Dilemmas of teaching arithmetical notation to young learners

Fraser Gobede[1] and Reidar Mosvold[2]

[1]University of Malawi, Zomba, Malawi; fgobede@unima.ac.mw

[2] University of Stavanger, Stavanger, Norway; reidar.mosvold@uis.no

This study takes the analysis of a Malawian Grade 1 teacher's mediation of mathematics as a starting point for discussing dilemmas that might be entailed in the teaching of arithmetical notation to young learners. Two exemplar episodes are selected from six video-recorded lessons that were analysed using the Mediating Primary Mathematics Framework. The teacher introduced the writing of numbers and mathematical symbols with their corresponding hand movements and used these movements as the criteria for enabling learners to assess the correctness of their written inscriptions. Two inherent dilemmas of this complex work of teaching are identified and discussed.

Keywords: Early years, mediation, gesture, inscriptions, interactions.

Introduction and theoretical background

It has recently been suggested to distinguish between considering teaching as something teachers do and as work to be done. The latter corresponds with Ball's (1993) investigation of dilemmas that arise in mathematics teaching and the more recent conceptualization of the special work of teaching mathematics (Ball, 2017). Our study further explores this distinction. As a case, we use the introduction of elementary mathematical notation in a Malawian Grade 1 classroom.

When learners are introduced to new mathematical concepts and notation during the early years of primary school, they often make errors. The teacher's reaction to a learner's error made during whole-class activities has implications for the individual learner and the whole class (Bass & Mosvold, 2019). For instance, when a learner has made a writing error, the teacher may just compare the wrong inscription or notation made by the learner with the correct one presented on a chart or workbook. However, as observed by Venkat and Askew (2018), young learners may not have yet developed the mental faculties for distinguishing features of seemingly related representations and may require appropriate teacher's mediating talk and gesture to make these features apparent.

Handling learner errors is complex (Sapire et al., 2016). For persistent errors and misconceptions, learners may need to be equipped with strategies for checking the correctness of their work even in the absence of the teacher. One possible strategy is to embody some mathematical concepts and processes that learners often find difficult to remember. The embodiment of mathematical concepts enables learners to view the subject as an activity involving physical actions and gestures (Edwards et al., 2014). This makes the association of mathematical concepts and processes with their corresponding physical actions essential, especially to young and inexperienced learners, who are just being inducted into school mathematics. Eventually, the teacher is supposed to help the learners progress from the embodied physical representations to their corresponding abstract mental structures (Venkat & Askew, 2018). If the teacher sticks to the physical representations, the learners may no longer see the need to look for mental conceptual structures to make some necessary connections and generalisations (Askew, 2019; Wilson, 2002). This implies that teaching can either enhance or

constrain what is made available to learn in a lesson—thus constituting numerous dilemmas. In this paper, we share the same understanding of a dilemma as Ball (1993) to mean paradoxical situations where the teacher has several alternative choices to make, each having varied consequences, and oftentimes the decision is required instantly.

Drawing on these insights from research on early mediation of mathematics, and on the perspective of considering the teaching of mathematics as work to be done, we ask:

What dilemmas can be entailed in the work of teaching mathematical notation to young learners?

To answer this question, we first apply the Mediating Primary Mathematics (MPM) framework by Venkat and Askew (2018) in the analysis of data to enable careful description of the mediating work that is performed by the teacher, before we discuss dilemmas entailed in this work. Like Bass and Mosvold (2019), we focus only on a small slice of the work of teaching mathematics here, namely what may be involved in attending to learners' errors.

Analytic framework

The MPM framework is guided by Vygotsky's sociocultural view of teaching as a set of mediated transactions, with the teacher as the main mediating agent in the classroom—who works with a set of sociocultural tools of mediation (Kozulin, 2003). We adopted the MPM framework to understand the sociocultural tools for mediating mathematics in the early years of primary school. The MPM framework identifies four strands or means of mediating primary mathematics, namely: Tasks and examples, artefacts, inscriptions, talk and gesture. The framework further subdivides talk and gesture into three sub-strands: Talk and gesture for generating solutions to problems, talk and gesture for making mathematical connections, as well as talk and gesture for building learning connections. Even though the teacher works with all the four means of mediation when teaching, we were particularly interested in the teacher's mediating talk and gesture for building learning connections (see Table 1). As shown in Table 1, teachers' mediating talk for building learning connections is manifested when handling errors from learners' offers. Venkat and Askew (2018) observed that learners' errors provide a context for richer mediation as teachers are prompted to make "responsive moves" to address the errors. In some cases, the learners' errors provide an ideal moment (or "teachable moment") where they could be more receptive to the teacher's remedial actions than if the error was just ignored or the teacher's response was deferred to a later time (Muir, 2008).

Research design and methodology

A qualitative case study design was adopted, which enabled an in-depth inquiry into the complex task of teaching mathematics to young children. Our case is a Grade 1 teacher with an overall teaching experience of seven and half years after graduating from a two-year teacher training programme. The teacher had been teaching mathematics to different cohorts of Grade 1 learners for four consecutive years. The teacher was selected as a paradigmatic case (Flyvbjerg, 2006), exemplifying outstanding learner achievement in resource-limited settings. The school consistently outperformed other primary schools in the same geographical area both during the standardised end of primary school examinations as well as during quiz competitions with nearby schools. The school was based in a remote village where learners mostly relied on the teacher as the sole source of mathematical

instruction, as learners had limited access to extra tuition through books, parents and relatives, or educational television programs. The rural setting increased the possibility of attributing the school's exemplary learner achievement to the classroom practices of its teachers.

Data collection was scheduled for the week when the teacher was introducing the addition of whole numbers to the Grade 1 learners. This was done across six lessons that were observed and video recorded. Unstructured interviews were conducted at the end of each lesson to seek clarification on some observations made in the classroom. An in-depth video-stimulated recall interview was conducted with the teacher after a preliminary analysis of the lesson transcripts. Interview data were analysed thematically. Themes were centred around the reasons for the teacher's choices made in the use of the means of mediation observed during the lesson. Analysis of the affordances that were made possible through the teacher's presentation of the chalkboard inscriptions was done through the application of variation theory (Kullberg et al., 2017), which is one of the theoretical foundations of the MPM framework.

The recorded videos were segmented into instructional episodes that are considered as a unit of analysis in the MPM framework. Even though this paper reports on the observations that were made while the teacher was working with written inscriptions, the analytical focus was on the teacher's mediating talk and gesture for building learning connections that accompanied the inscriptions in each episode. We referred to the indicators for mediating talk and gesture for building learning connections provided by the MPM framework, as shown in Table 1 that follows:

Table 1: Mediating talk and gesture for building learning connections. Adapted from Venkat and Askew (2018, p. 90)

Indicators for mediating talk and gesture for building learning connections	Level
Pull back to naïve methods OR No evaluation of offers (correct or incorrect).	0
Accepts/evaluates offers Accepts learner strategies or offers a strategy OR Notes or questions incorrect offer.	1
Advances or verifies offers. Builds on, acknowledges or offers a more sophisticated strategy OR Addresses errors/misconceptions through some elaboration e.g., "can it be….?" Would this be correct, or this? Non-example offers.	2
Advances and explains offers. Explains strategic choices for efficiency moves OR provides rationales in response to learner offers related to common misconceptions OR Provides rationale in anticipation of a common misconception.	3

As indicated in Table 1, analysis of the teacher's mediating talk for building learning connections examines the extent to which a teacher handles the evaluation of learners' offers in a lesson episode. A learner's offer could be a correct or an incorrect response to the teacher's question or a strategy for solving a problem. In some cases, the teacher may verify an offer or build on it in response to "teachable moments" (Muir, 2008). Ultimately, the teacher may make a "responsive move" (Venkat

& Askew, 2018) or an "asset oriented response" (Bass & Mosvold, 2019) in the form of explaining strategic choices made or providing the rationale for each option while taking into account the common misconceptions.

Findings and discussion

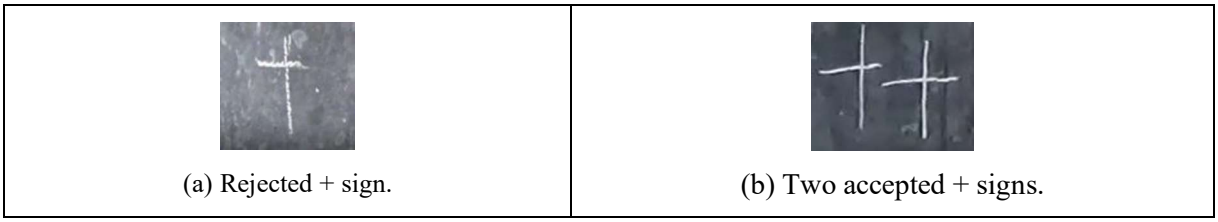
During the study, the teacher was introducing the addition of two whole numbers with a sum not exceeding 5. By this time, the learners had been in Grade 1 for ten weeks, and the teacher had taught them how to write the numbers 0 to 5. In the Malawi context, about 60 per cent of learners do not attend pre-school education before starting Grade 1 (Robertson et al., 2017), hence the preceding ten weeks were their first school experience in life. During classwork, the teacher asked learners to write the worked-out solutions on pre-written papers and the chalkboard, thereby opening up more opportunities for making "responsive moves" (Venkat & Askew, 2018) to the "teachable moments" (Muir, 2008) made possible through the writing errors made by the learners.

Verbalisation of hand movements when working with the plus sign

The teacher introduced the writing of the plus (+) sign during the first lesson of the study by demonstrating how to write the sign in the air while verbalising the hand movements as "Dot! Down! Cut-in-the-middle!" She then asked the learners to do the same:

Teacher: Aa-aah! We have not yet started writing! Just raise your hand and get ready to write *[inaudible],* alright? Everybody use your right hand! Begin!
Class and Teacher: *[Verbalise the movement of the hand while tracing the + sign in the air]* Dot! Down! Cut-in-the-middle!
Teacher: Again!
Class: *[Verbalise the movement of the hand while tracing the + sign in the air]* Dot! Down! Cut-in-the-middle!
Teacher: Again!
Class: *[Verbalise the movement of the hand while tracing the + sign in the air]* Dot! Down! Cut-in-the-middle!

After teaching the hand-movement for the plus sign, the teacher asked learners to suggest the hand movement for the equal sign. The hand movements verbalised by the teacher acted as the basis for the justification of the correctness or incorrectness of the notations made by the learners during the subsequent lessons. For instance, when reviewing how to write the plus sign in the introduction of the fourth lesson in the study, one learner wrote a plus sign that was not accepted by the class (see part (a) of Figure 1). The learner required a convincing explanation on why the written sign was rejected by the class. The teacher reasoned with the learner the original hand movement that was verbalised when the plus sign was introduced for the first time; that is, "Dot! Down! Cut-in-the-middle!" To remediate the error, the teacher asked a second learner to write the plus sign, but this time making sure that the downward stroke of the line making the + sign is cut in the middle by the horizontal stroke. The explanation of the hand movement clarified the contrasting feature of the sign offered by the first learner and the expected sign. After a second learner had written the correct sign, the teacher asked a third learner to re-write the correct notation for the plus sign (Figure 1, part b).



(a) Rejected + sign.

(b) Two accepted + signs.

Figure 1: Correction of an incorrectly written plus sign

The teacher used the verbalised hand movements as the rationale for either rejecting or justifying the offers provided by the three learners. We coded this as the highest level of the MPM framework's mediating talk and gesture for building learning connections (see Table 1) – where a teacher "provides rationales in response to learner offers related to common misconceptions."

The teacher's approach to verbalize hand movements when introducing new inscriptions was not only done for the plus sign. In the interview excerpt that follows, the teacher explained that even when teaching her learners to write numerals before the observed lessons, the same strategy of verbalising hand movements was used:

214 Teacher: How to write? We have several ways. Aah, first, we start to write in the air.
215 Researcher: Okay?
216 Teacher: If you had come when I was teaching numbers you could see that. Because when we say: "Let's write four!" We say: "Dot! Then down! Then right! Then…." Those things. We first start in the air, then after in the air, it's when we go on the ground, before they write in the exercise book.

In the interview excerpt above, the teacher referred to an example of how she introduced the writing of the number 4 based on learners' aptitude. We notice towards the end of Utterance 216 that the teacher must decide when it is appropriate to present the notation as verbal, gestural, written, or any combination of these forms, without losing the mathematical meaning of the notation—thus constituting a dilemma to the teacher.

Using verbalised hand movements when remediating errors related to the writing of 4

The teacher demonstrated the third level of the MPM framework's mediating talk and gesture for building learning connections (Table 1) when responding to learners' errors related to the writing of 4 during the fifth observed lesson. The teacher gave strategic explanations targeting the main source of the observed errors. In that lesson, learners were given pre-written problems on pieces of paper and were asked to find the sum in their groups. The worked-out solutions were then pasted on the chalkboard. One group wrote the answer on their paper as shown in Figure 2.

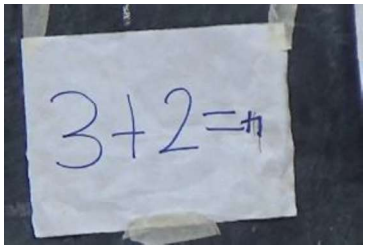


Figure 2: A wrong answer that was written as flipped 4

The representative of the group that was assigned the task shown in Figure 2 read the statement as "three plus two answer four". Rather than quickly dismissing the answer by looking at the expected sum of the given addends 3 and 2, the teacher approached the remediation in phases. Firstly, the teacher asked the class if 4 was supposed to be written as shown in Figure 2, and the class was divided. This posed another dilemma to the teacher—whether to quickly dismiss the incorrectly written numeral and work out the correct sum with the class or consider this as the right moment for remediating the inscription error first, even though the lesson's focus was on the addition of numbers.

The teacher started with remediating the writing error in Figure 2 by asking another learner to write 4 on the chalkboard (see part (a) of Figure 3) and asked the class if the 4 was written correctly. Instead of applying common logic or sentimentality, the teacher reminded the class of the verbalised hand movements for writing 4 ("Dot! Down! Turn-right! Cut-in-the-middle!") while moving a pointing stick. Thus, the teacher used the verbalised hand movement as the basis for justifying the correctness of the 4 offered by the group. The teacher repeated the verbalisation of the hand movement while writing another 4 above the one written by the learner as shown in part (b) of Figure 3.

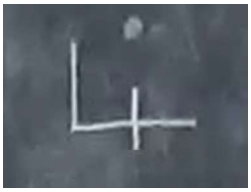
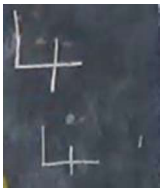
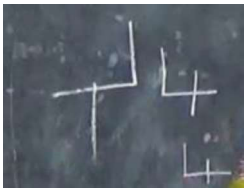
		
(a) A correct 4 written by a learner.	(b) Another correct 4 written by the teacher above the one written in part (a) by a learner.	(c) An incorrect 4 written by the teacher alongside the two correct 4s in (b).

Figure 3: Remediating errors related to the writing of 4 using similarity and contrast

The teacher employed similarity to emphasize the correct way of writing 4 (Figure 3, b), and she continued the discussion by providing a plausible contrast of an incorrectly written 4 alongside the two correctly written 4s (Figure 3, c). The teacher wrote the wrong "4" while simultaneously verbalising its hand movement as: "Dot! Down! Turn-left! Cut-in-the-middle". Rather than providing the contrast by rewriting the disoriented 4 shown in Figure 2, the teacher probably noted from the learners' inscriptions that the common error was the horizontal direction of the hand. After remediating the inscription error of 4 in Figure 2, the teacher then prompted the class to check if 4 was the correct answer to 3 + 2 as initially proposed by the assigned group. After working out the expected sum for 3 + 2, the next group of learners had been assigned to find 1 + 3 and wrote their answer (Figure 4, a).

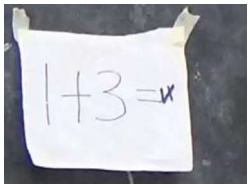
	
(a) Distorted 4 written by learners.	(b) Teacher's emulation of the error.

Figure 4: A distorted 4 written by one group and emulated by the teacher

The teacher used her technique of verbalising hand movements to check if the 4 shown in Figure 4 was correctly written. The teacher expressed the hand movement "Dot! Down! Go-up! Cut-in-the-middle!" while simultaneously writing the movements on the chalkboard (Figure 4, b). Next, the teacher isolated the feature that made the just written 4 incorrect, that is, the expected angular turn in the acceptable hand movement. This was verbalised by the teacher with an emphasis on the turn as "Dot! Down! Turn-right!" while simultaneously writing the hand movements on the chalkboard.

Concluding discussion

The findings indicate how the teacher worked with the MPM framework's mediating talk and gesture for building learning connections (shown in Table 1) related to the writing of arithmetic notations. After receiving an offer from the learners, the teacher first checked with the whole class whether the offer was correct or not. When the error was not apparent, the class was not sure if the offered notation was correct. By providing the rationales (verbalised hand movements) for justifying the learners' offers, the teacher achieved the highest level of the MPM framework's mediating talk and gesture for building learning connections (Venkat & Askew, 2018). Whereas the findings highlight remediation of notations written on the chalkboard, the teacher explained in an interview that the remediation starts with writing in the air, followed by writing on the ground, before using notebooks.

Analysis of data from this Malawian classroom illustrates two common dilemmas entailed in the work of teaching early mathematics. The first dilemma relates to deciding on how to establish correct mathematical notation in a way that is suitable for the learners' age and development (Ball, 1993), while at the same time maintaining mathematical integrity. In the episode analysed, the teacher used similarity and contrast (cf. Kullberg et al., 2017) to help learners identify key characteristics of correct mathematical notation, and she also used verbalised hand movement. Still, there is a risk of confusing learners about the underlying mathematical idea in the process. Instead of just telling learners if the offered inscriptions were correct, the teacher in this study attempted to justify the acceptance or rejections. This may provide the learners with an opportunity to learn about the importance of justification in mathematics. However, young learners may lack the necessary understanding on which the teacher can base justifications for actions taken during lessons (Venkat & Askew, 2018), and this provides a risk that the teacher must attend to. In this episode, the teacher used gestures to justify the correctness of the written arithmetic notations. This use of bodily based resources such as hands and fingers can make the learners feel competent to work out mathematical tasks anywhere anytime (Wilson, 2002).

A second dilemma relates to identifying and interpreting student errors on the fly and deciding on what errors to attend to first when several errors are present (Muir, 2008). The reasoning that is required for probing learners' errors on the fly tends to be one of the highest and complex forms of teacher knowledge (Sapire et al., 2016). In the second episode of the lesson, a learner presented an incorrectly written 4 as the sum of 3 + 2. The teacher then had to decide on whether to attend to the error, or to use other pedagogic moves that do not attend directly to the errors—like assigning competence to learners and positioning them as contributors (Bass & Mosvold, 2019). Deciding on

whether a situation constitutes a teachable moment (Muir, 2008), and deciding on how to act in ways that provide opportunities for learning, constitutes a common dilemma for mathematics teachers.

The MPM framework provides a useful lens to describe observations in mathematics classrooms, and to evaluate the level of teacher mediation, and it provides a useful language to describe what teachers do. This can be useful to some point, and it provides a simplification of teaching that can be useful to teachers as well as researchers. However, shifting attention toward the work of teaching opens the way to understanding what is actually involved in carrying out the complex, dynamic, and situated work of teaching (Ball, 1993; Ball, 2017). This does not simplify the picture, and it does not provide immediate solutions for how to act, but it approves of the real nature of the complex work of teaching mathematics. The dilemmas of this special work of teaching cannot be easily solved, and their management requires professional knowledge and judgment.

Acknowledgment

This study is kindly funded by the Norwegian Programme for Capacity Building in Higher Education and Research for Development (NORHED) through the Strengthening Numeracy in Early Years of Primary Education Through Professional Development of Teachers Project (ref: QZA-0498 MWI 16/0020).

References

Askew, M. (2019). Mediating primary mathematics: Measuring the extent of teaching for connections and generality in the context of whole number arithmetic. *ZDM, 51*(1), 213–226. https://doi.org/10.1007/s11858-018-1010-9

Bass, H., & Mosvold, R. (2019). Teacher responses to public apparent student error: A critical confluence of mathematics and equitable teaching practice. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Eleventh congress of the European society for research in mathematics education* (pp. 3596–3603). European Society for Research in Mathematics Education.

Flyvbjerg, B. (2006). Five misunderstandings about case-study research. *Qualitative Inquiry, 12*(2), 219–245.

Kozulin, A. (2003). Psychological tools and mediated learning. In A. Kozulin, B. Gindis, V. S. Ageyev, & S. M. Miller (Eds.), *Vygotsky's educational theory in cultural context* (pp. 15–38).

Kullberg, A., Kempe, U. R., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM, 49*(4), 559–569. https://doi.org/10.1007/s11858-017-0858-4

Muir, T. (2008). Zero is not a number: Teachable moments and their role in effective teaching of numeracy. *MERGA, 2*, 361–367.

Robertson, S., Cassity, E., & Kunkwenzu, E. (2017). *Girls' primary and secondary education in Malawi: Sector review: Final report.* Australian Council for Educational Research (ACER).

Sapire, I., Shalem, Y., Wilson-Thompson, B., & Paulsen, R. (2016). Engaging with learners' errors when teaching mathematics. *Pythagoras, 37*(1), 1–11.

Venkat, H., & Askew, M. (2018). Mediating primary mathematics: Theory, concepts, and a framework for studying practice. *Educational Studies in Mathematics, 97*(1), 71–92. https://doi.org/10.1007/s10649-017-9776-1

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin & Review, 9*(4), 625–636. https://doi.org/10.3758/BF03196322