Mathematics Discourse in Instruction: How it helps us think about research on mathematics teaching

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Mathematics Discourse in Instruction (MDI) is a framework developed to describe, interpret, and support development of mathematics teaching. Since its inception, it has been successfully used as a tool for research and professional development in South Africa, where it was developed, but it has been less used in other contexts. In this paper, we use shared data as a starting point to explore how the MDI framework can contribute to thinking about research on mathematics teaching. We elaborate on the conception of teaching underlying the framework, describe its elements and their rationale, and show how these can illuminate four core tasks of teaching mathematics.

Keywords: Exemplification, explanatory communication, learner participation, discourse, mathematics.

Introduction

In the past decades, multiple frameworks have been developed, each offering different perspectives on mathematics teaching. The underlying conceptualization of teaching is often not made explicit in these frameworks, and it can be challenging to communicate across theories and frameworks. To stimulate communication and move the group forward, sets of shared data were provided for participants of thematic working group 19 (TWG19) to use at CERME11. Papers applied different theoretical and analytical lenses on these data. For instance, Nic Mhuirí's (2019) application of the Teaching for Robust Understanding (TRU) framework enabled her to compare the quality of teaching across datasets according to the pedagogical norms of the TRU framework, but it also raised questions about underlying values of frameworks and usability across cultural contexts. Bass and Mosvold (2019) applied the instructional triangle as a conceptual framework, adding theoretical perspectives on agency, authority and identity. This allowed them to explore how various teacher moves may influence student agency and position. These examples illustrate how frameworks can offer different perspectives and allow researchers to notice different aspects of the data. Having shared data available in the group thus provided a productive space for discussion. In the present paper, we continue this effort by applying the Mathematics Discourse in Instruction (MDI) framework on one of the shared datasets to investigate what the framework might help us see, what underlying conception of teaching it has, and how a framework like this can contribute to research on mathematics teaching.

The framework[1]

MDI was developed in the Wits Maths Connect Secondary (WMCS) research and professional development project working with teachers in schools serving low-income communities in one province in South Africa. The goal was to improve teaching and learning. Mathematics teaching in

[1] References to 'we' and 'our' in this section are to the WMCS research team, and not to the authors of this paper.

secondary school classrooms is predominantly direct instruction, and typically described as 'traditional'. Our initial observation was that within this similarity were both important teaching differences and incoherence in the mathematical messages across lessons (Adler, 2017). In this context, we sought a framework that describes what mathematically is made available (or not) to learn, irrespective of pedagogical norms, and provide for developmental trajectories that we could use with teachers to improve the coherence of teaching and opportunities for learning mathematics. In its initial form MDI was used analytically to describe and compare lessons (see Adler & Ronda, 2015). Between 2015 and 2019 the project focused on using it to develop practice. It is thus a living framework and has functioned as a boundary object, shifting flexibly across teaching and research practices in the project (Adler, 2017). This paper contributes to current work of refining MDI.

MDI is underpinned by key tenets of sociocultural theory. Briefly, these include an orientation to mathematics as coherent and connected scientific knowledge (Vygotsky, 1978); and to mathematics teaching as goal directed with mediation towards learners' appropriation of increasingly sophisticated and increasingly general ways of thinking to progress in the discipline. Critically, teaching is always about something – an object of learning (Marton, 2015) – and the coherent mediation of that 'something' is the teacher's work. To focus the project's work, we foregrounded what we considered were high leverage practices in this work, and specifically in preparing for and teaching a lesson.

If teaching is always about 'something', a first core task of teaching is to identify the object of learning – that which students are to come to know and be able to do in a lesson. Our analysis below will show that while this task is obvious, its enactment is not trivial. Key next core tasks are selecting and sequencing examples, their related tasks, and representational forms (*exemplification*), attending to explicit mathematical word use and justifications/substantiations (*explanatory communication*) and to what learners are invited to do, say and write (*learner participation*). Our conceptualization of each of these tasks has been informed by two literature strands: on exemplification and variation in mathematics and mathematics education (e.g. Al-Murani et al., 2019), and on language as a resource in mathematics teaching and learning – including attention to lexicalisation (Planas, 2021) – and explicit criteria for mathematical explanations (Prediger, 2019). Our sociocultural perspective sees tasks of teaching as mediational, drawing in cultural tools that shape and are shaped by contexts (see Figure 1). The further salience of the four elements (tasks of teaching) in the framework was their resonance with practice and possibilities for connection and developmental work with teachers.

Task 1: Identifying the object of learning requires both mathematical and curriculum analysis. The 'object' in the lesson we study here is placing a fractional number on the number line (for more information about the context and task, see Ball, 2017). This task requires analysing (1) where this 'content' is located in the curriculum (and so on the trajectory of student learning), (2) its mathematical meaning and what other mathematical concepts, procedures and practices are connected and entailed. In analysis of shared data, we only had access to the enacted object (what is actually taught) and can only infer what was intended. Below, we show that the intended object of learning is provided in the description of the lesson, enabling mathematical and curriculum analysis.

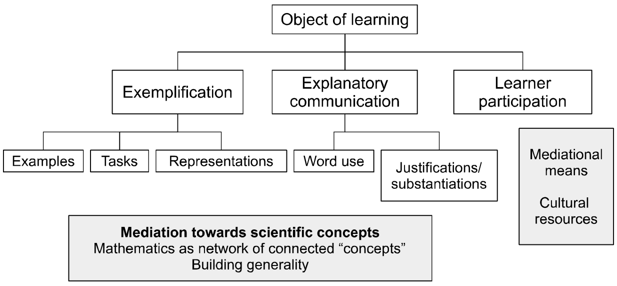

Figure 1. The MDI conceptual framework, adapted and refined from Adler & Ronda (2015).

Task 2: There is considerable literature foregrounding exemplification in mathematics education – as specialised knowledge for teaching, and as teaching and learning through discerning variation (for a review, see Adler & Pournara, 2020). Watson and Mason (e.g., 2006) brought these two strands together in mathematics. Al-Murani et al. (2019) build on this earlier work and use principles of variation to analyse example sets in mathematics lessons. They argue that discerning similarities and differences in an example set – what is changing over a stable background, or what is invariant as features change – is particularly useful in illustrating opportunities created for building generality and/or recognizing structure in mathematics learning. Resonant with this approach, two key principles of variation were recruited into MDI: similarity and contrast. We analyse variation amidst invariance (what is the same and what is different) and contrast (what is and what is not) across example sets to interpret opportunities made available for mathematics learning.

Task 3: Research in mathematics education has attended to important language practices – referred to as language responsive mathematics teaching (e.g., Prediger, 2019). That such teaching needs to extend beyond the communicative function of language to its epistemic function has long been recognised (e.g., Pimm, 1987). The epistemic function includes both vocabulary work, explaining words and phrases, thus requiring teaching to create opportunities for students to participate in (hear, see and use) specialised discourse (e.g., Planas, 2021). This is well illustrated in Planas' (2021) study of teaching equation solving, where she argues for the specific language practice of lexical elaboration. She shows how deliberate attention to the equivalence relation, and the appropriate words to support this meaning create opportunities for learning specialised discourse. The two elements of *explanatory communicatio*n (word use and justifications) in MDI link with this work.

Task 4: Learners obviously need to engage with mathematics to learn. MDI focuses on how students are invited to hear, see, write and speak mathematics, the latter beyond chorusing and one-word answers to teacher questions, and so to use specialised discourse themselves.

Exploration of framework with shared data

The MDI framework was developed for analysis of a lesson. Adler and Ronda (2015) operationalise key constructs of each task, with codes developed to distinguish, for example, similar and contrasting examples, differing task demands, colloquial and mathematical word use, criteria for explanations and learner participation. They describe how lessons are chunked into episodes and the analysis of episodes accumulate to describe the example set and other elements in terms of levels of a trajectory towards more coherent teaching. The exploration of the shared data below cannot do this kind of analysis as we only have one episode. Given the coherence of the overall lesson, the tasks would likely all be coded the highest level 3. As a coherent lesson, it thus is more useful to elaborate our exploration descriptively, and this is how we proceed.

The shared data we use – the Hoover data set – is a short episode (video recording and transcript) within a lesson together with a context document describing the class (rising fifth graders who attend a summer school in the United States), the motive for the lesson, its mathematical focus and student responses to a warm-up problem at the start of the lesson, and a similar exit task at the end. We use all of this for our analysis and discussion of what the MDI framework allows us to see.

The first step in MDI analysis is to identify the "ostensible" object of learning: analysis of the example set and related tasks, and the explanatory communication are all in relation to this 'object'. The warm-up problem asks: "What number does the orange arrow point to? Explain how you figured it out". The arrow points to the position of $\frac{1}{3}$ on a number line that extends from just before 0 to just past $2\frac{1}{3}$, with thirds marked on the line. In the context document accompanying this shared data set, the lesson is described as *marking a turning point in mathematical work* from *"naming fractions as parts of areas to identifying fractions as points on the number line"* (p. 1), with a critical aspect of the object of learning being the important *"shift" in understanding* that "on the number line the whole is defined as the distance from 0 to 1. With area models, the whole can be greater than 1" (p. 2). This is something students are to know. What they need to be able to do – the focus of the lesson – is to carry out "steps for naming a fraction correctly". This key procedure is evident in the chart provided. It is presented as the interpretation for teaching of *an extract from curriculum documents* (p. 7). The intensive mathematical and curriculum analysis highlighted in the italicized phrases that was done in preparation for this lesson are made visible. These foci for the lesson are not trivial and require specialized teacher knowledge of fractions as part of an area, and fraction as a number, where and how the progression from one to the other is located in the curriculum, and consequently what prior learning brings to the shift learners need to make. The next analytic step is to analyze the mediation of these object(s).

Exemplification – examples, tasks and representations

The focal example in the episode is the fraction $\frac{1}{3}$ and the task for the students is high demand. They are required to name the fraction being pointed to on the given number line which extends from 0 and past 2, and to explain their reasoning. There are no other given examples in the episode. The example in the concluding task is the fraction $\frac{2}{3}$, and the task is the same. The fraction $\frac{1}{7}$ is also discussed in the episode, bringing in a second fraction example. We see other fractions ($\frac{1}{4}$, $\frac{1}{5}$) written on the board and possibly offered prior to the $\frac{1}{7}$, and still other responses to the warm-up task such as $\frac{2}{4}$. We do not have access to other examples of fractions and their location on the number line that were likely to have been discussed later in the lesson. Specifically, we do not know whether different 'thirds' were discussed – where the 'whole' in thirds (the denominator) remained invariant, and the number of 'parts' (numerator) varied (e.g. $\frac{2}{3}$, $\frac{3}{3}$, $\frac{5}{3}$, $\frac{7}{3}$) – or whether there was discussion of examples with a different denominator, with a similar range of numerators (e.g. $\frac{1}{5}$, $\frac{2}{5}$ ...). There are thus limits to what can be said about the example set across the lesson and variation with respect to similarity, and so what of the written fractions were varying and invariant and how these might have been publicly discussed. The intended generality is the procedure for naming a fraction on a number line. What is observable is the chart that outlines the steps for the procedure in general terms, indicating that there were probably more examples of fractions with varying numerators and denominators, with the generalized fraction name written as $\frac{n}{d}$. The extent of how these varied, and what was invariant are what is salient to what is made available to learn.

What is observable in the episode is variation through contrast through the selection of $\frac{1}{7}$ for public discussion. This brings 'what is not' the name of the fraction, into focus i.e. specifically naming the fraction by counting the hash marks, or the 'parts' without reference to the unit whole. The selection of $\frac{1}{7}$ (as opposed to $\frac{1}{4}$, *or* $\frac{1}{5}$), is salient, provoked by the number line extending past 2. Unfortunately, the episode does not include the discussion of $\frac{1}{7}$ only some clarifications of the thinking that produced this, and so counting all the parts, making available a discussion following on the unit whole and not the length given (as in the case of the area model). Other student questions about starting at 'zero' and what was counted draw attention to the hash marks (as opposed to spaces). In MDI terms, and the application of principles of variation of similarity and contrast, we would conclude from what is available in the video, transcript and the context document, that the lesson provided opportunities for learning to discern the correct place of a fraction between 0 and 2 on the number line, specifically for identifying the whole, and focusing on counting spaces not hash marks.

Looking at the students' successful responses at the end of the lesson leads one to speculate that the whole lesson included discussion of a varied example set, with purposefully selected similar and contrasting examples, making available opportunity for generalizing the mathematical steps for marking a fraction on the number line (and through contrast not over-generalizing).

What is made possible to learn is, of course, not only a function of the selection and sequencing of examples, tasks and representations[2], but how the example set is discursively mediated.

Explanatory communication – word use and justifications

As the episode is restricted to the presentation and clarification only of the answer $\frac{1}{7}$, and questions posed by other students (the teacher's task here is focused on students hearing the explanation offered), there is limited possible analysis of word use towards the object of learning in the transcript. However, the description of the teachers' work through the rest of the lesson, and the extracts of students' explanations of their responses to the exit task in the context document enable analysis of what was or what was likely to have been made available in the classroom discussion.

From the selected written student explanations in the document, words they used initially to explain their answer included "count", "equal parts", "count from zero" indicating that they associated fractions with equal parts that needed to be counted, but not what or where these equal parts were nor the whole they were referring to. In contrast, in the exit task, we see the following: "You have to

[2] Of course, in this lesson and episodes the shift in representation from area to number line is built into the lesson and not focused on here.

count the space"; "all the spaces have to be equal"; "… because I saw what the howl (whole) was. Then I made sure it had equal parts. Last I counted the spaces."; "I counted the spaces"

The students' use of specialized discourse relevant to fraction as number is evident. Contributions to how this was made available to learn in the lesson is observable in the chart at the end of the context document that emphasizes the words: "Whole", "equal parts", "unit fraction", "count parts"; as well as in the extract below that elaborates the discussion (though we are not privy to who said what, when). From the context document (p. 4):

> … the discussion emphasizes the importance of partitioning the *unit interval* in *equal part*s and being sure to *count spaces (i.e. the intervals, not hash marks)* to *determine the distance from* 0 for a given point on the line. The students practice naming points on the line and also explaining carefully with *reference to the whole* and to *the equal parts* and to *counting spaces to determine the number*.

We have italicized the specialized discourse and interpret that these words and aspects of justifications for the fraction name were used in the lesson by both teacher and students. Moreover, as indicated in the underlined sections, we observe that students practiced this specialized word use and justifications, probably repeated on a range of different fraction examples.

Using MDI, we would analyze all episodes in a lesson for specialised discourse coding word use by distinguishing colloquial from mathematical, and coding justifications by distinguishing those that are non-or pseudo-mathematical (assertions – e.g. because the textbook says, or visual cues – e.g. the hash marks), from those that are local, partial, and then full explanations. Our data here is different and coding the given episode would put word use as colloquial and mathematical in name only. However, specialized mathematical word use coherent with the object of learning (whole, equal parts, counting spaces) was made available, as was a full explanation of the steps for correctly placing a number on the number line. If we had the full lesson transcript, MDI analysis by episode would show what and how word use and justifications build up through the lesson, and so through the temporal flow of the lesson, as well as who says what, and so what the teacher inserts, repeats, revoices and reinforces (as this is her work), and what students get to hear as well as say for themselves. This last point is apposite, for specialized discourse (word use and criteria for valid mathematical argument) is not spontaneously available, and thus a crucial aspect of knowledge for teaching and task of teaching (Planas, 2021; Prediger, 2019).

Learner participation

Given the traditional teaching context in which MDI was developed, the focus of research in the WMCS project was on the extent to which students themselves had opportunities to hear, speak and write mathematically. These opportunities are extensive even in this one episode. We discuss this further below.

Concluding discussion

Different frameworks enable scholars to notice different things about mathematics teaching. For instance, the TRU framework allowed Nic Mhuiri (2019) to evaluate and compare the quality of teaching across contexts in analysis of shared data. Theoretical constructs like agency and authority enabled Bass and Mosvold (2019) to explore different aspects of the same data sets. Their framework allowed them to notice how the pedagogical moves that teachers make can influence the agency and position of students in mathematics classrooms, and they argue that this is particularly important to notice when responding to apparent student errors. While these perspectives are indeed salient, the MDI framework was developed with the aim of describing the *mathematics* that is made available to learn. The emphasis is thus primarily on the mathematics as it is made available through examples, tasks and representations, and through word use and justifications. As we have tried to show in our discussion of shared data, selection and sequencing of examples is a key task in mathematics teaching, which is not straightforward, yet often overlooked. One thing the MDI framework allows for, is to highlight exemplification and how it may or may not contribute to the set learning goal.

A potential weakness of the framework is that the underlying conception of teaching is left implicit, linked only with being goal directed and involving mediation. We highlight two aspects of the conception of teaching underlying the MDI framework. First, like Ball (2017), it useful to think about mathematics teaching as a special mathematical work. Others have suggested a distinction between considering teaching as something teachers do, as opposed to a work to be done. This distinction was not considered in developing the MDI framework. However, the framework operationalizes four core mathematical tasks of teaching, and so the kind of work to be done: 1) deciding a mathematical object of learning, 2) selecting and sequencing examples, tasks, and representational forms, 3) attending to mathematical word use and justifications, and 4) considering what learners are invited to do, say, and write. Second is an underlying issue of teaching methods. Some frameworks of mathematics teaching are bound to specific teaching methods or pedagogical values. Nic Mhuiri (2019) highlights this in her discussion of the TRU framework, which appears to value an approach to mathematics teaching that corresponds with ongoing reform efforts in the United States. In contrast, the MDI framework aims at being useful across methods or pedagogies of teaching. It was developed in a context that is dominated by traditional teaching, but our effort to apply it to Deborah Ball's teaching in the shared data set indicate that it can be useful also for analysis of nontraditional mathematics teaching.

In the call for papers to TWG19, five domains were identified to facilitate communication and collaboration. When considering the potential contribution of MDI in relation to these domains, we first notice that MDI has a primary emphasis on extending mathematics to learners. Whereas the organization of interactions is less in focus, the framework does focus on the mathematical discourse. The MDI framework was not developed with an emphasis on responding to students, but equity was an underlying issue of concern. Although social, cultural, and political issues are not directly visible in use of the framework, access to mathematics *is* an issue of equity in many countries. Since learners in areas with significant poverty often do not get the opportunity to learn mathematics, improving their access to mathematics in the classroom is thus an equity issue – and a key motivation for the development and use of the MDI framework.

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