ is a smooth map between manifolds. Then for a differential form $\omega \in \Omega^k(M)$, we have that $\mathrm{d}f^*\omega = f^*\mathrm{d}\omega$.

Earlier we defined what a short exact sequence was when talking about a sequence of vector spaces. In much the same way we define a short exact sequence of cochain complexes.

Definition 31 (SES of cochain complexes, [2] chap. 24.3)**.** The following sequence of cochain complexes

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

is called *short exact* if i and j are cochain maps, and for every k we have that the sequence of vector spaces;

$$0 \to A^k \xrightarrow{i_k} B^k \xrightarrow{j_k} C^k \to 0$$

is a short exact sequence of vector spaces, i.e. if i_k is injective, and j_k is surjective, and $im(i_k) = \ker(j_k)$.

We then have the following very useful lemma for calculating the cohomology of a cochain complex.

Lemma 5 (Snake Lemma, [2] Theorem 24.7)**.** *Given a short exact sequence of cochain complexes*

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

it gives rise to the following long exact sequence (LES) in cohomology

$$\begin{aligned} 0 \to &H^0(\mathcal{A}) \xrightarrow{i^*} H^0(\mathcal{B}) \xrightarrow{j^*} H^0(\mathcal{C}) \xrightarrow{d^*} H^1(\mathcal{A}) \xrightarrow{i^*} H^1(\mathcal{B}) \\ &\xrightarrow{j^*} H^1(\mathcal{C}) \xrightarrow{d^*} ... \xrightarrow{d^*} H^k(\mathcal{A}) \xrightarrow{i^*} H^k(\mathcal{B}) \xrightarrow{j^*} H^k(\mathcal{C}) \xrightarrow{d^*} ... \end{aligned},$$

i and j are cochain maps that induces the maps i^* and j^* in cohomology, and d^* is a so-called connecting homomorphism.

4.3 The Mayer-Vietoris sequence

(Based on [2] chap. 25)

Our final goal for this chapter has been to develop a way of calculating the de Rham cohomology of a manifold. This type of calculation is, in general, a pretty difficult task, which is why we have spent so much time discussing these mathematical tools related to cochain complexes and sequences. Because now, we will finally introduce what is known as the Mayer-Vietoris sequence, which is a powerful tool when it comes to calculating de Rham cohomology.

The idea of calculating the de Rham cohomology of a manifold using Mayer-Vietoris is as follows. Although it is often hard to calculate the cohomology of the entire manifold M, it is often a simpler task to calculate the cohomology of open subsets of M, U_i. So for simplicity let's say that two subsets of M, U_1 and U_2, cover M, i.e. $U_1 \cup U_2 = M$. We then have a map $i = (i_1^*(a), i_2^*(a))$ where $i_{1/2} : U \to M$, $i_{1/2}(a) = a$. And a map $j = j_1^*(a) - j_2^*(b)$, where $j_{1/2} : U_1 \cap U_2 \to U_{1/2}$, $j_{1/2}(a) = a$. Then the following sequence is short exact:

$$0 \to \Omega^k(M) \xrightarrow{i} \Omega^k(U_1) \oplus \Omega^k(U_2) \xrightarrow{j} \Omega^k(U_1 \cap U_2) \to 0 \tag{4.2}$$

And then from the "Snake lemma" above, we get that this SES, induces a LES in cohomology.

Theorem 2 (Mayer-Vietoris, [2] chap. 25)**.** *Let M be a connected smooth manifold, with open subsets U_1 and U_2 s.t. $M = U_1 \cup U_2$. Then we have the following SES;*

$$0 \to \Omega^k(M) \xrightarrow{i} \Omega^k(U_1) \oplus \Omega^k(U_2) \xrightarrow{j} \Omega^k(U_1 \cap U_2) \to 0$$

gives rise to the LES in cohomology:

$$\begin{aligned} 0 \to & H^0(M) \xrightarrow{i^*} H^0(U_1) \oplus H^0(U_2) \xrightarrow{j^*} H^0(U_1 \cap U_2) \xrightarrow{d^*} \\ & H^1(M) \xrightarrow{i^*} H^1(U_1) \oplus H^1(U_2) \xrightarrow{j^*} H^1(U_1 \cap U_2) \xrightarrow{d^*} \ldots \\ & \xrightarrow{d^*} H^k(M) \xrightarrow{i^*} H^k(U_1) \oplus H^k(U_2) \xrightarrow{j^*} H^k(U_1 \cap U_2) \xrightarrow{d^*} \ldots \end{aligned}$$

with d^* being the connecting homemorphism; $d^* : H^k(U_1 \cap U_2) \to H^{k+1}(M)$.

4.4 Computations of de Rham cohomology

Finally, we are ready to perform some calculations of cohomology groups for some specific manifolds. We will look at the circle, S^1, and the sphere, S^2, as our examples. Before we start the calculations, we briefly mention the following version of the Poincaré lemma.

Lemma 6 (Poincaré's Lemma, [3] theorem 6.3 and example 6.4)**.** *If U is contractible, meaning it can be continuously deformed to a point. Then the cohomology of U is;*

$$H^p(U) \cong \begin{cases} \mathbb{R} & , p = 0 \\ 0 & , otherwise. \end{cases}$$

Example 18 (Cohomology of S^1)**.** Let $S^1 = U_1 \cup U_2$ where $U_1 = \{x^2 + y^2 = 2 | y > -1\}$ and $U_2 = \{x^2 + y^2 = 2 | y < 1\}$.

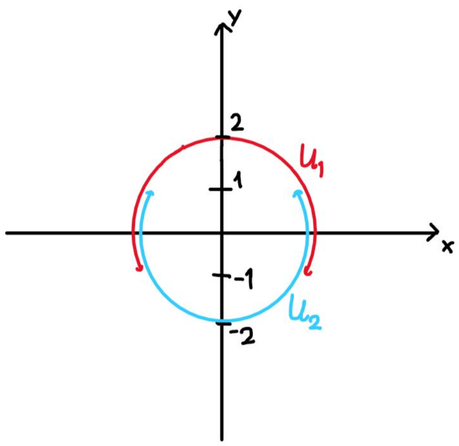

Figure 4.1: Open subsets U_1 and U_2 constructing S^1.

We then have the following LES form Mayer-Vietoris:

$$\begin{aligned} 0 \to & H^0(S^1) \to H^0(U_1) \oplus H^0(U_2) \to H^0(U_1 \cap U_2)) \to \\ & H^1(S^1) \to H^1(U_1) \oplus H^1(U_2) \to H^1(U_1 \cap U_2) \to 0 \end{aligned}$$

Using proposition 5 we have that $H^0(S^1) \cong \mathbb{R}$ and $H^0(U_1 \cap U_2) \cong \mathbb{R}^2$. We then have from the Poincare lemma above, that $H^0(U_1) \cong H^0(U_2) \cong \mathbb{R}$, and that $H^1(U_1) \cong H^1(U_2) \cong 0$, and that $H^1(U_1 \cap U_2) \cong 0$. Hence we are left with;

$$0 \to \mathbb{R} \xrightarrow{\alpha} \mathbb{R}^2 \xrightarrow{\beta} \mathbb{R}^2 \xrightarrow{\gamma} H^1(S^1) \to 0,$$

for some maps α, β and γ. Then $b_1(S^1) = \dim(H^1(S^1)) = \dim(im(\gamma))$. We also have

$$\begin{aligned}
& 2 = \dim(\ker(\gamma)) + b_1(S^1) \\
\implies & 2 = \dim(im(\beta)) + b_1(S^1) \\
\text{and } & 2 = \dim(\ker(\beta)) + \dim(im(\beta)) \\
\implies & 2 = \dim(im(\alpha)) + \dim(im(\beta)). \\
& \alpha \text{ is injective, so } \dim(im(\alpha)) = 1, \text{ and} \\
& 2 = 1 + \dim(im(\beta)) \\
& \quad \implies \dim(im(\beta)) = 1 \\
& \quad \implies b_1(S^1) = 1.
\end{aligned}$$

Hence the cohomology of S^1 is:

$$H^p(S^1) \cong \begin{cases} \mathbb{R}, & p = 0 \\ \mathbb{R}, & p = 1. \end{cases}$$

Example 19 (Cohomology of S^2)**.** We let $S^2 = U_1 \cup U_2$, where U_1 and U_2 are the open northern and southern hemisphere respectively, s.t. $U_1 \cap U_2$ are contractible to S^1 (See figure 4.2).

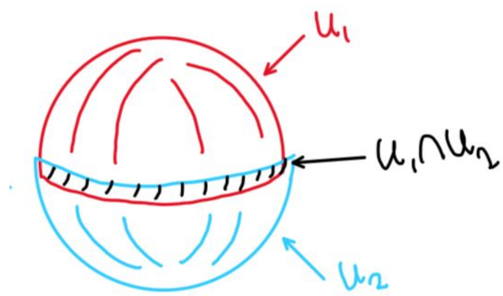

Figure 4.2: Open northern and southern hemisphere constructing S^2.

Then we have the following LES for from Mayer-Vietoris

$$\begin{aligned}
0 \to & H^0(S^2) \to H^0(U_1) \oplus H^0(U_2) \to H^0(U_1 \cap U_2) \to \\
& H^1(S^2) \to H^1(U_1) \oplus H^1(U_2) \to H^1(U_1 \cap U_2) \to \\
& H^2(S^2) \to H^2(U_1) \oplus H^2(U_2) \to H^2(U_1 \cap U_2) \to 0
\end{aligned}$$

Then since U_1 and U_2 are contractible, we can again use Poincare lemma: $H^0(U_1) \cong H^0(U_2) \cong \mathbb{R}$, and $H^i(U_1) \cong H^i(U_2) \cong 0$, for $i = 1, 2$. Also, since

$U_1 \cap U_2$ is contractible to S^1, we get $H^0(U_1 \cap U_2) \cong \mathbb{R}$, $H^1(U_1 \cap U_2) \cong \mathbb{R}$ and $H^2(U_1 \cap U_2) = 0$ We then have the following sequence:

$$0 \to \mathbb{R} \xrightarrow{\alpha} \mathbb{R}^2 \xrightarrow{\beta} \mathbb{R} \xrightarrow{\gamma} H^1(S^2) \xrightarrow{\delta} 0 \xrightarrow{\epsilon} \mathbb{R} \xrightarrow{\eta} H^2(S^2) \to 0,$$

for maps α, β, γ, δ, ϵ, η. We can then quickly see that since this is an exact sequence, then we know that the last part of the sequence:

$$0 \xrightarrow{\epsilon} \mathbb{R} \xrightarrow{\eta} H^2(S^2) \to 0$$

implies that $H^2(S^2) \cong \mathbb{R}$ since η has to be an isomorphism. Hence $b_2(S^2) = \dim(\mathbb{R}) = 1$.

Now to find $b_1(S^2)$ we have to do a similar calculation as in the example above with S^1.

We have that $b_1(S^2) = \dim(H^1(S^2)) = \dim(im(\gamma))$.

$$\begin{aligned}
&\implies 1 = \dim(\ker(\gamma)) + b_1(S^2) \\
&\implies 1 = \dim(im(\beta)) + b_1(S^2) \\
\text{Furthermore } &2 = \dim(\ker(\beta)) + \dim(im(\beta)) \\
&\implies 2 = \dim(im(\alpha)) + \dim(im(\beta)) \\
&\implies 2 = 1 + \dim(im(\beta)) \\
&\implies \dim(im(\beta)) = 1.
\end{aligned}$$

The second line then gives us that: $b_1(S^2) = 0$.

Hence the cohomology of S^2 is:

$$H^p(S^2) \cong \begin{cases} \mathbb{R} & , p = 0, \\ 0 & , p = 1, \\ \mathbb{R} & , p = 2. \end{cases}$$

One can show that this similarly generalizes to the following cohomology of the n-sphere:

$$H^p(S^n) \cong \begin{cases} \mathbb{R} & , p = 0, \\ 0 & , p = 1, \\ \vdots & \\ 0 & , p = n - 1 \\ \mathbb{R} & , p = n. \end{cases}$$

As mentioned, cohomology groups can show whether solutions exist to certain differential equations, particularly equations of motion. E.g., the two sourceless equations of Maxwell's equations in electromagnetism can be expressed as the following $\mathrm{d}F = 0$ where F is the electromagnetic field tensor, expressed as a two form $F = \mathrm{d}A \in \Omega^2(M)$, and A is locally defined (but not global in general). The other two of Maxwell's equations in vacuum can be written as $\mathrm{d}^\dagger F = 0$, where $\mathrm{d}^\dagger$ will be defined in the next chapter. Solutions are then given by so-called harmonic two-forms $\mathcal{H}^2$, and as we will see, $\mathcal{H}^2 \cong H^2(M)$. Now, as we see $H^2(S^2) = \mathbb{R}$, i.e. there exist solutions to Maxwell's equations on the 2-sphere even if we do not know its explicit form. However, there do not exist solutions on the 3-sphere S^3 since $H^2(S^3) = 0$.

Chapter 5

Hodge theory

(This chapter is based on [3] chap. 7, [5] chap. 4 and 5, and [10])

We have gone through some fundamental concepts in differential geometry and also looked at cohomology groups, which are examples of topological invariants, as our final computation also will be. Before we can do the one-loop computation, we have to look at a few more topics that will give us the tools needed to tackle it. In this chapter, we take a look at something called Hodge theory. We introduce an operator called the "Hodge star" and use this to define the adjoint of the exterior derivative; $\mathrm{d}^\dagger$. Then we will define the Laplacian operator in terms of d and $\mathrm{d}^\dagger$. This Laplacian operator will be a key idea, and the solution to our final computation will be expressed in terms of the determinants of different Laplace operators. We will also look at the Hodge decomposition, which, together with the Laplacian, will be the key topics for this chapter.

Before we can take a look at all of this, let us start by briefly mentioning what Hodge theory is. Hodge theory, developed by W. V. D. Hodge, is a way to study cohomology groups of a smooth manifold by using PDE's. The main idea in Hodge theory is that given a Riemannian manifold with a Riemannian metric, then every cohomology class has a particular representative. Such a representative is a differential form that, when acted upon by the Laplacian operator, will vanish. We call these forms "harmonic", they solve the Laplace equation and are in 1-1 correspondence with the classical topological invariants counted by the cohomology groups. So let us start this chapter by defining what a Riemannian manifold is and what a Riemannian metric is.

5.1 Riemannian manifold

(Based on [3] chap. 7.1 and [5] chap. 4 and 5)

Riemannian manifolds and the study of them called Riemannian geometry is a vast field in mathematics, and a proper introduction to this topic could easily be a thesis on its own. For our purpose in this thesis, we will only cover the very basics. Briefly mentioned, a metric, g, is something that allows us to define distance on a manifold. Since we are looking at topological theories here, we do not care too much about metrics. However, we need a metric for the final computations for technical reasons. For instance, to show that our result is topological, we have to perform a small variation w.r.t. the metric and show that it vanishes. We also need a metric to define a Riemannian manifold, which in turn also makes us able to define Hodge theory. We also need the metric to define the adjoint operator to the exterior derivative.

In the geometry you know from $\mathbb{R}^n$ we define the inner product for two vectors U and V as $U \cdot V = \sum_i^n U_i V_i$. However, on a manifold, the inner product has to be defined at every tangent space. In order to define such an inner product, we first have to define the metric tensor. A Riemannian metric, often denoted g, is a type $(0, 2)$-tensor field on a manifold M, i.e. g is a linear map; $g_p : T_pM \otimes T_pM \to \mathbb{R}$. The metric also satisfies the following axioms:

Let U, $V \in T_pM$

1. $g_p(U, V) = g_p(V, U)$
2. $g_p(U, U) \geq 0$, equality if and only if $U = 0$.

A Riemannian manifold is then simply a smooth manifold M equipped with a metric, (M, g).

Example 20. An example of a Riemannian manifold is $\mathbb{R}^n$. As we have shown, $\mathbb{R}^n$ is a smooth manifold. The usual metric that $\mathbb{R}^n$ is equipped with is δ_{ij}, which when represented as a matrix is, of course, just the identity matrix.

General Relativity (GR) is the field within physics where Riemannian geometry is used the most. Strictly speaking, one is not using Riemannian geometry but what is known as pseudo-Riemannian geometry. The pseudo comes from the fact that one also includes time. This time coordinate has to have the opposite sign to the coordinates regarding space. So on a pseudo-Riemannian manifold, the second axiom above is swapped out with: If $g_p(U, V) = 0$ for any vector $U \in T_p(M)$, then vector $V = 0$.

GR describes spacetime as a 4-dimensional pseudo-Riemannian manifold. The simplest version of this is what is known as Minkowski Space and is represented by $\mathbb{R}^4$ equipped with the Minkowski metric (setting c=1)

$$\eta_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

GR is essentially a study of Einstein's equations

$$\mathbf{R}_{ab} - \frac{1}{2} g_{ab} \mathbf{R} = 8\pi G \mathbf{T}_{ab},$$

which is a set of non-linear, coupled partial differential equations. For anyone with some experience with differential equations, we see that these are tough equations to solve. A huge part of the area of GR is finding solutions to these equations, imposing various assumptions and approximations. The solution to these equations is metrics. Probably one of the most famous solutions is what is known as the *Schwarzchild metric*, which is the unique solution to Einstein's equations if one assumes Spherical symmetry and vacuum ($\mathbf{T}_{ab} = 0$). The fact that this solution is unique is stated by the *Jebsen-Birkhoff theorem* ([11], chap. 10.15). The Schwarzchild metric takes the form

$$g_{ab} = \begin{bmatrix} -(1 - \frac{2GM}{r}) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2GM}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin(\theta) \end{bmatrix}.$$

This solution is essentially an empty universe with only one static black hole.

We end our digression into GR here; this was only meant as a glimpse into a field within physics where one uses pseudo-Riemannian geometry. For our purpose, we will only look at Riemannian geometry from here on out. Many results apply to Riemannian geometry, but fall apart when switching to pseudo-Riemannian geometry. For example, one of the things that is much better understood in Riemannian geometry is Cohomology. We have discussed Riemannian manifolds and metrics in this thesis on topological field theory because we will look at our final theory over a Riemannian manifold. Nevertheless, our final result will be invariant under the metric, i.e. invariant to curvature and distance on the manifold.

5.2 Invariant top-forms

(Based on [3] chap.7.9)

When we introduced integration on a manifold, we saw that in order to perform an integration over an m-dimensional manifold we needed a top-form, i.e. a non-vanishing m-form. If a manifold M is orientable and equipped with a metric g, then the following top-form exist

$$\chi_M := \sqrt{|g|}dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m,$$

which is invariant under coordinate transformations: Here g denotes the determinant, $g = \det g_{ab}$, and x^a is just the local coordinates of the chart (U, ϕ). Let's show that this top-form is in fact invariant under coordinate transformations. For instance take another chart (V, ψ), with $U \cap V \neq \emptyset$. Then w.r.t to the local coordinates y^d of (V, ψ), the invariant top-form takes the form:

$$\sqrt{\left|\det\left(\frac{\partial x^a}{\partial y^c}\frac{\partial x^b}{\partial y^d}g_{ab}\right)\right|}dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n.$$

We know that we can express $dy^d = \frac{\partial y^d}{\partial x^a}dx^a$, hence we get the following;

$$\begin{aligned}\left|\det\left(\frac{\partial x^a}{\partial y^c}\right)\right|&\sqrt{|g|}\det\left(\frac{\partial y^d}{\partial x^b}\right)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m \\ &= \pm\sqrt{|g|}dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m.\end{aligned}$$

If x^a and y^c define the same orientation, then the Jacobian $\det(\partial x^a/\partial y^c)$ is positive definite on $U \cap V$ implying that the top-form χ_M is invariant under coordinate transformations. With an invariant top-form, we can define integration of a function $f \in C^\infty(M)$ over M as:

$$\int_M f\chi_M := \int_M f\sqrt{|g|}dx^1dx^2 \ldots dx^m$$

This type of expression is very familiar in physics since many objects can be expressed as these kinds of volume integrals. In fact, by doing ordinary calculus, you have actually performed these integrals. It is just that your manifold has usually been $\mathbb{R}^n$ or something embedded in $\mathbb{R}^n$ for which the metric is just the identity matrix which results in $\sqrt{|g|} = 1$.

5.3 Operators in Hodge theory

(Based on [3] chap.7.9)

Back in chapter 3 we introduced differential forms, and along with this, the de Rham operator, also known as the exterior derivative d. We saw that d gave us a way to define derivation of forms and was, in one sense, a generalization of the derivative. The exterior derivative gave us the following map: $d : \Omega^p(M) \to \Omega^{p+1}(M)$. Hence you might wonder if a similar map exists from p-forms to $(p-1)$-forms? Yes, is the short answer; this map is called the adjoint of the derivative.

On an m-dimensional manifold M one can show that $\Omega^p(M)$ is isomorphic to $\Omega^{m-p}(M)$. For a Riemannian manifold M with a metric g, we can define such an isomorphism known as the Hodge star, $*$. $*$ is then simply a linear map: $* : \Omega^p \to \Omega^{m-p}$. To properly define it, we have to define the following totally anti-symmetric tensor:

$$\epsilon_{a_1 a_2 \dots a_m} = \begin{cases} +1 & \text{, if } (a_1 a_2 \dots a_m) \text{is an even permutation of } (12\dots m), \\ -1 & \text{, if } (a_1 a_2 \dots a_m) \text{ is an odd permutation of } (12\dots m), \\ 0 & \text{, otherwise.} \end{cases}$$

If we then take a look at a p-form

$$\omega = \frac{1}{p!} \omega_{a_1 a_2 \dots a_p} dx^{a_1} \wedge dx^{a_2} \dots dx^{a_p} \in \Omega^p(M),$$

then the Hodge star acts in following way

$$*\omega = \frac{\sqrt{|g|}}{p!(m-p!)} \omega_{a_1 a_2 \dots a_p} \epsilon^{a_1 a_2 \dots a_p}_{b_{p+1} b_{p+2} \dots b_m} dx^{b_{p+1}} \wedge dx^{b_{p+2}} \dots \wedge dx^{b_m}$$

Notice that the invariant top-form is just the Hodge star acting on 1;

$$*1 = \frac{\sqrt{|g|}}{m!} \epsilon_{a_1 a_2 \dots a_m} dx^{a_1} \wedge dx^{a_2} \dots \wedge dx^{a_m} = \sqrt{|g|} dx^1 \wedge dx^2 \dots \wedge dx^m.$$

For the Hodge star one also have the following lemma

Lemma 7 ([3] Theorem 7.4)**.** *If (M, g) is a Riemannian manifold of dimension m with $\omega \in \Omega^p(M)$ then*

$$**\omega = (-1)^{p(m-p)} \omega.$$

Hence $**$ returns ω (with either a plus or minus sign).

One helpful feature of the Hodge star is that it allows us to define an inner product for differential forms. We take two forms ω and χ, where

$\omega, \chi \in \Omega^p(M)$. Then we can create an m-form $\omega \wedge *\chi$ with the exterior product in the following way;

$$\begin{aligned}
\omega \wedge *\chi =& \frac{1}{(p!)^2} \omega_{a_1 \dots a_p} \chi_{b_1 \dots b_p} \frac{\sqrt{|g|}}{(m-p)!} \epsilon^{b_1 \dots b_p}_{a_{p+1} \dots a_m} \\
& \times dx^{a_1} \wedge \dots dx^{a_p} \wedge dx^{a_{p+1}} \wedge \dots \wedge dx^{a_m} \\
&= \frac{1}{p!} \sum_{ab} \omega_{a_1 \dots a_p} \chi^{b_1 \dots b_p} \frac{1}{p!(m-p)!} \epsilon_{b_1 \dots b_p a_{p+1} \dots a_m} \\
& \times \epsilon_{a_1 \dots a_p a_{p+1} \dots a_m} \sqrt{|g|} dx^1 \wedge \dots \wedge dx^m \\
&= \frac{1}{p!} \omega_{a_1 \dots a_p} \chi^{a_1 \dots a_p} \sqrt{|g|} dx^1 \wedge \dots \wedge dx^m .
\end{aligned}$$

From this expression we see that the product has the property of symmetry:

$$\omega \wedge *\chi = \chi \wedge *\omega .$$

Since $\omega \wedge *\chi$ is a top-form we can use it to integrate over M. Thus, we can define the following inner product of two p-forms:

$$\begin{aligned}
(\omega, \chi) :=& \int \omega \wedge *\chi \\
=& \frac{1}{p!} \int_M \omega_{a_1 \dots a_p} \chi^{a_1 \dots a_p} \sqrt{|g|} dx^1 \wedge \dots \wedge dx^m .
\end{aligned} \tag{5.1}$$

This inner product is symmetric, $(\omega, \chi) = (\chi, \omega)$ since $\omega \wedge *\chi = \chi \wedge *\omega$. It is also positive definite if (M, g) is Riemannian, $(\omega, \omega) \geq 0$, where the equality holds if and only if $\omega = 0$.

We are now finally ready to define the adjonit map of the exterior derivative.

Definition 32 ([3] Definition 7.6.)**.** Recall that $\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)$. The adjoint exterior derivative operator $\mathrm{d}^\dagger : \Omega^p(M) \to \Omega^{p-1}(M)$ is defined as

$$\mathrm{d}^\dagger = (-1)^{mp+m+1} * \mathrm{d} *$$

when (M, g) is Riemannian.

Now that we have looked at the operators d, $*$ and $\mathrm{d}^\dagger$ we can summarize all these operators in a single diagram as

$$\begin{tikzcd}
\Omega^p(M) \arrow[r, "\mathrm{d}^\dagger"] \arrow[d, "*"'] & \Omega^{p+1}(M) \\
\Omega^{m-p}(M) \arrow[r, "\pm \mathrm{d}"] & \Omega^{m-p+1}(M) \arrow[u, "*"']
\end{tikzcd} .$$

An important feature about $\mathrm{d}^\dagger$ is that it also squares to zero,

$$\mathrm{d}^{\dagger 2} = *\mathrm{d} * *\mathrm{d}* \propto *\mathrm{d}^2* = 0.$$

The adjoint also satisfies an important equation stated in the following theorem.

Theorem 3 ([3] Theorem 7.5)**.** *Letting (M, g) be a Riemannian manifold that is orientable and compact without boundary, and we have forms $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^{p-1}(M)$, then the following equation is satisfied*

$$(d\alpha, \beta) = (\alpha, d^\dagger \beta). \tag{5.2}$$

5.4 Laplacian and Hodge decomposition

(Based on [3] chap. 7.9)

We are finally ready to define the Laplacian.

Definition 33 ([3] definition 7.7.)**.** The *Laplacian* for differential forms on a Riemannian manifold M is defined as follows: $\Delta : \Omega^r(M) \to \Omega^r(M)$, where

$$\Delta = (\mathrm{d} + \mathrm{d}^\dagger)^2 = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}.$$

Example 21. Let's have a look at how the Laplacian acts on a smooth function, $f \in C^\infty(\mathbb{R})$. The Laplacian is then the following map: $\Delta : \Omega^0(\mathbb{R}) \to \Omega^0(\mathbb{R})$ on 0-forms. Since f is a scalar function we have that $\mathrm{d}\mathrm{d}^\dagger f = 0$ since there are no such things as (-1)-forms. We therefore get

$$\Delta f = \mathrm{d}^\dagger \mathrm{d} f = *\mathrm{d} * (\partial_a f dx^a).$$

Which after some further calculation we end up with

$$\Delta f = -\frac{1}{\sqrt{|g|}} \partial_b(\sqrt{|g|} g^{ba} \partial_a f).$$

However, since we are on $\mathbb{R}$, the metric is just 1. Hence we get the final familiar form of the Laplacian in 1-dimension

$$\Delta f = -\frac{\partial^2 f}{\partial x^2}.$$

Example 22 ([3] Example 5.11 and 7.16)**.** In electromagnetism one can show that we can express the electromagnetic vector potential as a one form, $A = A_a dx^a$. The electromagnetic field tensor is a two form and can be written in the following way, $F = \mathrm{d}A$, allowing us to rewrite Maxwell's equations in the following way: The two source-less equations $\nabla \cdot B = 0$ and $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ becomes the following equation in terms of differential forms;

$$\mathrm{d}F = \mathrm{d}^2 A = 0.$$

For the other two equations we first let ρ be the electric charge density and $\mathbf{j}$ the electric current density. Together ρ and $\mathbf{j}$ constructs the current one form $j = \eta_{ab} j^b dx^a = -\rho \mathrm{d}t + \mathbf{j} \cdot \mathrm{d}\mathbf{x}$. The two remaining equations with a source $\nabla \cdot \mathbf{E} = \rho$ and $\partial \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}$ can then be expressed as the following in terms of differential forms;

$$\mathrm{d}^\dagger F = \mathrm{d}^\dagger \mathrm{d}A = j.$$

We can also write it as $(\mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger \mathrm{d})A = \Delta A = j$. Because we can always choose an A that satisfies the *Lorentz condition* $\mathrm{d}^\dagger A = 0$ since the electromagnetic potential has a large gauge degree of freedom: $A \to A + \mathrm{d}\epsilon$. We will return to study such gauge theories and gauge symmetry later when we come to field theory.

Remember from the last chapter where we learned about cohomology, we defined $\mathrm{d}\omega = 0$ to be a closed form. In the same way we can now define a *coclosed* form as $\mathrm{d}^\dagger \omega = 0$. Finally a p-form is called harmonic if $\Delta \omega = 0$. Note that

$$\Delta \omega = \mathrm{d}^\dagger \mathrm{d}\omega + \mathrm{d}\mathrm{d}^\dagger \omega = 0$$

implies that $\mathrm{d}\omega = 0$ and $\mathrm{d}^\dagger \omega = 0$. This is because you can think of the components $\mathrm{d}^\dagger \mathrm{d}\omega$ and $\mathrm{d}\mathrm{d}^\dagger \omega$ as orthogonal vectors. So $\Delta \omega = 0$ this implies on a compact manifold

$$\begin{aligned} \mathrm{d}^\dagger \mathrm{d}\omega = 0 &\implies \mathrm{d}\omega = 0, \\ \mathrm{d}\mathrm{d}^\dagger \omega = 0 &\implies \mathrm{d}^\dagger \omega = 0. \end{aligned}$$

Hence a form is harmonic if and only if it is closed and coclosed. We will denote the set of harmonic p-forms as $\mathcal{H}^p$. In cohomology we were interested in constructing these cohomology groups where we had closed forms and factored out the exact forms. In a similar way a p-form is called *coexact* if it is written globally as;

$$\omega_p = \mathrm{d}^\dagger \alpha_{p+1}$$

where $\alpha_{p+1} \in \Omega^{p+1}(M)$. The set of exact forms we denote $\mathrm{d}\Omega^{p-1}(M)$ and coexact as $\mathrm{d}^\dagger \Omega^{p+1}(M)$. Now we are ready to introduce the Hodge decomposition theorem.

Theorem 4 (Hodge decomposition theorem, [3] Theorem 7.7)**.** *Let (M, g) be a Riemannian manifold that is orientable and compact, without boundary. Then p-forms, $\Omega^p(M)$, can be uniquely decomposed:*

$$\Omega^p(M) = \mathcal{H}^p \oplus d\Omega^{p-1}(M) \oplus d^\dagger \Omega^{p+1}(M)$$

I.e. any p-form ω_p can be written globally as

$$\omega_p = \gamma_p + d\alpha_{p-1} + d^\dagger \beta_{p+1}$$

where $\gamma_p \in \mathcal{H}^p(M)$, $\alpha_{p-1} \in \Omega^{p-1}(M)$ and $\beta_{p+1} \in \Omega^{p+1}(M)$. Note that $\mathcal{H}^p \cong H^p(M)$.

This theorem, combined with the Laplacian, will be the most essential takeaways from this chapter. They will both be heavily used in the final one-loop computation of a partition function.

Chapter 6

Physics tools

(This chapter is based on [3] chap. 1 and [4] chap. 1)

We have finally covered what we need for the final computation regarding mathematical knowledge and tools. This chapter will be devoted to some tools and concepts from physics that will play an important role in the next chapter. We start by giving a very rough introduction to classical/analytical mechanics and field theory, and we only focus on a few ideas and concepts that will be used later. The interested reader may consult e.g. [12]. After that, we will pay quantum field theory (QFT) a short visit and, more specifically, look at the path integral. This is a convenient computational technique used in QFT all the time. The partition function, which we will do some explicit computations of, is defined in terms of the path integral in QFT.

6.1 Lagrangian formalism

(Based on [3] chap. 1.1)

Field theory is a framework heavily used within physics that resolves many problems arising from ordinary Newtonian mechanics. Field theory is used in analytical mechanics, QFT, and particle physics, to mention a few branches of physics. For our purpose in this thesis, we will be interested in what is known as the action. Before we introduce the action, let us briefly recap some basic concepts from Newtonian mechanics.

We start by looking at a single particle. The particle has mass m, and we denote the particle simply as m. Then we want to describe the motion of m in three-dimensional space, let $x(t)$ denote the position of m at some time t. If the particle is being acted upon by some external force $F(x)$, then $x(t)$

will satisfy the following differential equation;

$$m\frac{d^2x}{dt^2} = F(x(t)).$$

which we know as Newton's second law or simply Newton's equation. In physics, this is an example of what is known as the *equation of motion*. Recall, that if the force $F(x)$ is conservative, we can express it in terms of a scalar function, $V(x)$, in the following way; $F(x) = -\nabla V(x)$. We call this scalar function, $V(x)$, the potential energy. Whenever the force F is conservative, then the total energy is conserved;

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x)$$

which is the sum of kinetic and potential energy.

Now that we have recalled some basic knowledge from Newtonian mechanics, let us jump into analytical mechanics. Two ways are heavily used and often denoted as analytical/classical mechanics. The first one is called the Lagrangian formalism, which is the most relevant formalism for our purpose; hence it is the only one we will discuss in detail. The other is called the Hamiltonian formalism and has quite a few similarities. However, the two differ in how to obtain the equation of motion for a system. Therefore it often depends on what problem you are looking at to determine which formalism is more convenient. For example, the Hamiltonian formalism is often used in particle physics and QFT. Nevertheless, let us get into the Lagrangian formalism. We need this because our final computation will rely on a theory expressed in terms of differential forms over a 6-dimensional manifold. This theory is called an action, expressed in terms of the Lagrangian.

The Lagrangian is quite similar to the sum of the energy of a system. The sum of energy and the Lagrangian only differ by a sign. The Lagrangian is defined as the kinetic energy minus the potential energy, $L = T - U$. This quantity is so fundamental in theoretical physics that if someone proposes some new physical theory, the first question one will get is: "What is the Lagrangian?". The Lagrangian formalism uses something known as generalized coordinates. Say that the state of our system (e.g. the positions of particles) is described by the coordinates $\{q_i : 1 \leq i \leq N\}$, where N is the total number of coordinates. The coordinates $\{q_i\}$ is whats known as the *generalized coordinates*, and they are elements of a manifold, $\{q_i\} \in M$. M is known as the *configuration space* for the generalized coordinates.

Example 23. An example of such a generalized coordinate is, for instance, if you imagine your particle moving along a circle, S^1, which will be our configuration space, then the generalized coordinate q is the angle θ.

Another example can be to take S^2 as your configuration space. The position of the particle moving on the surface can be completely determined by the two angles θ and ϕ, where θ and ϕ are the polar and azimuthal angles, respectively. Hence our generalised coordinates $\{q_i : i = 1, 2\}$ will be $q_1 = \theta$ and $q_2 = \phi$.

Notice that we do not need any radial coordinate r to describe the position since S^2 is only the shell of a sphere. Hence, we restrict the particle to move strictly on the surface, keeping r constant.

In terms of the generalized coordinate q_i one can define the generalized velocity as $\dot{q}_i = \frac{dq_i}{dt}$. The Lagrangian is a function of the generalized coordinate and generalized velocity, $L(q, \dot{q})$. Now we can introduce the action $\mathcal{S}$ which will lead us to a method of finding the equation of motion. Consider our particle moving in a one-dimensional space to keep it simple (but generalizing to higher dimensions is rather straight forward). The particle follows the trajectory $q(t)$ in the time span $t \in [t_i, t_f]$, with conditions $q(t_i) = q_i$ and $q(t_f) = q_f$. The *action* is then defined to be the following functional;

$$\mathcal{S}[q(t), \dot{q}(t)] = \int_{t_i}^{t_f} L(q, \dot{q}) dt. \tag{6.1}$$

The action $\mathcal{S}[q(t), \dot{q}(t)]$ is a functional that takes in some trajectory in terms of the generalized coordinates and velocity and returns a real number. It turns out that to obtain the equation of motion, one does not simply solve the integral. To find the equation of motion, we must use what is known as Hamilton's principle. *Hamilton's principle* also often referred to as the *principle of least action*, says that: "the physically realized trajectory corresponds to an extremum of the action"[3]. Hence the Lagrangian has to be chosen in such a way that it satisfies Hamilton's principle. The simplest way to deal with Hamilton's principle is in a local form as a differential equation. Now let $q(t)$ be some path that obtains an extremum of the action. To find an extremum in ordinary calculus, we usually find the derivative of our path/function and set this equal to zero. It is, in principle, the same thing we want to do here with the action functional. We start of by considering a small variation $\delta q(t)$ of the particles trajectory with $\delta q(t_i) = \delta q(t_f) = 0$. The action then takes the following form under the variation:

$$\begin{aligned} \delta\mathcal{S} &= \int_{t_i}^{t_f} L(q + \delta q, \dot{q} + \delta\dot{q}) dt - \int_{t_i}^{t_f} L(q, \dot{q}) dt. \\ &= \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q dt. \end{aligned} \tag{6.2}$$

Since we are dealing with an extremum this integral must vanish. This result of course holds for any variation δq hence the integrand must equal zero. We then obtain the following result;

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0. \tag{6.3}$$

This is known as the *Euler-Lagrange equation* and is one of the most important results of analytical mechanics. If one has N degrees of freedom, then the Euler-Lagrange equation takes the form;

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0, \ (1 \leq k \leq N). \tag{6.4}$$

These are the equations of motion of the system. Let's take a look at few familiar examples from Newtonian mechanics to see how this new formalism works.

Example 24. Let us see how we can obtain Newton's equation of motion using the Lagrangian formalism. Consider a particle with mass m moving around under a conservative force $F = -\nabla V$. Meaning that we can express the potential $V(q)$ as a scalar function for some generalized coordinate q. This particle also follows trajectories of ordinary mechanics, such that we can express the kinetic energy as $\frac{1}{2}m\dot{q}^2$. The Lagrangian then takes the form $L = \frac{1}{2}m\dot{q}^2 - V(q)$. Putting this into the Euler-Lagrange equation, it is simple to show that it reduces to the following

$$m\ddot{q}_k + \frac{\partial V}{\partial q_k} = 0. \tag{6.5}$$

Since we were dealing with a conservative force, we can rewrite this as $F = m\ddot{q}$. Hence we obtained the ordinary equation of motion from Newtonian mechanics.

Example 25. Another familiar example is, for instance, a mass m moving under a restoring force. We might take a block of mass m attached to a spring sliding on a friction-less table. Here we can choose the familiar Cartesian coordinates. Since this block will only move back and forth, it is just a one-dimensional motion. Hence we can choose our generalized coordinate to be x, the direction along the motion. The kinetic energy then just takes the familiar form $\frac{1}{2}m\dot{x}^2$. Since we are dealing with a restoring force $F = -kx$, we can express the the potential as $\frac{1}{2}kx^2$, giving us the Lagrangian $L = \frac{1}{2}m\ddot{x} - \frac{1}{2}kx^2$. Using the Euler-Lagrange, this reduces to

$$m\ddot{x} + kx = 0. \tag{6.6}$$

This is just the familiar equation of motion for a one-dimensional harmonic oscillator.

Remark. *It is interesting to see that the equation of motion comes from an extremum of the action. However, the action is not only defined for these trajectories but for any trajectory. In other words, the action is defined for any imaginable trajectory for a particle moving from A to B.*

The fact that the action is defined for any path will play a key role when discussing the integral path. The path integral is essentially an integral used in QFT where one integrates over all the possible paths the particle can take. As you might guess, some of these paths will be more probable than others. Hence, the answer is often called a probability amplitude. We will discuss the path integral in greater detail in the next section. Before we head over to QFT, we want to briefly discuss what kind of roles geometry and topology play in physics. We mostly argue a bit about why it might be of interest to study topological field theories, as we have been setting ourselves up for this entire thesis.

Geometry and Topology

In physics, one cares both about the geometry and the topology. Geometry is what gives you local information about the motion of the particles. For instance, how a particle is moving on the surface of the earth comes down to aspects regarding geometry. If the geometry is stretched, scaled, or twisted in any way, this changes the geometry and hence the physics. The changes to the geometry are devoted to the metric. One of the main theories in physics is Einstein's General Relativity (GR). GR is a theory about the geometry and curvature of spacetime. It describes gravity not as a force but as the curvature of spacetime. If you start with empty space (vacuum), the spacetime will be what one calls flat. Then if one imagines placing an object there, like a star, this will curve the space, which is what we feel as gravity. This is one of the many ways within physics where geometry and curvature are used to describe the physics.

When it comes to looking at topology within physics, it might seem like a somewhat counter-intuitive thing to study. Topology, of course, does not care about curvature and distance, meaning topological properties are left invariant under the stretching and deforming of your space (as long as you do not tear it). Therefore, studying topological field theories will not give you any local information, only global information. Hence we cannot talk about the exact paths a particle might trace out in a given space. Also, in almost every physics course at the undergraduate level and graduate level, one is looking at physics on spaces where the topology is trivial; therefore, there is not much point in studying topology in this context. However, there are a few theoretical theories within physics, such as String Theory, where topology becomes non-trivial. Therefore it makes sense to invest time into exploring quantities regarding the topology. Furthermore, if a theory has non-trivial topology, it can be challenging to study both the geometry and the topology simultaneously. Therefore, it is a more manageable task to study them separately. In a nutshell, when we are studying only the topological properties, we are doing what is known as topological field theory.

The thing one is usually most interested in when it comes to topological theories is finding and computing topological invariants. I.e. topological properties that are left invariant under continuous deformation or stretching. This can then contribute as a handy tool when trying to distinguish complicated mathematical objects. String Theory is one example of such a theory within theoretical physics which has many problems arising when trying to distinguish objects and theories from each other. Moreover, topological invariants might also help simplify the theory for certain calculations within the theory. Topological invariants are a massive field of study in mathematics, so-called enumerative geometry. It can be approached from different branches of mathematics like algebraic geometry and differential geometry, for example. Two examples are Knot theory and Donaldson-Thomas invariants [13],[14]. Our approach in this thesis is to use mathematical physics. We have covered all the tools needed from mathematics in previous chapters. Only a few more physics tools are needed to compute the topological invariants in the form of a partition function of a topological theory on a manifold. We have already looked at some topological invariants in the form of cohomology. Within topological invariants, there are two main types of invariants; Classical invariants and Quantum invariants. Classical invariants arise from boundary conditions or cohomology related to PDE's such as Maxwell's equations. Topological cohomology is, for instance, used to classify solutions of local geometric differential equations. Hence there is a relationship between topology and geometry where the classical invariants can help to find and count solutions to classical equations of motion. The other type, Quantum invariants, has been our main goal for this thesis. These topological invariants are computed by the partition function, and just as for the classical invariants, these types of invariants also give a relationship between topology and geometry. The topological quantum invariant is related to quantum geometry aspects of local QFT. That is, the quantum geometry of the moduli space of solutions.

6.2 Path integral and partition function

(Based on [4] chap. 1)

The Path integral is one of the most useful tools one encounters in QFT, and it is being used all the time. The Path integral also shows up in other branches of physics like particle physics. Although there does not exist a completely rigorous definition of the path integral, it is also used within a few branches of mathematics, but mainly within mathematical physics (like this thesis). In mathematical physics, one uses tools and computation techniques from various branches of physics to find results within formal mathematics. As mentioned, the path integral is used, for instance, to find topological invariants.

For those unfamiliar with the path integral, you can think of it as doing an integral and summing over all the possible paths a particle can take from a point A to a point B. The path integral was first used by Dirac, but the method was completed by Richard Feynman. He also developed what is known as Feynman diagrams which are perturbation techniques used to calculate more complicated path integrals with multiple interactions between different particles. We will not discuss these diagrams since the computation we are interested in is a one-loop computation of a partition function, for which the use of Feynman diagrams is unnecessary. The goal and reason for the path integral formalism are to generalize the action principle from classical mechanics.

Before we look at the path integral in more depth, we take a little story from Zee [4] chapter 1.2, giving an intuitive way of thinking of the path integral. The story takes place during a quantum mechanics class, and the Professor is teaching the students about the famous double-slit experiment, with no other than Feynman in the class as a student. The experiment goes in the following way: We have a source S from where our particle is emitted, at time $t = 0$. It then passes through either one of the two holes A_1 or A_2 drilled in a screen, and then the particle is detected by a detector located at D at time $t = T$. The superposition principle then gives the probability amplitude of detection at D. I.e. the amplitude of detecting the particle at D is given by the sum of the amplitude from the particle passing through A_1 and going to point D, and the amplitude of the particle going through A_2 and then going to point D.

After the Professor had introduced this idea, the student, Feynman, asked what would happen if one drilled a third hole into the screen. The Professor then replied with the answer that the amplitude would now be the sum of three amplitudes. I.e. the particle passing through either A_1, A_2 or A_3 and going to point D, and then sum over these three amplitudes. As the Professor was about to continue, Feynman asked again about what would happen if one drilled a fourth and fifth hole. Starting to get frustrated, the Professor answered that it then would obviously be the sum over all the holes.

In a slightly more rigorous way, we can denote the amplitude of the particle going from source S at time $t = 0$ passing through hole A_i then going to D as $\mathcal{A}(S \to A_i \to D)$. The amplitude of the particle being detected at D is given as

$$\mathcal{A}(\text{detected at } D) = \sum_i \mathcal{A}(S \to A_i \to D).$$

Feynman continued asking; now, he wondered what would happen if one put another screen behind the first one with some number of holes in it. The Professor started to get very irritated this time; he then rambled on that it would be the sum of all the paths; $\sum_{i,j} \mathcal{A}(S \to A_i \to B_j \to D)$. Nevertheless, Feynman continued, wondering what would happen if one put in a fourth and a fifth screen. And then he asked the rather deep question: "What if I put in a screen and drill an infinite number of holes in it so that the screen is no longer there?"[4]. The Professor then just ignored Feynman's question and continued with the class.

This is a really deep idea and very amusing, although the Professor didn't give this much attention. Feynman basically showed that if we would just have empty space between the source and the detector, the amplitude would still be a sum of paths over an infinite amount of screens, each with an infinite amount of holes. I.e. the amplitude will be the sum of all the possible paths the particle can take from the source to the detector.

$$\begin{aligned}&\mathcal{A}(\text{particle moving from } S \text{ to } D \text{ in time } T) = \\ &\quad \sum_{\text{(paths)}} \mathcal{A}(\text{Particle going from } S \text{ to } D \text{ following a specific path in time } T).\end{aligned}$$

In order to make this more rigorous, Feynman started following Newton and Leibniz. That is taking a path, and then we approximate it by straight line segments and then letting the segments go to zero. We see that this is basically the same as Feynman's argument of filling the empty space with an infinite amount of screens, each with an infinite amount of holes, and putting them infinitesimally close to each other. Although the path integral isn't completely rigorously defined, the idea of Feynman's thought process behind it intuitively makes sense.

Now that we have discussed the path integral more intuitively without any mathematical expressions let's construct the amplitude in a more mathematical way. Expressing the amplitude more mathematically is done using the unitarity of quantum mechanics. That is, if we know the amplitude of each of the infinitesimal segments, we can obtain the amplitude of the path by multiplying together the amplitudes of the segments. From quantum mechanics, we know that the amplitude for a particle propagating from a point q_I to a point q_F in a time T is given by the unitary operator e^{-iHT}, H here is the Hamiltonian. The Hamiltonian is defined as the kinetic energy plus the potential energy and usually takes the following form for a particle of mass m moving in a potential as

$$H = T + V = \frac{p^2}{2m} + V(q).$$

Here p is the generalized momentum, and q is our usual generalized coordinate. The corresponding familiar Hamiltonian operator used in quantum mechanics is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}).$$

Remark. *The hat on the Hamiltonian operator will be omitted from here on out, and later whenever we write Hamiltonian, we mean the operator.*

We will use the bra-ket notation developed by Dirac. The state of the particle at q we denote as a ket, $|q\rangle$, and the amplitude discussed above then becomes $\langle q_F|e^{-iHT}|q_I\rangle$. So the whole formalism of the path integral could have been written down mathematically, starting from only this quantity, just stating the amplitude as a postulate. In fact, this is how Dirac developed the path integral formalism before Feynman.

We will now follow Dirac's formulation to express the path integral mathematically. We start off by splitting the time T into N, giving us N time segments lasting a time $\delta t = T/N$. We then rewrite the amplitude in the following way

$$\langle q_F|e^{-iHT}|q_I\rangle = \langle q_F|e^{-iH\delta t}e^{-iH\delta t}\cdots e^{-iH\delta t}|q_I\rangle\,. \tag{6.7}$$

From quantum mechanics we know that $|q\rangle$ forms a complete set of of states, i.e. $\int dq\,|q\rangle\,\langle q| = 1$. In the previous equation, we insert 1's between all the factors of $e^{-iH\delta t}$ we then can rewrite it in the form

$$\begin{aligned}&\langle q_F|e^{-iHT}|q_I\rangle = \\ &(\Pi_{j=1}^{N-1}\int dq_j)\,\langle q_F|e^{-iH\delta t}|q_{N-1}\rangle\,\langle q_{N-1}|e^{-iH\delta t}|q_{N-2}\rangle\cdots \\ &\cdots\langle q_2|e^{-iH\delta t}|q_1\rangle\,\langle q_1|e^{-iH\delta t}|q_I\rangle\,.\end{aligned} \tag{6.8}$$

The result is as we see $N-1$ integrals from one state to the next. To be able to keep track of everything, we focus our attention on one of these individual factors $\langle q_{j+1}|e^{-iH\delta t}|q_j\rangle$. Starting with the case where the Hamiltonian takes the form $H = \hat{p}^2/2m$, i.e., the free particle case. The hat denotes that $\hat{p}$ is an operator. We then denote the eigenstate of $\hat{p}$ as $|p\rangle$ and p is just the eigenvalue, i.e. $\hat{p}p = p\,|p\rangle$. From quantum mechanics, we also have the following identity $\langle q|p\rangle = e^{ipq}$. Just like for q the momentum p is normalized such that $\int (dp/2\pi)\,|p\rangle\,\langle p| = 1$. We then do the same as above by inserting the complete set of states represented by 1's. The one factor we are focusing on then takes the form

$$\begin{aligned}\langle q_{j+1}|e^{-i\delta t(\hat{p}^2/2m)}|q_j\rangle &= \int \frac{dp}{2\pi}\,\langle q_{j+1}|e^{-i\delta t(\hat{p}^2/2m)}|p\rangle\,\langle p|q_j\rangle \\ &= \int \frac{dp}{2\pi} e^{-i\delta t(p^2/2m)}\,\langle q_{j+1}|p\rangle\,\langle p|q_j\rangle \\ &= \int \frac{dp}{2\pi} e^{-i\delta t(p^2/2m)} e^{ip(q_{j+1}-q_j)}.\end{aligned} \tag{6.9}$$

Here the hat has been removed from the momentum operator since it is acting upon one of its eigenstates and hence can be replaced by the eigenvalue p. This integral is known as a Gaussian integral. These types of integrals are often encountered in a course on quantum mechanics and in QFT. Doing the integral and putting it back into (6.8) with $q_I := q_0$ and $q_F := q_N$ we obtain

$$\langle q_F|e^{-iHT}|q_I\rangle = \left(\frac{-i2\pi m}{\delta t}\right)^{\frac{N}{2}} \Pi_{j=0}^{N-1} \int dq_j\, e^{i\delta t(m/2)\sum_{j=0}^{N-1}[(q_{j+1}-q_j)/\delta t]^2}. \tag{6.10}$$

Now we rewrite in a form that would follow Newton Leibniz, i.e. going to the continuum limit where $\delta t \to 0$. Replacing the $[(q_{j+1} - q_j)/\delta t]^2$ by the derivative $\dot{q}^2$, and the sum $\delta t \sum_{j=0}^{N-1}$ is replaced by an integral $\int_0^T dt$. Then it is common in QFT to rewrite the integral over all the paths as

$$\int \mathcal{D}q(t) = \lim_{N\to\infty} \left(\frac{-i2\pi m}{\delta t}\right)^{\frac{N}{2}} \Pi_{j=0}^{N-1} \int dq_j. \tag{6.11}$$

The path integral representation of the amplitude is then

$$\langle q_F|e^{-iHT}|q_I\rangle = \int \mathcal{D}q(t) e^{i\int_0^T dt \frac{1}{2}m\dot{q}^2}. \tag{6.12}$$

Hence we have a result showing us that if we want to calculate the amplitude for a particle, we have to integrate over all the possible paths $q(t)$ with $q(0) = q_I$ and $q(T) = q_F$. This was for the free particle case, but if we let the particle act under a potential, the Hamiltonian takes the form $H = \hat{p}^2/2m + V(\hat{q})$, the path integral then becomes

$$\langle q_F|e^{-iHT}|q_I\rangle = \int \mathcal{D}q(t)e^{i\int_0^T dt(\frac{1}{2}m\dot{q}^2 - V(q))}. \tag{6.13}$$

We recognize $\frac{1}{2}m\dot{q}^2 - V(q)$ as the Lagrangian $L(q, \dot{q})$ and the integral of the Lagrangian $\int_0^T dtL(q, \dot{q})$ is what we saw as the action $\mathcal{S}$. Hence the general representation of the path integral is

$$\langle q_F|e^{-iHT}|q_I\rangle = \int \mathcal{D}q(t)e^{i\mathcal{S}(q)}. \tag{6.14}$$

Here the path integral is represented in terms of it starting and ending at some initial and final position, q_I and q_F. One usually specifies the particle starting and ending in an initial state I, and ending in a final state F. The state which we are usually most interested in is the ground state; hence we will denote the initial and final state $|I\rangle$ and $|F\rangle$ as $|0\rangle$. Therefore, the amplitude we are interested in is $\langle 0|e^{-iHT}|0\rangle$, which is denoted just as $\mathcal{Z}$,

$$\mathcal{Z} := \langle 0|e^{-iHT}|0\rangle = \int \mathcal{D}q(t)e^{i\mathcal{S}(q)}. \tag{6.15}$$

This is what one usually refers to as a partition function. This version is sometimes denoted as the Minkowskian partition function. As mentioned, this path integral isn't perfectly rigorous in the mathematical sense. The main problem is that it intuitively spits out infinity as one is integrating over infinitely many paths. Therefore, we are counting on and assuming that the path integral converges due to some of the different paths canceling each other out and that some paths are more probable than others. In the cases where one still gets infinities, one can use mathematical tools like regularisation, which will be briefly discussed in the next chapter. To make the path integral more rigorous, it is common to do a so-called Wick rotation to Euclidean time. We slightly modify the derivation by doing a substitution $t \to -it$ and perform a rotation in the complex t plane. The integral then takes the form

$$\mathcal{Z} = \int \mathcal{D}q(t)e^{-\mathcal{S}_E(q)}. \tag{6.16}$$

This is called the *Euclidean Path integral*. The action here is the Euclidean action; $\mathcal{S}_E := \int dt(\frac{1}{2}m\dot{q}^2 + V(q))$. This path integral is only for a single particle. However, generalizing this to a path integral that holds for many particles is rather straightforward and will not be shown in-depth here. To change the path integral from being valid for a single particle to N particles, the Hamiltonian takes the form

$$H = \sum_a \frac{1}{2m_a}\hat{p}_a^2 + V(\hat{q}_1, \hat{q}_2, \dots, \hat{q}_N). \tag{6.17}$$

Then one basically goes through all the same steps that we just did above for a single particle and obtains a path integral that looks very much the same

$$\mathcal{Z} = \int \mathcal{D}q(t)e^{-\mathcal{S}(q)}. \tag{6.18}$$

However, now the action is no longer for a single particle, but for N particles. So the Euclidean action can be written in the following form

$$\mathcal{S} = \int_0^T dt\Big(\sum_a \frac{1}{2}m_a\dot{q}_a^2 + V(q_1, q_2, \dots, q_N)\Big). \tag{6.19}$$

The potential $V(q_1, q_2, \dots, q_N)$ contains energy caused by both the external potential and the energy caused by interactions between the individual particle. Now we have a description of the path integral in terms of many particles. However, we want a description in terms of fields. If we imagine each of the particles uniformly distributed in some space where each of the particles is a distance l from one another. We then go to the continuum limit by letting the distance $l \to 0$, creating what one refers to as a field in QFT. We are no longer talking about the individual particles but instead a sort of continuous jelly of some particle. Since we no longer have individual particles, the label a is replaced by a position vector $\mathbf{x}$; hence we write $q(t, \mathbf{x})$. When we talk about fields, it is common to substitute the generalized coordinates q with ϕ. A field is then the function $\phi(t, \mathbf{x})$. Then, to make the partition function in terms of fields and not particles, we substitute the sums in the action by integrals, $\sum_a \to \int d^D x$. And the individual positions of particles are replace by a field, $q_a \to \phi(x)$. This field can further be generalized from functions to differential forms, which will be the main type of field considered in this thesis. We end up with the final form of the Euclidean partition function;

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-\mathcal{S}(\phi)}. \tag{6.20}$$

This is the form of partition function we will look at in the final chapter. However, let us first take a look at an example with a partition function where the fields are differential forms.

Example 26. An example of such a partition function where our fields are differential forms can for instance be the following action of abelian Chern-

Simons theory [15]

$$\mathcal{S}(\alpha) = \int_{M_3} \alpha \wedge \mathrm{d}\alpha \tag{6.21}$$

where $\alpha \in \Omega^1(M_3)$. Here we can then think α as a field. The Partition function then takes the form

$$\mathcal{Z} = \int \mathcal{D}\alpha e^{-\mathcal{S}(\alpha)} \tag{6.22}$$

Since we are only doing a one-loop computation, we can evaluate the result using some useful integral results used in QFT and QM:

$$I_1 = \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \exp(-\sum_{ij} x^i S_{ij} x^j) \propto \frac{1}{\sqrt{\det(S)}}, \tag{6.23}$$

$$I_2 = \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \mathrm{d}y_1 \ldots \mathrm{d}y_n \exp(-\sum_{ij} x^i A_{ij} y^j) \propto \frac{1}{\det(A)}. \tag{6.24}$$

Then evaluating the path integral using I_1, we get

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\alpha e^{-\mathcal{S}(\alpha)} \\ &\propto \frac{\mathrm{Vol}(Z^1)}{\mathrm{Vol}(G)} \frac{1}{\det(\mathrm{d} : \Omega^1_{\mathrm{d}^\dagger} \to \Omega^2_{\mathrm{d}})^{\frac{1}{2}}}. \end{aligned} \tag{6.25}$$

Where $\Omega^1_{\mathrm{d}^\dagger}$ denotes the set of coexact 1-forms, and likewise Ω^2_{d} is the set of exact 2-forms. The $\mathrm{Vol}(Z^1)$ arises from

$$\begin{aligned} \int \mathcal{D}\alpha e^{-\mathcal{S}(\alpha)} &= \int_{Z^1} \mathcal{D}\alpha \int_{\Omega^1_{\mathrm{d}^\dagger}} \mathcal{D}\alpha e^{-\mathcal{S}(\alpha)} \\ &= \mathrm{Vol}(Z^1) \int_{\Omega^1_{\mathrm{d}^\dagger}} \mathcal{D}\alpha e^{-\mathcal{S}(\alpha)}, \end{aligned}$$

as $\mathcal{S}(\alpha)$ is independent of the closed part of α. You can think of $\mathrm{Vol}(Z^1)$ as the volume of the space of all the closed 1-forms sitting on the manifold. This is, in fact, something that is not only infinite but also infinite-dimensional. This is why we have this $\mathrm{Vol}(G)$ that we use to cancel out these other volumes. $\mathrm{Vol}(G)$ can be thought of as the "volume of all gauge-symmetries"(see below). We will see below how to rewrite (6.25) in terms of determinants of Laplacians.

Similar partition functions to this is what we will see in the last chapter.

Remark. *The "proportional to" symbol will be substituted with an equality sign when doing the one-loop computations of the partition functions later. An overall scaling of the partition function is irrelevant.*

6.3 Gauge symmetry

(Based on [3] chap. 1.8)

Gauge theory has to do with geometry and symmetry. In today's theoretical physics, basically, every branch is a gauge theory. EM, QM and QFT are some examples of gauge theories, and even GR can be described as a gauge theory. One of the main principles within gauge theory is that the "physics should not depend on how one chooses to describe it"[3]. For instance, choosing a different set of coordinates or different reference frames are things that should not affect the physics in any way.

In EM, for example, we could describe the electromagnetic field tensor F as $F = \mathrm{d}A$, where $A \in \Omega^1$ is the electromagnetic potential. However, we are free to do a so-called gauge transformation. Instead of choosing the electromagnetic potential to be A, we could do the following gauge symmetry transformation:

$$A \to A + \mathrm{d}\phi \tag{6.26}$$

Where $\phi \in \Omega^0$. We see that this other potential leaves the electromagnetic field tensor invariant;

$$F = \mathrm{d}(A + \mathrm{d}\phi) = \mathrm{d}A + \mathrm{d}^2\phi = \mathrm{d}A. \tag{6.27}$$

As the physics (i.e. Maxwell's equations) is given in terms of F, we see that (6.26) will not affect the physics, i.e. it is a gauge-symmetry. Gauge theories are, in principle, field theories where the forces that arise in the theories are related to such symmetries. Like in particle physics, one describes the forces that are acting between the particles as fields. If a theory is invariant under some gauge transformation of the fields, we say that it is *gauge invariant.*

Example 27. Let's have a look at a Topological field theory that will be gauge invariant. We take M_4 to be a $4-$dim manifold that is compact and orientable we have $\alpha \in \Omega^1$ and $\chi \in \Omega^2$ as our fields. We can then have a look at the following theory

$$\mathcal{S} = \int_{M_4} \alpha \wedge \mathrm{d}\chi. \tag{6.28}$$

We notice that here we can do a gauge transformation of χ that will leave $\mathcal{S}$ invariant. We can perform the following gauge transformation

$$\chi \to \chi + \mathrm{d}\kappa \tag{6.29}$$

where $\kappa \in \Omega^1$. Here we also have a gauge of gauge transformation, where

$$\kappa \to \kappa + \mathrm{d}\psi, \tag{6.30}$$

with $\psi \in \Omega^0$. Hence when looking at the theory, we don't have to require the field χ to be exactly the same over the manifold, but it might transform according to the transformations above.

In example 26 and 27 the field transformations of α that preserves the action, or gauge-transformations are

$$Z^1 = \mathcal{H}^1 \oplus \mathrm{d}\Omega^0.$$

$\mathrm{d}\Omega^0$ is the set of all globally exact 1-forms, while $\mathcal{H}^1$ are only locally exact (by Poincaré lemma). In the local field theory, we usually only take the globally exact forms $\mathrm{d}\Omega^0$ as our actual gauge transformations, while $\mathcal{H}^1$ is more related to global topology, equation of motion, and classical invariants.

Chapter 7

Topological field theory

We have now spent almost the entire thesis on developing a framework that makes us able to do an explicit one-loop computation of a partition function. The final ingredient is Zeta-regularised determinants. In mathematics and physics, one will sometimes run into infinities that one needs to deal with. In mathematics, for instance, one might wish to study ill-defined or divergent series that give infinities. Or in physics, for instance, in QFT, one runs into infinities when evaluating path integrals. One often takes advantage of the famous Riemann Zeta function to handle these infinities. Our answer to the partition function will be in terms of determinants of the Laplacian operator, which will be equivalent to an infinite product of the eigenvalues. I.e. it will diverge; hence one will have to use some type of regularization, and that's why one chooses to define the zeta regularized determinants. Now we will not do the explicit regularization for the final theory to get an explicit number out since that could have been a thesis on its own. Instead, we will only discuss regularization and show a simple example.

The theory we will look at will be over a six-dimensional manifold. This computation is inspired by String Theory. For String Theory to work, we need ten dimensions. Therefore in String Theory, we decompose the ten dimensions into four-dimensional spacetime and compactify the last six dimensions to a so-called Calabi-Yau manifold. So the partition function we will be looking at over a 6-dimensional manifold will be related to these six dimensions that one meets in String Theory.

Before we continue, we want to briefly mention more specifically what it means for a field theory to be topological. The theories we will look at will be in the form

$$\mathcal{S} = \int_X \alpha \wedge \mathrm{d}\beta,$$

where α and β are differential forms representing our fields. We see that this action is completely independent of any metric. For a topological field theory, all quantities extracted from the action should be topological and independent of the metric. In chap. 5 we discussed that we need a metric for technical reasons in the computation of the partition function. A metric is also needed to define the adjoint to the exterior derivative: $\mathrm{d}^\dagger$, which appears in the partition function. Since our theory is topological, it is independent of the metric we choose, but we need to choose one. This is what topological means in our context.

If we however take a look at the Einstein-Hilbert action [16] in GR:

$$\mathcal{S} = \int dx^4 \sqrt{-g}\mathbf{R},$$

we see that this action depends heavily on the metric. As mentioned in chap. 5, GR is a theory about the geometry and curvature of spacetime, so the metric plays the primary role in the theory.

We also mention that the computations in this chapter can be formalised through the BV-BRST formalism [17], [18]. This level of rigor is, however, not required for the one-loop computations in this thesis and might even serve to obscure the results.

7.1 Zeta regularised determinants

(Based on [19] Appendix B)

Here we will briefly discuss the zeta-regularized determinants just to get a glimpse at how this type of regularization works. In the computations of partition functions that will follow later in the chapter, we will get answers in terms of the determinants of the Laplacian. This will be equivalent to an infinite product of the eigenvalues. I.e. the answer will be infinite. However, using zeta-regularisation, one can extract meaningful finite answers from these infinite products. We begin by looking at the Riemann Zeta function.

The Riemann Zeta function is one of the most famous functions in all of mathematics and is host to many problems that are still unsolved today. The most well-known being the millennium problem of the Riemann Hypothesis. The Riemann Zeta function, also sometimes denoted as the Euler-Riemann Zeta function, is defined as follows. For an increasing sequence $A = \{a_1, a_2, \dots\}$ of positive real numbers, the Riemann Zeta function is defined as

$$\zeta_A(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s} = \frac{1}{a_1^s} + \frac{1}{a_2^s} + \frac{1}{a_3^s} + \dots \tag{7.1}$$

where s is a complex variable. It takes this form when $\mathrm{Re}(s) >> 1$, but by analytic continuation, it can be extended to being analytic over the entire complex plane $\mathbb{C}$. This function is important both within mathematics, physics, and even statistics. If one lets $s = 1$ and the sequence to be $B = \{1, 2, 3, \dots\}$, then this is the famous harmonic series that is well known for being divergent. For real numbers much larger than 1 the series will converge to some finite number. If one lets the sequence be equal to the natural numbers, $B = \{1, 2, 3, \dots\} = \mathbb{N}$ and choose $s = -1$ we get the following divergent form of the zeta function $\zeta_B(-1) = 1 + 2 + 3 + \dots$, which have the famous result $\zeta(-1)_{\mathbb{N}} = -\frac{1}{12}$ for the analytically continued function. The zeta-regularised product of a sequence A is defined as

$$\exp(-\zeta_A'(0)), \tag{7.2}$$

where the prime here denotes the derivative w.r.t. the complex variable s. It is easy to see that this becomes the products of the elements the sequence A when A is finite.

If we have a vector space V and some operator $O : V \to V$ with non-negative discrete real eigenvalues only, then one can define the zeta-regularised determinant as:

$$\tilde{\det} O := \exp(-\zeta_A'(0)). \tag{7.3}$$

Here the tilde is just to show that we are working with the regularised determinant, and A is the sequence of eigenvalues. A useful identity when it comes to computations is that if $P : V \to W$ where P is an operator, we have formally that

$$|\det P| := (\det P^\dagger P)^{\frac{1}{2}}. \tag{7.4}$$

Omitting the zero eigenvalues, we also have the following formal identity

$$\det P^\dagger P = \det P P^\dagger. \tag{7.5}$$

Here $P^\dagger : W \to V$ is the adjoint of P assuming a suitable inner-product. If we have two operators $P, Q : V \to W$ that satisfy $PQ = QP = 0$, one can show formally that

$$\det(P + Q) = \det(P) \det(Q) \tag{7.6}$$

These formal results will come in handy when we look at the determinants of the Laplacian on differential forms.

Recall that the Laplacian was defined as $\Delta = \mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger$. We have mentioned earlier that our result of the partition function will be in terms of determinants of the Laplacian. The reason to write the result in terms of the

Laplacian is that it is positive (semi-)definite and elliptic [20]. This means it has a discrete set of non-negative eigenvalues on a compact manifold, with a finite-dimensional eigenspace at each level. Hence we can regularize its determinant. We will ignore the zero-eigenvalues when computing $\det(\Delta)$. These are related to harmonic forms, cohomologies, and classical invariants. Recall, from Hodge decomposition, we have that

$$\Omega^p = \mathcal{H}^p \oplus \mathrm{d}\Omega^{p-1} \oplus \mathrm{d}^\dagger \Omega^{p+1}. \tag{7.7}$$

The harmonic forms $\mathcal{H}^p$ is the zero eigenspaces of the Laplacian, so these can be omitted when we compute the regularized determinant of the Laplacian, $\tilde{\det}(\Delta)$. We formally write it in in the following way, using (7.6)

$$\det(\Delta) = \det(\mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger) = \det(\mathrm{d}^\dagger \mathrm{d})\det(\mathrm{d}\mathrm{d}^\dagger). \tag{7.8}$$

Another useful identity we get by using (7.5) is

$$\det(\mathrm{d}\mathrm{d}^\dagger|_p) = \det(\mathrm{d}^\dagger \mathrm{d}|_{p-1}). \tag{7.9}$$

The final identity comes from that the Laplacian commutes with the Hodge star,

$$\tilde{\det}(\Delta^p) = \tilde{\det}(\Delta^{n-p}), \tag{7.10}$$

which is true both formally, and for the regularised Laplacian. These identities and results will come in handy when we do the computation of the partition function later.

Doing an explicit regularisation of the computations in the upcoming chapter could, as mentioned, be a thesis on its own. The reason for this section has been to discuss a possible way of obtaining finite results from the computations of the partition functions in upcoming sections. The following is a simple example of regularisation just to get a sense of how it is done.

Example 28. To get a feel for an explicit computation of zeta-regularized determinants, we take a look at one dimensional manifold equal to the line segment from 0 to π, where we identify 0 with π, giving the circle S^1. In one dimension the Laplacian is given as $\Delta = -\frac{\partial^2}{\partial x^2}$. The eigen-modes between 0 and π can be represented as sine functions, $f_n(x) = \sin(nx)$, where $n \in \mathbb{N}$.

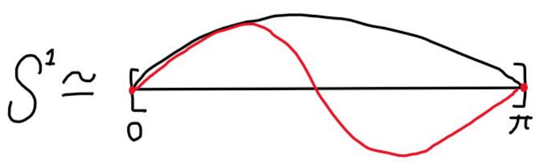

Figure 7.1: Visualisation of the line with modes represented as sine functions. Identifying the 0 with π gives an isomorphism between the line segment and S^1.

Acting with the Laplacian gives

$$\Delta \sin(nx) = n^2 \sin(nx). \tag{7.11}$$

Hence the eigenvalues are $\lambda_n = n^2$. So the determinant of the Laplacian is

$$|\Delta| = \Pi_{n=1}^{\infty} n^2 = (\Pi_{n=1}^{\infty} n)^2. \tag{7.12}$$

Where $|\Delta|$ is just a short hand notation for the determinant. We denote $|D| = \sqrt{|\Delta|} = \Pi_{n=1}^{\infty} n$. Here D is the "square-root" of Δ, i.e. the Dirac operator. Then we have that

$$\zeta_D(s) = \zeta_B(s), \tag{7.13}$$

where B is the sequence above of the natural numbers, $B = \{1, 2, 3, \dots\}$. The regularised determinant of D is then

$$|\tilde{D}| = \exp(-\zeta_B'(0)). \tag{7.14}$$

The derivative of the zeta function evaluated at zero is one of the results that has been explicitly computed for the zeta function and it is [21]

$$\zeta_B'(0) = -\frac{1}{2}\log(2\pi). \tag{7.15}$$

Hence we get the following result for the regularized determinant of D

$$|\tilde{D}| = \sqrt{2\pi}. \tag{7.16}$$

So the regularised determinant of the Laplacian on S^1 is

$$|\tilde{\Delta}| = 2\pi. \tag{7.17}$$

7.2 One-loop computation of Partition function

We have finally got all the tools we need to compute quantum invariants in terms of one-loop partition functions. Let's start off by considering the following toy-model example before moving to the main theory.

Remark. *In the computation below there will arise volumes of harmonic and exact forms related to different differential forms which will be treated as the same volume, e.g. the action $\mathcal{S} = \int_{M_3} \alpha \wedge d\beta$ will give rise to volumes of harmonic 1-forms $Vol(\mathcal{H}^1_\alpha)$ and $Vol(\mathcal{H}^1_\beta)$, for which we will think of as equal, thus writing $Vol(\mathcal{H}^1_\alpha)\,Vol(\mathcal{H}^1_\beta) = Vol(\mathcal{H}^1)^2$. The same holds for the other volumes. These subtleties can be ignored as the volumes will be canceled out by the volume of the overall gauge group in the end.*

Example 29. Let's have a look at the following topological field theory

$$\mathcal{S} = \int_{M_3} \alpha \wedge \mathrm{d}\beta, \tag{7.18}$$

for $\alpha, \beta \in \Omega^1(M_3)$. Recall our useful results from QFT (6.23) and (6.24). Here we will invoke (6.24)

$$I_2 = \int \mathrm{d}x_1 \dots \mathrm{d}x_n \mathrm{d}y_1 \dots \mathrm{d}y_n \exp(-\sum_{ij} x^i A_{ij} y^j) \propto \frac{1}{\det(A)}.$$

For our 1-forms α and β we have the following gauge-transformations leaving $\mathcal{S}$ invariant; $\alpha \to \alpha + \kappa_1$ where $\mathrm{d}\kappa_1 = 0$, and $\beta \to \beta + \kappa_2$, where $\mathrm{d}\kappa_2 = 0$.

From Hodge decomposition theorem 4 we have that the set of 1-forms can be decomposed into: $\Omega^1 = \mathcal{H}^1 \oplus \mathrm{d}\Omega^0 \oplus \mathrm{d}^\dagger\Omega^2$. Here $\mathrm{d}\Omega^0 = B^1(M_3)$, the exact forms. The closed forms is then, $Z^1(M_3) = \mathcal{H}^1 \oplus \mathrm{d}\Omega^0$.

$$\begin{aligned} \mathcal{Z} &= \frac{1}{\mathrm{Vol}(G)} \int \mathcal{D}\alpha\mathcal{D}\beta \exp(-\mathcal{S}) \\ &= \frac{\mathrm{Vol}(Z^1)\mathrm{Vol}(Z^1)}{\mathrm{Vol}(G)} \int \mathcal{D}\alpha\mathcal{D}\beta \exp(-\mathcal{S}) \\ &= \frac{\mathrm{Vol}(Z^1)^2}{\mathrm{Vol}(G)} \frac{1}{\det(\mathrm{d} : \Omega^1_{\mathrm{d}^\dagger} \to \Omega^2_{\mathrm{d}})} \end{aligned} \tag{7.19}$$

$$\det(\mathrm{d} : \Omega^1_{\mathrm{d}^\dagger} \to \Omega^2_{\mathrm{d}}) := \det(\mathrm{d}^\dagger\mathrm{d} : \Omega^1_{\mathrm{d}^\dagger} \to \Omega^2_{\mathrm{d}^\dagger})^{\frac{1}{2}} = |(\mathrm{d}^\dagger\mathrm{d})^1_{\mathrm{d}^\dagger}|^{\frac{1}{2}}. \tag{7.20}$$

For the shorthand notation the notation we will drop the subscript, $|(\mathrm{d}^\dagger\mathrm{d})^1|^{\frac{1}{2}}$. Here $(\mathrm{d}^\dagger\mathrm{d})^n$ is the operator $\mathrm{d}^\dagger\mathrm{d}$ on n-forms, not to the n'th power. The partition function we can be written as:

$$\mathcal{Z} = \frac{\mathrm{Vol}(Z^1)^2}{\mathrm{Vol}(G)} \frac{1}{|(\mathrm{d}^\dagger\mathrm{d})^1|^{\frac{1}{2}}}. \tag{7.21}$$

Recall that the Laplace operator was defined as $\Delta : \Omega^p(M) \to \Omega^p(M)$, where $\Delta := \mathrm{d}^\dagger\mathrm{d} + \mathrm{d}\mathrm{d}^\dagger$.

Note that if $\Delta\alpha = 0 \implies \mathrm{d}^\dagger\mathrm{d}\alpha + \mathrm{d}\mathrm{d}^\dagger\alpha = 0$. This means that α is harmonic, but if you struggle to see why you can think of the terms as orthogonal vectors. Hence $\mathrm{d}^\dagger\mathrm{d}\alpha = 0$ and $\mathrm{d}\mathrm{d}^\dagger\alpha = 0$. So we have that

$$\begin{aligned} 0 = (\alpha, \mathrm{d}^\dagger\mathrm{d}\alpha) = (\mathrm{d}\alpha, \mathrm{d}\alpha) = ||\mathrm{d}\alpha||^2 = 0 \implies \mathrm{d}\alpha = 0 \\ 0 = (\alpha, \mathrm{d}\mathrm{d}^\dagger\alpha) = (\mathrm{d}^\dagger\alpha, \mathrm{d}^\dagger\alpha) = ||\mathrm{d}^\dagger\alpha||^2 = 0 \implies \mathrm{d}^\dagger\alpha = 0 \end{aligned} \tag{7.22}$$

The determinant of the Laplace operator we can write as:

$$|\Delta| = |\mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger| = |\mathrm{d}^\dagger \mathrm{d}||\mathrm{d}\mathrm{d}^\dagger|$$

since the operators are orthogonal.

The partition function then becomes:

$$\mathcal{Z} = \frac{\mathrm{Vol}(Z^1)^2}{\mathrm{Vol}(G)} \frac{1}{|(\mathrm{d}^\dagger \mathrm{d})^1|^{\frac{1}{2}}} = \frac{\mathrm{Vol}(Z^1)^2}{\mathrm{Vol}(G)} \frac{|(\mathrm{d}\mathrm{d}^\dagger)^1|^{\frac{1}{2}}}{|(\mathrm{d}^\dagger \mathrm{d})^1|^{\frac{1}{2}} |(\mathrm{d}\mathrm{d}^\dagger)^1|^{\frac{1}{2}}} = \frac{\mathrm{Vol}(Z^1)^2}{\mathrm{Vol}(G)} \frac{|(\mathrm{d}\mathrm{d}^\dagger)^1|^{\frac{1}{2}}}{|\Delta^1|^{\frac{1}{2}}} \tag{7.23}$$

Now let's try to rewrite $|(\mathrm{d}\mathrm{d}^\dagger)^1|$. We have that $\mathrm{d}\mathrm{d}^\dagger \alpha = \lambda \alpha$, for $\mathrm{d}\alpha \neq 0$, where λ is the eigenvalues. Now this is iff $\mathrm{d}^\dagger \mathrm{d}\mathrm{d}^\dagger \alpha = \lambda \mathrm{d}^\dagger \alpha$, we let $\gamma = \mathrm{d}^\dagger \alpha$, hence $\gamma \in \Omega^0(M)$. We then have that $\mathrm{d}^\dagger \mathrm{d}\gamma = \lambda\gamma$, for $\mathrm{d}^\dagger \gamma = 0$. Which means that we can write $(\mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger)\gamma = \lambda\gamma$, but $(\mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger)$ in this case is just Δ^0. So $\Delta^0 \gamma = \lambda\gamma$. Hence the Partition function becomes:

$$\mathcal{Z} = \frac{\mathrm{Vol}(Z^1)^2}{\mathrm{Vol}(G)} \frac{|(\mathrm{d}\mathrm{d}^\dagger)^1|^{\frac{1}{2}}}{|\Delta^1|^{\frac{1}{2}}} = \frac{\mathrm{Vol}(Z^1)^2}{\mathrm{Vol}(G)} \frac{|\Delta^0|^{\frac{1}{2}}}{|\Delta^1|^{\frac{1}{2}}} \tag{7.24}$$

Let us work out $\mathrm{Vol}(Z^1)$. From Hodge decomposition we know that $Z^1 = \mathcal{H}^1 \oplus \mathrm{d}\Omega^0$, so $\mathrm{Vol}(Z^1) = \mathrm{Vol}(\mathcal{H}^1) \oplus \mathrm{Vol}(\mathrm{d}\Omega^0)$. We then have a general result for the volume of an operator O acting on a space A.

$$\mathrm{Vol}(OA) = \det(O) \frac{\mathrm{Vol}(A)}{\mathrm{Vol}(Ker(O))} \tag{7.25}$$

So $\mathrm{Vol}(\mathrm{d}\Omega^0)$ becomes

$$\mathrm{Vol}(\mathrm{d}\Omega^0) = \det(\mathrm{d} : \Omega^0 \to \Omega^1) \frac{\mathrm{Vol}(\Omega^0)}{\mathrm{Vol}(\mathcal{H}^0)} = |(\mathrm{d}^\dagger \mathrm{d})^0|^{\frac{1}{2}} \frac{\mathrm{Vol}(\Omega^0)}{\mathrm{Vol}(\mathcal{H}^0)} = |\Delta^0|^{\frac{1}{2}} \frac{\mathrm{Vol}(\Omega^0)}{\mathrm{Vol}(\mathcal{H}^0)} \tag{7.26}$$

Continuing with the computation of the partition function;

$$\begin{aligned}
\mathcal{Z} &= \frac{\mathrm{Vol}(Z^1)^2}{\mathrm{Vol}(G)} \frac{|\Delta^0|^{\frac{1}{2}}}{|\Delta^1|^{\frac{1}{2}}} \\
&= \frac{\mathrm{Vol}(\mathcal{H}^1)^2 \mathrm{Vol}(\mathrm{d}\Omega^0)^2}{\mathrm{Vol}(G)} \frac{|\Delta^0|^{\frac{1}{2}}}{|\Delta^1|^{\frac{1}{2}}} \\
&= \frac{1}{\mathrm{Vol}(G)} \frac{|\Delta^0|^{\frac{1}{2}}}{|\Delta^1|^{\frac{1}{2}}} \frac{(|\Delta^0|^{\frac{1}{2}})^2 \mathrm{Vol}(\mathcal{H}^1)^2 \mathrm{Vol}(\Omega^0)^2}{\mathrm{Vol}(\mathcal{H}^0)^2} \\
&= \frac{1}{\mathrm{Vol}(G)} \frac{|\Delta^0|^{\frac{3}{2}}}{|\Delta^1|^{\frac{1}{2}}} \frac{\mathrm{Vol}(\mathcal{H}^1)^2 \mathrm{Vol}(\Omega^0)^2}{\mathrm{Vol}(\mathcal{H}^0)^2}.
\end{aligned} \tag{7.27}$$

We have arrived at the final result of the partition function computation. In the result, we do not care too much about all these volumes. These are, in practice, usually factored out by the volume of the overall gauge group, G. The interesting components are these determinants of the Laplacian. The result we obtained here is a well known result and is often denoted as $T_{RS} = \frac{|\Delta^0|^{\frac{3}{2}}}{|\Delta^1|^{\frac{1}{2}}}$, called the Ray-Singer Torsion [22]. The result is still formal and infinite, but written in terms of elliptic Laplacians, it can be regularised to a finite number using the methods described above. As we are studying topological field theories and topological invariants, let us show that this result is topological.

To show that it is topological we perform a small variation of T_{RS} w.r.t. the metric. We remember that $\Delta^p : \Omega^p \to \Omega^p$, we then have a result that a small variation w.r.t. the metric of $\log(|\Delta^p|)$ is proportional to $\dim(R_p)$, where R_p is the point-wise vector space of p-form on an m-dimensional manifold. I.e. $\delta_g \log|\Delta^p| \propto \dim(R_p)$. See equation (8.5) in [23].

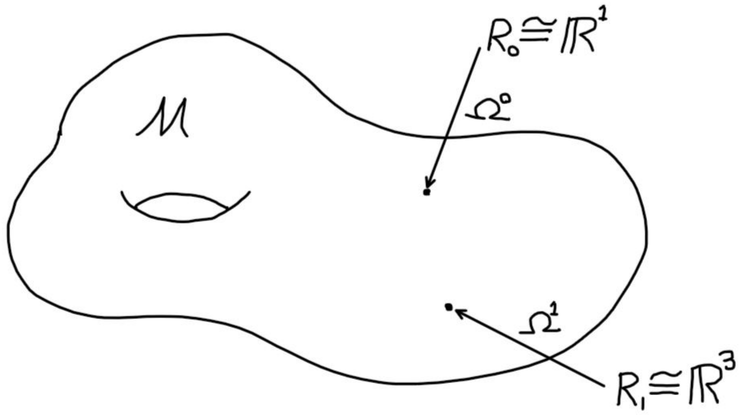

Figure 7.2: Point-wise vector space of 0- and 1-forms on a 3-manifold M

The dimension of R_0, which is the point-wise vector space of 0-forms on our 3-manifold, is $\dim(R_0) = \binom{3}{0} = 1$, and the dimension of R_1 is; $\dim(R_0) = \binom{3}{1} = 3$. So then we get:

$$\begin{aligned}\delta \log(T_{RS}) &\propto \frac{3}{2}\dim(R_0) - \frac{1}{2}\dim(R_1) \\ &= \frac{3}{2}\dim(\mathbb{R}) - \frac{1}{2}\dim(\mathbb{R}^3) \\ &= \frac{3}{2}\times 1 - \frac{1}{2}\times 3 = 0\end{aligned} \tag{7.28}$$

Hence $\delta \log(T_{RS}) = 0$ which means that T_{RS} is topological! It thus defines a quantum invariant for M_3.

7.3 Final theory

For our final computation we want to have a look at the following action inspired by String theory:

$$\mathcal{S} = \int_{M_6} \beta \wedge \mathrm{d}\gamma \tag{7.29}$$

where $\beta \in \Omega^2(M_6)$ and $\gamma \in \Omega^3(M_6)$. We quickly observe that $\Omega^2 \ncong \Omega^3$, therefore we instead look at the following theory:

$$\mathcal{S} = \frac{1}{2}\int_{M_6} (\beta + \gamma) \wedge \mathrm{d}(\beta + \gamma), \tag{7.30}$$

which when we expand it out gives us;

$$\begin{aligned}\mathcal{S} = &\frac{1}{2}\int_{M_6} \beta \wedge \mathrm{d}\beta + \frac{1}{2}\int_{M_6} \gamma \wedge \mathrm{d}\gamma \\ &+ \frac{1}{2}\int_{M_6} \beta \wedge \mathrm{d}\gamma + \frac{1}{2}\int_{M_6} \gamma \wedge \mathrm{d}\beta.\end{aligned} \tag{7.31}$$

We first observe that the first integral is the integral of a 2- wedge 3-form, hence a 5-form over a 6-dimensional manifold, which is not a top-form; hence this integral gives us zero. The same holds for the second integral. However, here we have a 7-form over a 6-dim manifold, giving us zero. We are therefore only left with the last two integrals

$$\mathcal{S} = \frac{1}{2}\int_{M_6} \beta \wedge \mathrm{d}\gamma + \frac{1}{2}\int_{M_6} \gamma \wedge \mathrm{d}\beta \tag{7.32}$$

Applying Stokes theorem and integration by parts for the second integral we get;

$$\begin{aligned}\mathcal{S} &= \frac{1}{2}\int_{M_6} \beta \wedge \mathrm{d}\gamma + \frac{1}{2}\left(\gamma \wedge \beta|_{\partial M_6} - \int_{M_6} \mathrm{d}\gamma \wedge \beta\right) \\ &= \frac{1}{2}\int_{M_6} \beta \wedge \mathrm{d}\gamma - \frac{1}{2}\int_{M_6} \mathrm{d}\gamma \wedge \beta \\ &= \int_{M_6} \beta \wedge \mathrm{d}\gamma\end{aligned} \tag{7.33}$$

We are left with the original theory, but now we are set to take advantage of (6.23). So we contemplate the action

$$\mathcal{S} = \frac{1}{2}\int_{M_6} (\beta+\gamma)\wedge \mathrm{d}(\beta+\gamma),$$

for the (2+3)−form $\beta+\gamma$. This then gives us the following partition function:

$$\mathcal{Z} = \frac{1}{\mathrm{Vol}(G)}\int \mathcal{D}\beta\mathcal{D}\gamma \exp\left(-\mathcal{S}\right) \tag{7.34}$$

Invoking (6.23) the partition function results in,

$$\begin{aligned}\mathcal{Z} &= \frac{\mathrm{Vol}(Z^3)\mathrm{Vol}(Z^2)}{\mathrm{Vol}(G)}\frac{1}{\det(\mathrm{d}:\Omega^{2+3}\to\Omega^{3+4})^{\frac{1}{2}}}\\ &= \frac{\mathrm{Vol}(Z^3)\mathrm{Vol}(Z^2)}{\mathrm{Vol}(G)}\frac{1}{|(\mathrm{d}^\dagger\mathrm{d})^2|^{\frac{1}{4}}|(\mathrm{d}^\dagger\mathrm{d})^3|^{\frac{1}{4}}}.\end{aligned} \tag{7.35}$$

Then using Hodge decomposition to expand out Z^2 and Z^3:

$$\mathcal{Z} = \frac{\mathrm{Vol}(\mathcal{H}^3)\mathrm{Vol}(\mathrm{d}\Omega^2)\mathrm{Vol}(\mathcal{H}^2)\mathrm{Vol}(\mathrm{d}\Omega^1)}{\mathrm{Vol}(G)}\frac{1}{|(\mathrm{d}^\dagger\mathrm{d})^2|^{\frac{1}{4}}|(\mathrm{d}^\dagger\mathrm{d})^3|^{\frac{1}{4}}} \tag{7.36}$$

Let's now have a look at $\mathrm{Vol}(\mathrm{d}\Omega^1)$ and $\mathrm{Vol}(\mathrm{d}\Omega^2)$ individually before we collect terms.

$$\begin{aligned}\mathrm{Vol}(\mathrm{d}\Omega^1) &= \det(\mathrm{d}:\Omega^1\to\Omega^2)\frac{\mathrm{Vol}(\Omega^1)}{\mathrm{Vol}(Z^1)}\\ &= \frac{|(\mathrm{d}^\dagger\mathrm{d})^1|^{\frac{1}{2}}\mathrm{Vol}(\Omega^1)}{\mathrm{Vol}(\mathcal{H}^1)\mathrm{Vol}(\mathrm{d}\Omega^0)}\\[1em] \mathrm{Vol}(\mathrm{d}\Omega^0) &= \det(\mathrm{d}:\Omega^0\to\Omega^1)\frac{\mathrm{Vol}(\Omega^0)}{\mathrm{Vol}(Z^0)}\\ &= \frac{|\Delta^0|^{\frac{1}{2}}\mathrm{Vol}(\Omega^0)}{\mathrm{Vol}(\mathcal{H}^0)}\\[1em] \Longrightarrow\ \mathrm{Vol}(\mathrm{d}\Omega^1) &= \frac{\mathrm{Vol}(\mathcal{H}^0)\mathrm{Vol}(\Omega^1)}{\mathrm{Vol}(\mathcal{H}^1)\mathrm{Vol}(\Omega^0)}\frac{|(\mathrm{d}^\dagger\mathrm{d})^1|^{\frac{1}{2}}}{|\Delta^0|^{\frac{1}{2}}}\\ \mathrm{Vol}(\mathrm{d}\Omega^2) &= \det(\mathrm{d}:\Omega^2\to\Omega^3)\frac{\mathrm{Vol}(\Omega^2)}{\mathrm{Vol}(Z^2)}\\ &= \frac{|(\mathrm{d}^\dagger\mathrm{d})^2|^{\frac{1}{2}}\mathrm{Vol}(\Omega^2)}{\mathrm{Vol}(\mathcal{H}^3)\mathrm{Vol}(\mathrm{d}\Omega^1)}.\end{aligned} \tag{7.37}$$

We collect terms and simplify:

$$\begin{aligned}\mathcal{Z} &= \frac{\mathrm{Vol}(\mathcal{H}^3)\mathrm{Vol}(\mathcal{H}^2)\mathrm{Vol}(\Omega^2)\mathrm{Vol}(\mathrm{d}\Omega^1)}{\mathrm{Vol}(G)\mathrm{Vol}(\mathcal{H}^2)\mathrm{Vol}(\mathrm{d}\Omega^1)} \frac{|(\mathrm{d}^\dagger\mathrm{d})^2|^{\frac{1}{2}}}{|(\mathrm{d}^\dagger\mathrm{d})^2|^{\frac{1}{4}}|(\mathrm{d}^\dagger\mathrm{d})^3|^{\frac{1}{4}}} \\ &= \frac{\mathrm{Vol}(\mathcal{H}^3)\mathrm{Vol}(\Omega^2)}{\mathrm{Vol}(G)} \frac{|(\mathrm{d}^\dagger\mathrm{d})^2|^{\frac{1}{4}}}{|(\mathrm{d}^\dagger\mathrm{d})^3|^{\frac{1}{4}}}.\end{aligned} \tag{7.38}$$

Now we want to express the determinants in terms of the determinants of the Laplacian.

$$|(\mathrm{d}^\dagger\mathrm{d})^1| = \frac{|\Delta^1|}{|(\mathrm{d}\mathrm{d}^\dagger)^1|} = \frac{|\Delta^1|}{|(\mathrm{d}^\dagger\mathrm{d})^0|} = \frac{|\Delta^1|}{|\Delta^0|} \tag{7.39}$$

In the first equality we multiply with $|(\mathrm{d}\mathrm{d}^\dagger)^1|/|(\mathrm{d}\mathrm{d}^\dagger)^1|$, and then rewrite to get the Laplacian on one forms over Laplacian on 0-forms in the denominator. Then doing something similar for the other determinants we get.

$$|(\mathrm{d}^\dagger\mathrm{d})^2| = \frac{|\Delta^2|}{|(\mathrm{d}\mathrm{d}^\dagger)^2|} = \frac{|\Delta^2|}{|(\mathrm{d}^\dagger\mathrm{d})^1|} = \frac{\Delta^2}{\Delta^1}|(\mathrm{d}\mathrm{d}^\dagger)^1| = \frac{|\Delta^2||\Delta^0|}{|\Delta^1|} \tag{7.40}$$

$$|(\mathrm{d}^\dagger\mathrm{d})^3| = \frac{|\Delta^3|}{|(\mathrm{d}\mathrm{d}^\dagger)^3|} = \frac{|\Delta^3|}{|(\mathrm{d}^\dagger\mathrm{d})^2|} = \frac{\Delta^3}{\Delta^2}|(\mathrm{d}\mathrm{d}^\dagger)^2| = \frac{|\Delta^3||\Delta^1|}{|\Delta^2||\Delta^0|}. \tag{7.41}$$

The Partition function then becomes

$$\begin{aligned}\implies \mathcal{Z} &= \frac{\mathrm{Vol}(\mathcal{H}^3)\mathrm{Vol}(\Omega^2)}{\mathrm{Vol}(G)} \left(\frac{|\Delta^2||\Delta^0||\Delta^2||\Delta^0|}{|\Delta^1||\Delta^3||\Delta^1|}\right)^{\frac{1}{4}} \\ &= \frac{\mathrm{Vol}(\mathcal{H}^3)\mathrm{Vol}(\Omega^2)}{\mathrm{Vol}(G)} \left(\frac{|\Delta^2|^2|\Delta^0|^2}{|\Delta^1|^2|\Delta^3|}\right)^{\frac{1}{4}} \\ &= \frac{\mathrm{Vol}(\mathcal{H}^3)\mathrm{Vol}(\Omega^2)}{\mathrm{Vol}(G)} \frac{|\Delta^2|^{\frac{1}{2}}|\Delta^0|^{\frac{1}{2}}}{|\Delta^1|^{\frac{1}{2}}|\Delta^3|^{\frac{1}{4}}}.\end{aligned} \tag{7.42}$$

Then choosing the $\mathrm{Vol}(G) = \mathrm{Vol}(\mathcal{H}^3)\mathrm{Vol}(\Omega^2)$ we obtain our final result

$$\mathcal{Z} = \frac{|\Delta^2|^{\frac{1}{2}}|\Delta^0|^{\frac{1}{2}}}{|\Delta^1|^{\frac{1}{2}}|\Delta^3|^{\frac{1}{4}}}. \tag{7.43}$$

The Laplacians can again be regularised to a finite answer. Let us then examine if this is topological: We have looked at a theory over a 6 dimensional manifold. Point-wise on this manifold;

Ω^3 has dimension $\binom{6}{3} = 20$,
Ω^2 has dimension $\binom{6}{2} = 15$,
Ω^1 has dimension $\binom{6}{1} = 6$,
Ω^0 has dimension $\binom{6}{0} = 1$.

A small variation with respect to the metric of the Partition function then gives us:

$$\implies \delta_g \log(\mathcal{Z}) \propto \left(\frac{1}{2} \cdot 15 + \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot 6 - \frac{1}{4} \cdot 20\right) = 0 \tag{7.44}$$

Hence the Partition function is topological as expected. This result you can think of as a six-dimensional version of the Ray-Singer Torsion.

Further discussion on the result

This discussion will require further knowledge than what we have covered in this thesis. More specifically, knowledge about complex manifolds. For example, chapter 8 in [3] will give the required background.

Usually, in String Theory, one looks at complex manifolds such as Calabi-Yau's, more specifically the three-dimensional complex version, which can be modeled as a six-dimensional real manifold as we did for our final computation. For a three-dimensional complex Calabi-Yau, we have the following Hodge diamond, with the determinants A, B, and C [24] given in figure 7.3.

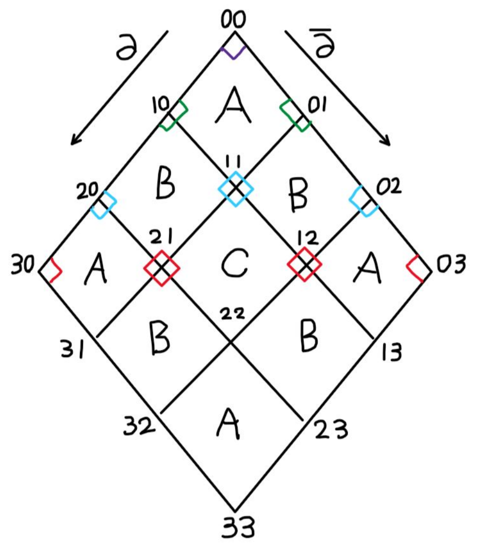

Figure 7.3: The Hodge diamond for a three-dimensional complex manifold. Here the numbers pq at each vertex refers to $\Omega^{(p,q)}$. The determinant of the Laplacian splits up into the factors A, B and C according to the figure [24]

We obtained the result

$$\mathcal{Z}(M_6) = \frac{|\Delta^2|^{\frac{1}{2}}|\Delta^0|^{\frac{1}{2}}}{|\Delta^1|^{\frac{1}{2}}|\Delta^3|^{\frac{1}{4}}}. \tag{7.45}$$

Now we want to express this result in terms of the determinants of a Calabi-Yau. $|\Delta^3|$ we express in terms of the red determinants, i.e. $|\Delta^3| = A^4B^4C^2$. $|\Delta^2|$ we express in terms of the blue determinants i.e. $|\Delta^2| = A^3B^4C$. $|\Delta^1|$ expressed in terms of the green determinants gives $|\Delta^1| = A^2B^2$. And finally, $|\Delta^0| = A$ in terms of the purple determinant. If we then have a look at our result, we get (squaring to get rid of the square roots);

$$\mathcal{Z}^2 = \frac{|\Delta^2||\Delta^0|}{|\Delta^1||\Delta^3|^{\frac{1}{2}}} = \frac{A^4B^4C}{A^2B^2A^2B^2C} = 1. \tag{7.46}$$

Hence our result is trivial on a Calabi-Yau! We can use this to test if a manifold X can be a Calabi-Yau or not;

$$\mathcal{Z}(X) \neq 1 \implies X \neq CY. \tag{7.47}$$

Hence X cannot be a Calabi-Yau. So we have a sophisticated method to check if a manifold X can be a Calabi-Yau. More speculatively, one can imagine cooking up a similar partition function $\tilde{\mathcal{Z}}(X)$ which is trivial if X is complex (but not Calabi-Yau in general). If one then computes this for the six-sphere S^6, and gets $\tilde{\mathcal{Z}}(S^6) \neq 1$ this would imply that S^6 cannot be complex, solving a famous open problem in differential geometry. This shows that topological QFT can be a powerful tool to use for distinguishing manifolds and geometry, in addition to its many applications in physics and String Theory.

References

[1] Michael B Green, John H Schwarz, and Edward Witten. *Superstring theory: volume 2, loop amplitudes, anomalies and phenomenology.* Cambridge university press, 2012.

[2] Loring W. Tu. *An introduction to manifolds.* Springer, 2007.

[3] Mikio Nakahara. *Geometry, topology and physics (2nd edition).* England: Institute of Physics Publishing, 2003.

[4] Anthony Zee. *Quantum field theory in a nutshell.* Princeton university press, 2003.

[5] Sean M Carroll. *Spacetime and geometry.* Cambridge University Press, 2019.

[6] John Renze, Eric Weisstein, and Todd Rowland. *Compact manifold.* MathWorld-A Wolfram Web Resource. URL: https://mathworld.wolfram.com/CompactManifold.html.

[7] Wikipedia. *Tangent space — Wikipedia, The Free Encyclopedia.* http://en.wikipedia.org/w/index.php?title=Tangent%20space&oldid=1087254603. [Online; accessed 22-May-2022]. 2022.

[8] Aleph0. *Stokes' theorem on manifolds.* May 2020. URL: https://www.youtube.com/watch?v=1lGM5DEdMaw.

[9] Loring W. Tu. *An introduction to manifolds.* Springer, 2011.

[10] Wikipedia. *Hodge theory — Wikipedia, The Free Encyclopedia.* http://en.wikipedia.org/w/index.php?title=Hodge%20theory&oldid=1092610500. [Online; accessed 12-June-2022]. 2022.

[11] Øyvind Grøn and Sigbjørn Hervik. *Einstein's General Theory of Relativity With Modern Applications in Cosmology.* Springer, 2007.

[12] Herbert Goldstein. *Classical Mechanics.* Addison-Wesley, 1980.

[13] Edward Witten. "Quantum field theory and the Jones polynomial". In: *Communications in Mathematical Physics* 121.3 (1989), pp. 351–399.

[14] Simon K Donaldson and Richard P Thomas. "Gauge theory in higher dimensions". In: (1998).

[15] Shiing-Shen Chern and James Simons. "Characteristic forms and geometric invariants". In: *Annals of Mathematics* 99.1 (1974), pp. 48–69.

[16] Sean M Carroll. "A no-nonsense introduction to general relativity". In: *Enrico Fermi Institute and Department of Physics, University of Chicago* (2001).

[17] IA Batalin and GA Vilkovisky. "Gauge algebra and quantization". In: *Phys. Lett., B;(Netherlands)* 102.1 (1981).

[18] Albert Schwarz. "Topological quantum field theories". In: *arXiv preprint hep-th/0011260* (2000). URL: https://arxiv.org/abs/hep-th/0011260.

[19] Anthony Ashmore et al. "Topological G2 and Spin(7) strings at 1-loop from double complexes". In: *Journal of High Energy Physics* 2022.2 (Feb. 2022). DOI: 10.1007/jhep02(2022)089. URL: https://doi.org/10.1007%2Fjhep02%282022%29089.

[20] Lawrence C Evans. *Partial differential equations*. Vol. 19 (2nd ed.) American Mathematical Soc., 2010.

[21] Wikipedia. *Particular values of the Riemann zeta function — Wikipedia, The Free Encyclopedia*. http://en.wikipedia.org/w/index.php?title=Particular%20values%20of%20the%20Riemann%20zeta%20function&oldid=1057709423. [Online; accessed 23-May-2022]. 2022.

[22] Daniel B Ray and Isadore M Singer. "R-torsion and the Laplacian on Riemannian manifolds". In: *Advances in Mathematics* 7.2 (1971), pp. 145–210.

[23] Jan de Boer et al. "G2 Hitchin functionals at one loop". In: *Classical and Quantum Gravity* 25.7 (2008), p. 075006. DOI: 10.1088/0264-9381/25/7/075006. URL: https://doi.org/10.1088/0264-9381/25/7/075006.

[24] Vasily Pestun and Edward Witten. "The Hitchin Functionals and the Topological B-Model at One Loop". In: *Letters in Mathematical Physics* 74.1 (Oct. 2005), pp. 21–51. DOI: 10.1007/s11005-005-0007-9. URL: https://doi.org/10.1007%2Fs11005-005-0007-9.