Robust observer-based leader-following consensus for a class of nonlinear multi-agent systems: application to UAV formation control

Juan A. Vazquez Trejo[1,2], Damiano Rotondo[3], Manuel Adam Medina[2], Didier Theilliol[1]

Abstract— **This paper presents the design of a robust observer-based control for a class of nonlinear multi-agent systems. The leader-following consensus problem is solved in order for all the agents to follow the trajectory of a virtual leader in spite of a nonlinear input which depends on the local and neighboring agents. The main contribution of this paper is to guarantee the stability and robustness of the estimated states and the synchronization error for each agent. Linear matrix inequality (LMI)-based sufficient conditions are obtained for computing the controller and observer gains. The effectiveness of the proposed approach is shown considering a formation control problem in a fleet of unmanned aerial vehicles under a simulation setting.**

Index Terms— **Linear matrix inequalities, leader-following consensus, nonlinear multi-agent systems, networked swarms, observer-based control.**

I. INTRODUCTION

Networked control of unmanned aerial vehicles (UAVs) has been studied extensively in the literature considering problems such as formation control [1], [2], [3], containment control [4], event-triggered control [5], flocking [6], just to mention a few works. Multi-agent systems have been of interest due to their potential in accomplishing missions that a single UAV cannot perform. Multi-agent systems are integrated by individual dynamical systems called agents that can communicate their states through networks. Each individual agent has sensors and actuators for interacting with their environment [7]. The focus on the consensus problem of multi-agent systems has been described by different dynamic models such as: single integrators [8], double integrators [9], high-order integrators [10], linear [11], and nonlinear dynamics [12].

Consensus is an important problem in the context of coordination of multi-agent systems where all the agents reach an agreement in relation to the states of the other agents [13]. A consensus algorithm or consensus protocol is an interaction rule that specifies the exchange of information between an agent and its neighbors on the network [14]. Leader-following consensus problem is considered when all the trajectories of the agents must converge to a physical or virtual leader agent [15]. In the last decades, the leader-following consensus problem has been employed in fleets of UAVs when the control objective of the collective group is to follow the trajectories and velocities described by a physical or virtual agent such as reported in [16], [17] considering second-order multi-agent systems, or multi-UAV-systems [18], [19]. As reported in [3], consensus algorithms and angle transformations can be applied to manipulate the UAVs. In this light, this paper takes into consideration the leader-following control problem in order to synchronize a fleet of UAVs collectively.

An interesting topic in the literature is the leader-following consensus problem of nonlinear multi-agent systems with a nonlinearity in the dynamics of the agents. Centralized and decentralized event-triggered leader-following consensus for a class of switched nonlinear multi-agent are investigated in [20]. The distributed leader-following consensus problem for a class of first-order and second-order Lipschitz nonlinear multi-agent systems with unknown control directions and unknown bounded external disturbances is addressed in [21]. The distributed event-triggered consensus problem for a class of nonlinear multi-agent systems subject to actuator saturation considering an uncertain nonlinear function is presented in [22]. A sampled-data leader-following consensus of nonlinear multi-agent systems with random switching network topologies and communication delay is studied in [23]. In [20] and [21], heterogeneous nonlinear multi-agent systems have been considered where, in [20], the nonlinearity is different for each agent and [21] considers only different values in the agents' parameters. [22] and [23] present high-order nonlinear homogeneous multi-agent systems with a linear and an additive nonlinear part. However in all works cited [20], [21], [22], [23], the additive nonlinearity is Lipschitz and only dependent on the local states of each agent.

Compared to the aforementioned works, an additive nonlinearity dependent on the neighboring and local states of the agents, which can influence the performance of the consensus, is taken under consideration in this paper. The main contributions of this paper can be listed as follows. 1) Based on the Lyapunov approach, a robust leader-following control is designed to follow the trajectories of a physical or virtual leader agent despite the additive nonlinearity. 2) Linear matrix inequality (LMI)-based sufficient conditions are provided in order to guarantee the existence of the controller and observer gains. 3) To illustrate this strategy, an extension in a fleet of UAVs which can be considered as second-order multi-agent system solving the

[1]University of Lorraine, CRAN UMR 7039 CNRS, France. juan-antonio.vazquez-trejo@univ-lorraine.fr, didier.theilliol@univ-lorraine.fr

[2]Electronic engineering department, TECNM/Cenidet, Cuernavaca Morelos, México. manuel.am@cenidet.tecnm.mx

[3]Department of Electrical and Computer Engineering (IDE), University of Stavanger, Stavanger, Norway. damiano.rotondo@uis.no

leader-following formation control problem with a nonlinear function is presented.

The paper is organized as follows. Notation and basic graph theory are reported in Section 2. The problem statement is defined in Section 3. The robust observer-based leader-following control is designed in Section 4. Simulations of the formation control problem are used to show the effectiveness of the proposed strategy in Section 5. Finally, the main conclusions are presented in Section 6.

II. Preliminaries

Given a matrix X, X^T denotes its transpose, $X > 0 (< 0)$ denotes a symmetric positive (negative) definite matrix. The set $\mathbb{R}$ denotes the real numbers. $\|.\|$ denotes the Euclidean norm. For simplicity, the symbol $*$ within a symmetric matrix represents the symmetric entries. The Hermitian part of a square matrix X is denoted by $\text{He}\{X\} := X + X^T$. The symbol $\otimes$ denotes the Kronecker product.

Lemma 1 ([24]): For real matrices A, B, C, and D with appropriate dimensions, the Kronecker product $\otimes$ has the following properties:

1) $(A+B)\otimes C = A\otimes C + B\otimes C$;
2) $(A\otimes B)^T = A^T \otimes B^T$;
3) $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$.

Lemma 2 (Schur complements [25]): For a given symmetric matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} < 0$ the following statements are equivalent:

1) $A < 0$, $C - B^T A^{-1} B < 0$;
2) $C < 0$, $A - B^T C^{-1} B < 0$.

Lemma 3 (Young's relation [26]): For given matrices A and B with appropriate dimensions, for any invertible matrix S and scalar $\mu > 0$, the following holds:

$$\text{He}\left\{A^T B\right\} \le \mu A^T S A + \mu^{-1} B^T S^{-1} B.$$

A directed graph $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_N\}$ is a non-empty finite node set and $\mathcal{E} = \{(i,j) : i,j \in \mathcal{V}\} \subseteq \mathcal{V}\times\mathcal{V}$ is an edge set of ordered pairs of N nodes (total agents). The neighbors of node i are denoted as $j \in \mathcal{N}_i$. The adjacency matrix $\mathcal{A} = \left[a_{ij}\right] \in \mathbb{R}^{N\times N}$ associated with the graph $\mathcal{G}$ is defined such that $a_{ii} = 0$, $a_{ij} > 0$ if and only if $(i,j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L} = \left[\mathcal{L}_{ij}\right] \in \mathbb{R}^{N\times N}$ of the graph $\mathcal{G}$ is defined as $\mathcal{L}_{ii} = \sum_{j\neq i} a_{ij}$ and $\mathcal{L}_{ij} = -a_{ij}, i \neq j$.

III. Problem statement

Consider the following nonlinear multi-agent system:

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) + \omega_i\left(x_i(t), x_j(t)\right), \\ y_i(t) &= Cx_i(t), \end{aligned} \tag{1}$$

where $x_i(t) \in \mathbb{R}^{n_x}$ is the vector state, $u_i(t) \in \mathbb{R}^{n_u}$ is the input vector, $y_i(t) \in \mathbb{R}^{n_y}$ is the measurement output vector with matrices $A \in \mathbb{R}^{n_x\times n_x}$, $B \in \mathbb{R}^{n_x\times n_u}$, and $C \in \mathbb{R}^{n_y\times n_x}$; $\omega_i : \mathbb{R}^{Nn_x} \to \mathbb{R}^{n_x}$ is an additive exogenous nonlinearity which is dependent on the neighboring and local states of the agents. The following assumptions hold in this paper.

Assumption 1: The pair (A, B) is stabilizable.

Assumption 2: The pair (A, C) is detectable.

Assumption 3: The graph $\mathcal{G}$ is strongly connected.

Assumption 4: $\omega_i\left(x_i(t), x_j(t)\right)^T N_\omega \omega_i\left(x_i(t), x_j(t)\right) \le \gamma^2 \delta_i^T H^T H \delta_i$ where $\gamma > 0$ is the bounding parameter, $N_\omega \in \mathbb{R}^{n_x\times n_x}$ is a symmetric positive definite matrix to be calculated, $H \in \mathbb{R}^{n_x\times n_x}$ is a constant matrix, and δ_i is the synchronization error between the agent i and the leader agent.

Remark 1: Note that $\omega_i\left(x_i(t), x_j(t)\right)$ can represent a malfunction in the consensus such as false-data cyber-attacks which depend on the neighboring and local states [27].

An observer-based leader-following control for the multi-agent system in (1) is designed such that all the agents follow the reference trajectories in a consensus despite $\omega_i\left(x_i(t), x_j(t)\right)$. The leader dynamics are described as follows:

$$\dot{x}_r(t) = Ax_r(t) + Bu_r(t), \tag{2}$$

where $x_r(t) \in \mathbb{R}^{n_x}$ is the leader's state vector, $u_r(t) \in \mathbb{R}^{n_u}$ is the leader's input vector. The leader agent can be considered as an exogenous system or a command generator which generates the desired trajectory to be followed by all agents [28]. Let us define the synchronization error for each agent i as follows:

$$\delta_i(t) = x_i(t) - x_r(t), \tag{3}$$

then, the synchronization error dynamics are obtained:

$$\begin{aligned} \dot{\delta}_i(t) &= \dot{x}_i(t) - \dot{x}_r(t) \\ &= Ax_i(t) + Bu_i(t) - Ax_r(t) - Bu_r(t) + \omega_i\left(x_i(t), x_j(t)\right), \\ &= A\delta_i(t) + B\left(u_i(t) - u_r(t)\right) + \omega_i\left(x_i(t), x_j(t)\right). \end{aligned} \tag{4}$$

Definition 1 ([29]): The leader-following consensus problem is achieved if the multi-agent (1) satisfies:

$$\lim_{t\to\infty} \|x_i(t) - x_r(t)\| \to 0,\ \forall i = 1, 2, \ldots, N, \tag{5}$$

for any initial condition despite $\omega_i\left(x_i(t), x_j(t)\right)$.

As reported in [29], the standard observer-based leader-following control is given as follows:

$$\begin{aligned} u_i(t) =& K_c \sum_{j\in\mathcal{N}_i} a_{ij}\left(\hat{x}_i(t) - \hat{x}_j(t)\right) \\ &+ \alpha_i K_c\left(\hat{x}_i(t) - x_r(t)\right) + u_r(t), \end{aligned} \tag{6}$$

where $K_c \in \mathbb{R}^{n_u\times n_x}$ is the control gain to be designed, α_i is defined to be 1 whenever the agent i is a neighbor of the leader and 0 otherwise, $\hat{x}_i(t)$ is the estimated state vector of the agent i, and $\hat{x}_j(t)$ is the estimated state vector of its neighbors j. Based on (4) and (6), it is obtained:

$$\begin{aligned} \dot{\delta}_i(t) =& A\delta_i(t) + BK_c \sum_{j\in\mathcal{N}_i} a_{ij}\left(\hat{\delta}_i(t) - \hat{\delta}_j(t)\right) \\ &+ \alpha_i BK_c \hat{\delta}_i(t) + \omega_i\left(x_i(t), x_j(t)\right). \end{aligned} \tag{7}$$

Let us define $\delta = \left[\delta_1(t)^T, \delta_2(t)^T, \ldots, \delta_N(t)^T\right]^T$, $\hat{\delta} = \left[\hat{\delta}_1(t)^T, \hat{\delta}_2(t)^T, \ldots, \hat{\delta}_N(t)^T\right]^T$, the nonlinear input $\omega =$

$\left[\omega_1\left(x_i(t),x_j(t)\right)^T,\omega_2\left(x_i(t),x_j(t)\right)^T,\ldots,\omega_N\left(x_i(t),x_j(t)\right)^T\right]^T$, and $\Lambda = diag\left(\alpha_1,\alpha_2,\ldots,\alpha_N\right)$, thus, $\bar{\mathcal{L}} = \mathcal{L}+\Lambda$. Then, the synchronization error dynamics are rewritten using the Kronecker product as follows[1]:

$$\dot{\delta} = \left(I_N\otimes A\right)\delta + \left(\bar{\mathcal{L}}\otimes BK_c\right)\hat{\delta} + \omega. \tag{8}$$

Lemma 4: The matrix $\bar{\mathcal{L}}$ has nonnegative eigenvalues ($0<\lambda_j = \lambda_j(\bar{\mathcal{L}}), \forall i = 1,2,\ldots,N$). The matrix $\bar{\mathcal{L}}$ is positive definite if and only if the graph $\mathcal{G}$ is strongly connected [29].

Remark 2: It is worth mentioning if the synchronization error is zero ($\delta = 0, t\to\infty$) despite ω, then it is said all the agents reach the consensus.

In this paper, the leader-following control uses the estimation provided by the following observer:

$$\begin{aligned}&\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + Bu_i(t) + \omega_i\left(x_i(t),x_j(t)\right) + L_o\left(y_i(t)-\hat{y}_i(t)\right),\\ &\hat{y}_i(t) = C\hat{x}_i(t),\end{aligned} \tag{9}$$

where $\hat{x}_i(t)\in\mathbb{R}^{n_x}$ is the estimated state vector, $\hat{y}_i(t)\in\mathbb{R}^{n_y}$ is the estimated output vector, and the matrix $L_o\in\mathbb{R}^{n_x\times n_y}$ is the observer gain to be designed. Let us define $\hat{x} = \left[\hat{x}_1(t)^T,\hat{x}_2(t)^T,\ldots,\hat{x}_N(t)^T\right]^T$, then equation (9) can be expressed as follows:

$$\dot{\hat{x}} = \left(I_N\otimes A\right)\hat{x} + \left(\bar{\mathcal{L}}\otimes BK_c\right)\hat{x} + \omega + \left(I_N\otimes L_oC\right)\left(x-\hat{x}\right). \tag{10}$$

Let $z_1 = \delta$, $z_2 = \delta-\hat{\delta} = x-\hat{x}$, $z = \left[z_1^T,z_2^T\right]^T$, then (8) and (10) can be rewritten as follows:

$$\dot{z} = \tilde{A}z + \begin{bmatrix}\omega\\0\end{bmatrix}, \tag{11}$$

where $\tilde{A} = \begin{bmatrix}I_N\otimes A+\bar{\mathcal{L}}\otimes BK_c & -\bar{\mathcal{L}}\otimes BK_c\\ 0 & I_N\otimes\left(A-L_oC\right)\end{bmatrix}$.

Given the nonlinear multi-agent system (1), the problem under consideration in this paper is to synthesize a robust observer-based leader-following controller K_c and observer L_o gains described in (6) and (9) such that all the agents follow the virtual leader (2) in a consensus despite the nonlinearities.

IV. Robust observer-based leader-following control

In the following subsections, sufficient conditions are presented in order to guarantee a robust leader-following consensus for the nonlinear multi-agent system (1). Then, LMI-based conditions are presented in order to compute the controller and observer gains.

A. Leader-following consensus analysis

Given the closed-loop nonlinear multi-agent system (11), the following theorem provides sufficient conditions to guarantee a robust leader-following consensus.

Theorem 1: Assume that Assumptions 1-4 holds; $z = 0$ is an asymptotically stable equilibrium state if there exist matrices $P_1>0$, $P_2>0$, $N_\omega>0$, and scalars $\beta>0$, $\gamma>0$ such that the following holds $\forall j = 1,2,\ldots,N$:

$$\begin{bmatrix}Q_{1_j} & -\lambda_jP_1BK_c & P_1\\ * & Q_2 & 0\\ * & * & -N_\omega\end{bmatrix}<0, \tag{12}$$

where $Q_{1_j} = \mathrm{He}\left\{P_1\left(A+\lambda_jP_1BK_c\right)\right\}+\beta\gamma^2H^TH$ and $Q_2 = \mathrm{He}\left\{P_2\left(A-L_oC\right)\right\}$.

[1]Due to space limitation, explicit dependence of signals on time are omitted.

Proof: Let us calculate the time derivative of the Lyapunov function $V = z^T\tilde{P}z$ along any solution of the system (11) given by:

$$\dot{V} = 2z^T\tilde{P}\dot{z} = 2z^T\tilde{P}\left(\tilde{A}z+\begin{bmatrix}\omega\\0\end{bmatrix}\right), \tag{13}$$

where $\tilde{P} = diag\left(I_N\otimes P_1, I_N\otimes P_2\right)$. From (13), it is obtained:

$$\begin{aligned}\dot{V} =& 2z_1^T\left(I_N\otimes P_1A+\bar{\mathcal{L}}\otimes P_1BK_c\right)z_1\\ &-2z_1^T\left(\bar{\mathcal{L}}\otimes P_1BK_c\right)z_2 + 2z_1^TP_1\omega\\ &+2z_2^T\left(I_N\otimes P_2(A-L_oC)\right)z_2.\end{aligned} \tag{14}$$

Let us perform a spectral decomposition similarly as reported in [30], [29], such that $\bar{\mathcal{L}} = TJT^{-1}$ with a matrix $T\in\mathbb{R}^{N\times N}$ and a diagonal matrix $J = diag\left(\lambda_1,\lambda_2,\ldots,\lambda_N\right)\in\mathbb{R}^{N\times N}$. Define a change of coordinates as follows:

$$\begin{aligned}\varphi_1 &= \left(T^{-1}\otimes I_N\right)z_1,\\ \varphi_2 &= \left(T^{-1}\otimes I_N\right)z_2,\\ \phi &= \left(T^{-1}\otimes I_N\right)\omega.\end{aligned} \tag{15}$$

Replacing (15) in (14) leads to:

$$\begin{aligned}\dot{V} =& 2\varphi_1^T\left(I_N\otimes P_1A+J\otimes P_1BK_c\right)\varphi_1\\ &-2\varphi_1^T\left(J\otimes P_1BK_c\right)\varphi_2 + 2\varphi_1^TP_1\phi\\ &+2\varphi_2^T\left(I_N\otimes P_2(A-L_oC)\right)\varphi_2.\end{aligned} \tag{16}$$

By Lemma 4 all the eigenvalues λ_j are positive, then (14) can be rewritten as follows:

$$\begin{aligned}\dot{V} =& \sum_{j=1}^N\varphi_{1_j}^T\mathrm{He}\left\{P_1A+\lambda_jP_1BK_c\right\}\varphi_{1_j}\\ &-2\sum_{j=1}^N\varphi_{1_j}^T\lambda_jP_1BK_c\varphi_{2_j} + 2\sum_{j=1}^N\varphi_{1_j}^TP_1\phi_j\\ &+\sum_{j=1}^N\varphi_{2_j}^T\mathrm{He}\left\{P_2(A-L_oC)\right\}\varphi_{2_j}.\end{aligned} \tag{17}$$

Let $\psi_j = \left[\varphi_{1_j}^T,\varphi_{2_j}^T,\phi_j^T\right]^T$, then (17) is rewritten as follows:

$$\begin{aligned}&\dot{V} = \sum_{j=1}^N\psi_j^T\Theta_j\psi_j,\\ &\Theta_j = \begin{bmatrix}\mathrm{He}\left\{P_1\left(A+\lambda_jP_1BK_c\right)\right\} & -\lambda_jP_1BK_c & P_1\\ * & \mathrm{He}\left\{P_2\left(A-L_oC\right)\right\} & 0\\ * & * & 0\end{bmatrix}.\end{aligned} \tag{18}$$

According to Assumption 4, the following constraint is holding:

$$\omega_i^TN_\omega\omega_i\le\gamma^2\delta_i^TH^TH\delta_i. \tag{19}$$

Then, using the Kronecker product, it is obtained:

$$\omega^T\left(I_N\otimes N_\omega\right)\omega\le\gamma^2\delta^T\left(I_N\otimes H^TH\right)\delta. \tag{20}$$

Replacing (15) in (20) leads to::

$$\phi^T\left(I_N\otimes N_\omega\right)\phi\le\gamma^2\varphi_1^T\left(I_N\otimes H^TH\right)\varphi_1. \tag{21}$$

Similarly, (21) can be rewritten as follows:

$$\sum_{j=1}^N\phi_j^TN_\omega\phi_j\le\sum_{j=1}^N\gamma^2\varphi_{1_j}^TH^TH\varphi_{1_j} \tag{22}$$

which is equivalent to the quadratic inequality

$$\sum_{j=1}^N\psi_j^T\begin{bmatrix}-\gamma^2H^TH & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & N_\omega\end{bmatrix}\psi_j\le 0. \tag{23}$$

When (23) is satisfied, by using the S-procedure [31], then (18) is equivalent to the existence of $P_1 > 0$, $P_2 > 0$, $N_\omega > 0$, and a scalar $\beta > 0$ such that:

$$\begin{aligned} \dot{V} &\leq \sum_{j=1}^{N} \psi_j^T \Omega_j \psi_j, \\ \Omega_j &= \begin{bmatrix} Q_{1_j} & -\lambda_j P_1 B K_c & P_1 \\ * & Q_2 & 0 \\ * & * & -N_\omega \end{bmatrix}, \end{aligned} \tag{24}$$

where $Q_{1_j} = \text{He}\{P_1(A + \lambda_j P_1 B K_c)\} + \beta\gamma^2 H^T H$ and $Q_2 = \text{He}\{P_2(A - L_o C)\}$. If the matrix $\Omega_j < 0, \forall j = 1, 2, \ldots, N$, then $\dot{V} < 0$, which completes the proof. ■

Due to the products between decision variables, condition (12) represents a bilinear matrix inequality, which is much harder to handle computationally due to lack of convexity [32]. The following theorem provides LMI-based conditions in order to design the controller and observer gains.

B. Controller K_c and observer L_o gains synthesis

Theorem 2: The equilibrium state $z = 0$ for the system (11) is asymptotically stable if there exist matrices $\bar{P}_1 > 0$, $P_2 > 0$, $N_\omega > 0$, N_c, M_o, and scalars $\rho > 0$, $\mu > 0$ such that the following holds $\forall j = 1, 2, \ldots, N$:

$$\begin{bmatrix} \mathcal{Q}_{1_j} & 0 & I & \bar{P}_1 H & -\lambda_j B N_c & 0 \\ * & \mathcal{Q}_2 & 0 & 0 & 0 & I \\ * & * & -N_\omega & 0 & 0 & 0 \\ * & * & * & -\rho I & 0 & 0 \\ * & * & * & * & -\mu^{-1}\bar{P}_1 & 0 \\ * & * & * & * & * & -\mu\bar{P}_1 \end{bmatrix} < 0, \tag{25}$$

where $\mathcal{Q}_{1_j} = \text{He}\left\{A\bar{P}_1 + \lambda_j B N_c\right\}$ and $\mathcal{Q}_2 = \text{He}\{P_2 A - M_o C\}$. The controller and observer gains are calculated as $K_c = N_c \bar{P}_1^{-1}$ and $L_o = P_2^{-1} L_o$ respectively.

Proof: Pre- and post-multiplying (12) by $diag\left(P_1^{-1}, I, I\right)$, the following is obtained:

$$\begin{bmatrix} \mathscr{F}_{1_j} & -\lambda_j B K_c & I \\ * & \mathcal{F}_2 & 0 \\ * & * & -N_\omega \end{bmatrix} < 0, \tag{26}$$

where $\bar{P}_1 = P_1^{-1}$, $\mathscr{F}_{1_j} = \text{He}\left\{A\bar{P}_1 + \lambda_j B K_c \bar{P}_1\right\} + \beta\gamma^2 \bar{P}_1 H^T H \bar{P}_1$, and $\mathcal{F}_2 = \text{He}\{P_2(A - L_o C)\}$. Using Lemma 2 and selecting $\rho = (\beta\gamma^2)^{-1}$ (26) becomes:

$$\begin{bmatrix} \mathcal{F}_{1_j} & -\lambda_j B K_c & I & \bar{P}_1 H \\ * & \mathcal{F}_2 & 0 & 0 \\ * & * & -N_\omega & 0 \\ * & * & * & -\rho I \end{bmatrix} < 0. \tag{27}$$

where $\mathcal{F}_{1_j} = \text{He}\left\{A\bar{P}_1 + \lambda_j B K_c \bar{P}_1\right\}$. Note that (27) can be rewritten as follows:

$$\begin{aligned} &\begin{bmatrix} \mathcal{F}_{1_j} & 0 & I & \bar{P}_1 H \\ * & \mathcal{F}_2 & 0 & 0 \\ * & * & -N_\omega & 0 \\ * & * & * & -\rho I \end{bmatrix} \\ &+ \text{He}\left\{ \begin{bmatrix} -\lambda_j B K_c \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix} \right\} < 0. \end{aligned} \tag{28}$$

Applying Lemma 3 in (28), the following inequality is obtained:

$$\begin{aligned} &\begin{bmatrix} \mathcal{F}_{1_j} & 0 & I & \bar{P}_1 H \\ * & \mathcal{F}_2 & 0 & 0 \\ * & * & -N_\omega & 0 \\ * & * & * & -\rho I \end{bmatrix} \\ &+ \mu \begin{bmatrix} -\lambda_j B K_c \\ 0 \\ 0 \\ 0 \end{bmatrix} \bar{P}_1 \begin{bmatrix} -\lambda_j (B K_c)^T & 0 & 0 & 0 \end{bmatrix} \\ &+ \mu^{-1} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} \bar{P}_1^{-1} \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix} < 0, \end{aligned} \tag{29}$$

where $\mu > 0$. Using Lemma 2 in (29), selecting $N_c = K_c \bar{P}_1$ and $M_o = P_2 L_o$, the inequality (25) is obtained, which completes the proof. ■

In order to show the effectiveness of the proposed strategy, the following example presents a comparison between the classical leader-following formation control and the proposed strategy.

V. Example: Application to formation control

According to [3], a fleet of unmanned aerial vehicles (UAVs) can be represented as a double integrator multi-agent system manipulating the angles of each UAV. Let us consider the following second order nonlinear multi agent system:

$$\begin{aligned} \dot{p}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= \bar{u}_i(t), \end{aligned} \tag{30}$$

where $p_i(t), v_i(t), \bar{u}_i(t) \in \mathbb{R}^{n_d}$ are respectively the position, velocity, and acceleration input ($\forall i = 1, 2, \ldots, N$) in an n_d dimensional Euclidean space, and the leader's dynamics are considered:

$$\begin{aligned} \dot{p}_r(t) &= v_r(t), \\ \dot{v}_r(t) &= u_r(t). \end{aligned} \tag{31}$$

Two components are considered in the acceleration input $\bar{u}_i(t) = u_i(t) + \mathbf{w}_i(p_i(t), p_j(t))$ where $u_i(t)$ is the control input and $\mathbf{w}_i(p_i(t), p_j(t)) \in \mathbb{R}^{n_d}$ is a nonlinear input. Let us define h_i and $h_j \in \mathbb{R}^{n_d}$ as the rigid desired-position formation which is considered to be constant. The synchronization error between the agent i and the virtual agent is defined as $\bar{p}_i(t) = p_i(t) - p_r(t)$, $\bar{v}_i(t) = v_i(t) - v_r(t)$, and $\bar{\delta}_i(t) = \left[\bar{p}_i(t)^T - h_i^T, \bar{v}_i(t)^T\right]^T$; thus, the synchronization dynamics (30) and (31) can be represented as follows:

$$\begin{aligned} &\dot{\bar{\delta}}_i(t) = \bar{A}\bar{\delta}_i(t) + \bar{B}u_i(t) + \bar{\omega}_i(p_i(t), p_j(t)), \\ &\bar{y}_i(t) = \bar{C}\bar{\delta}_i(t) \\ &\bar{A} = \begin{bmatrix} 0_{n_d \times n_d} & I_{n_d} \\ 0_{n_d \times n_d} & 0_{n_d \times n_d} \end{bmatrix}, \ \bar{B} = \begin{bmatrix} 0_{n_d \times n_d} \\ I_{n_d} \end{bmatrix}, \\ &\bar{C} = \begin{bmatrix} I_{n_d} & 0_{n_d \times n_d} \end{bmatrix}, \ \bar{\omega}_i(p_i(t), p_j(t)) = \bar{B}\mathbf{w}_i(p_i(t), p_j(t)). \end{aligned} \tag{32}$$

In order to design a formation control based on the leader-following approach developed in this paper, the control (6) is modified as follows:

$$u_i = \bar{K}_c \sum_{j \in \mathcal{N}_i} a_{ij}\left(\hat{\bar{x}}_i - \hat{\bar{x}}_j\right) + \alpha_i \bar{K}_c\left(\hat{\bar{x}}_i - \bar{x}_r\right) + u_r(t). \tag{33}$$

where $\bar{K}_c$ is the robust control gain to be calculated, $\hat{\bar{x}}_i = \left[\hat{\bar{p}}_i(t)^T - h_i^T, \hat{\bar{v}}_i(t)^T\right]^T$ is the estimated state vector of the second-order agent i, $\hat{\bar{x}}_j = \left[\hat{\bar{p}}_j(t)^T - h_j^T, \hat{\bar{v}}_j(t)^T\right]^T$ is the estimated state vector of the second-order agent j, and $\bar{x}_r = \left[p_r(t)^T, v_r(t)^T\right]^T$. The formation control (33) can be rewritten as follows:

$$u_i = \bar{K}_c\left[\sum_{j \in \mathcal{N}_i} a_{ij}\left(\hat{\bar{\delta}}_i - \hat{\bar{\delta}}_j\right)\right] + \alpha_i \bar{K}_c \hat{\bar{\delta}}_i. \tag{34}$$

TABLE I

DESIRED FINAL SHAPE, INITIAL POSITIONS, AND INITIAL VELOCITIES OF EACH AGENT.

Desired positions	Initial position	Initial velocity
$h_1 = [0,0,0]^T$	$p_1(0) = [-4,-1,0]^T$	$v_1(0) = [0,0,0]^T$
$h_2 = [4,0,0,]^T$	$p_2(0) = [-1,1,0]^T$	$v_2(0) = [0,0,0]^T$
$h_3 = [6,2\sqrt{3},0]^T$	$p_3(0) = [-2,-2,0]^T$	$v_3(0) = [0,0,0]^T$
$h_4 = [4,4\sqrt{3},0]^T$	$p_4(0) = [-2,4,0]^T$	$v_4(0) = [0,0,0]^T$
$h_5 = [0,4\sqrt{3},0]^T$	$p_5(0) = [3,2,0]^T$	$v_5(0) = [0,0,0]^T$
$h_6 = [-2,2\sqrt{3},0]^T$	$p_6(0) = [2,3,0]^T$	$v_6(0) = [0,0,0]^T$

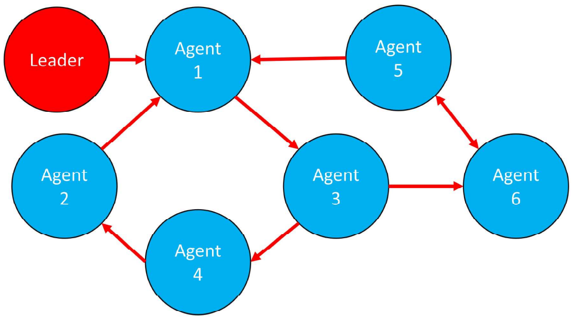

Fig. 1. Communication topology between agents.

After connecting (32) and (34), the closed-loop system can be represented as in (8). The nonlinear input is defined as follows:

$$\mathbf{w}_i(t) = -\frac{b}{\mathcal{L}_{ii}} \sum_{j\in\mathcal{N}_i} a_{ij} \sin\left(\|\mathbf{z}_{ij}\|\right) \left(\frac{\mathbf{z}_{ij}}{\sqrt{1+\|\mathbf{z}_{ij}\|^2}}\right) \tag{35}$$
$$\mathbf{z}_{ij} = p_j(t) - p_i(t) - h_j + h_i,$$

where $b > 0$.

Remark 3: According to [33], [34], [35], [36], [7], the reachable formation has to satisfy the constraints $Ah_i = 0,\ \forall i = 1,2\ldots,N$. By satisfying the above constraints, the controller and observer gains can be calculated with Theorem 2 to solve the leader-following formation control problem using the control law in (33) when Assumption 4 is fulfilled.

LMI in Theorem 2 is solved with the following parameters: $\mu = 0.01$, $H = I$ (H is selected in such a way Assumption 4 is fulfilled), and $\Lambda = diag\,(1,0,0,0,0,0)$. Six agents are considered shaping a hexagon with the parameters in Table I. The leader's control input is $u_r = [0,0,0.5t]^T - v_r(t)$ and for testing the nonlinearity has $b = 4$. The communication topology is shown in Fig. 1 with the following Laplacian matrix:

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}.$$

Two simulations are considered. In Simulation, A the classical formation control as reported in [7] is designed adding the nonlinear input and in Simulation B the proposed strategy is used. In order to calculate the controller and observer gains in Simulation A the separation principle is used obtaining $K_c = -[I_{n_d}, I_{n_d}]$, and $L_o = \left[1.0006 I_{n_d}^T, 0.2504 I_{n_d}^T\right]^T$. In Figs. 2 and 3, the agents' trajectories are plotted in order to show the difference in the final shape of the agents throughout the simulations. Fig. 2 shows the profile of the agents' trajectories in Simulation A and it is clear that the agents cannot reach the desired formation due to the nonlinearity. In contrast, Fig. 3 illustrates how all the agents can achieve the desired formation in spite of the nonlinearity.

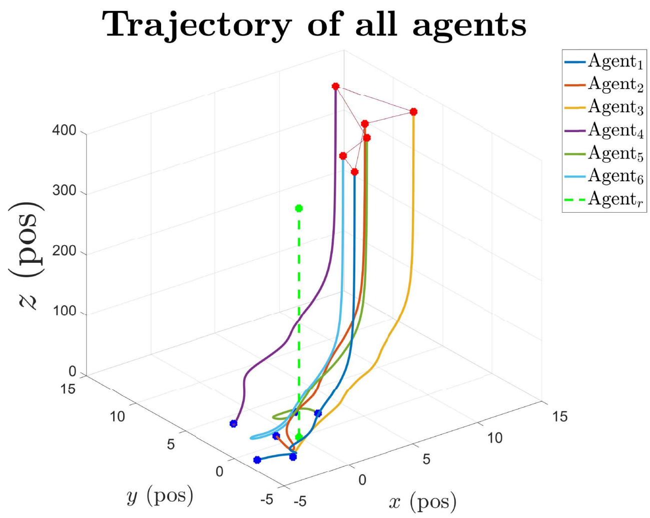

Fig. 2. Profile of the agents' trajectories (Simulation A, classical formation control).

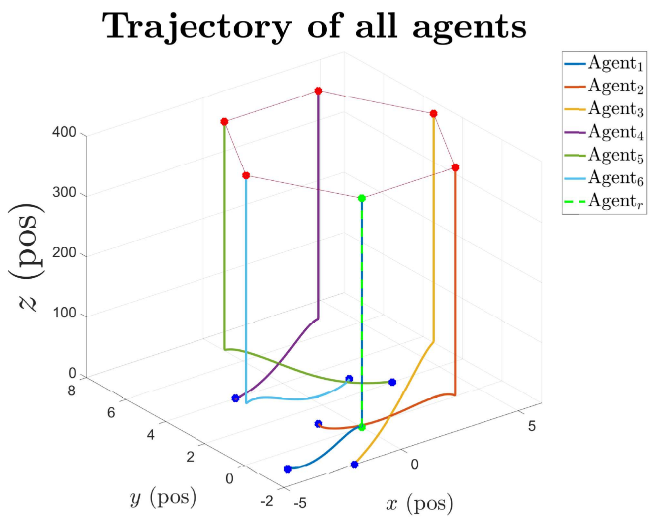

Fig. 3. Profile of the agents' trajectories (Simulation B, proposed strategy).

Figs. 4 and 5 illustrate the performance of the consensus of the final positions between the agent i and its neighbors j. Let us define $d_{ij} = \|p_j - p_i - h_j + h_i\|$. In Simulation A (Fig. 4), since the agents cannot achieve the desired formation, the consensus between the agents is not achieved. In Simulation B (Fig. 5), the agents reach the desired formation despite the nonlinearity and the synchronization error converges to zero.

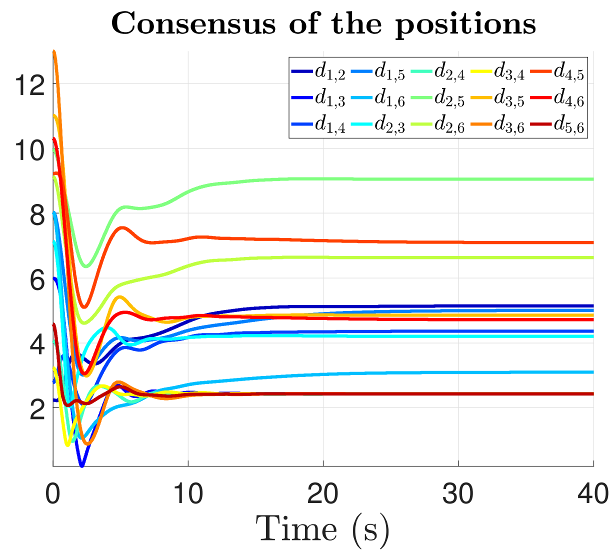

Fig. 4. Consensus of the agents' positions (Simulation A, classical formation control).

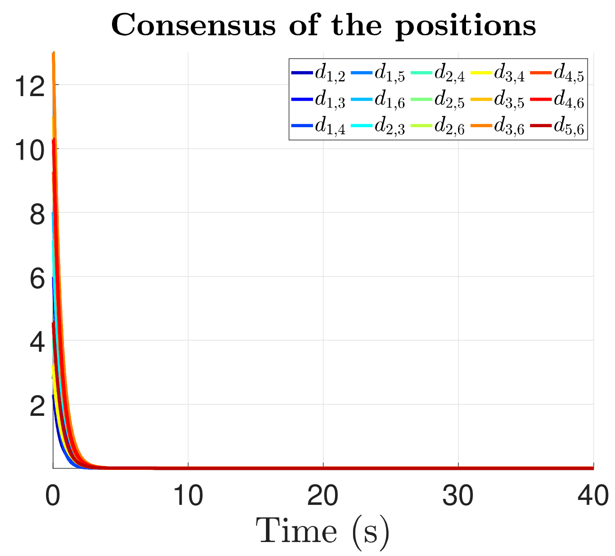

Fig. 5. Consensus of the agents' positions (Simulation B, proposed strategy).

Figs. 6 and 7 illustrate the profile of the difference between the velocities of each agent and the target velocities described by the leader agent. The agents can follow the velocities of the leader agent in both simulations because the objective of the malfunction is only affecting the agents' positions.

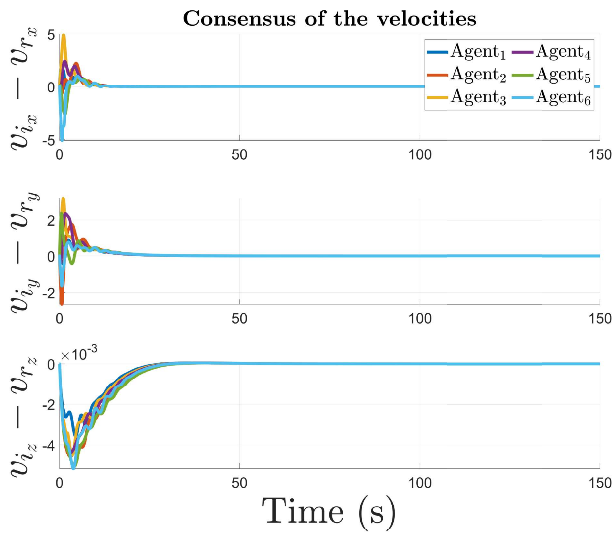

Fig. 6. Consensus of the agents' velocities (Simulation A, classical formation control).

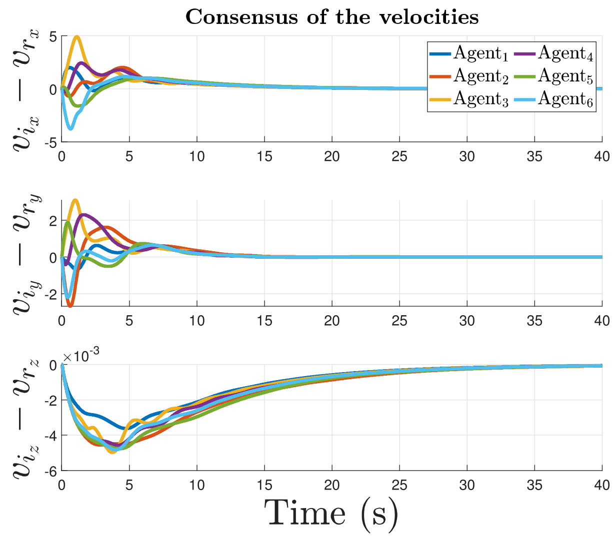

Fig. 7. Velocities of the agents (Simulation B, proposed strategy).

Fig. 8 presents the evaluation of $\omega_i(x_i(t), x_j(t))^T N_\omega \omega_i(x_i(t), x_j(t))$ and $\gamma^2 \delta_i^T H^T H \delta_i$ in order to show that the constraint in Assumption 4 is always fulfilled all over Simulation B. The dotted line represents $\omega_i(x_i(t), x_j(t))^T N_\omega \omega_i(x_i(t), x_j(t))$, whereas the solid line represents $\gamma^2 \delta_i^T H^T H \delta_i$, and different colors are used to denote different agents.

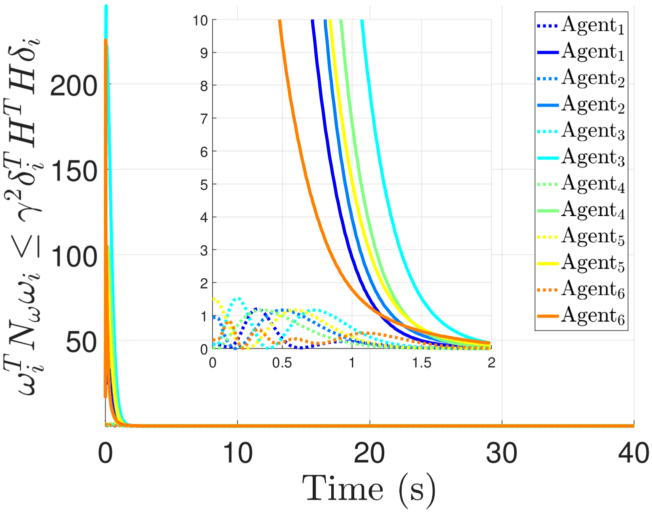

Fig. 8. Profile of the nonlinear input and the upper bound (Simulation B, proposed strategy).

VI. Conclusions

In this paper, the leader-following consensus problem for a class of nonlinear multi-agent system has been studied. Sufficient conditions are established that guarantee the computation of the controller and observer gains in order to calculate a control law such that achieves the control goal. The general strategy has been exemplified using the formation control problem in second-order multi-agent systems adding a nonlinear input. A comparison between two simulations has been presented in order to show the effectiveness of the proposed strategy. In particular, it has been shown that if the Assumptions 1-4 are fulfilled then all the agents converge to the trajectories of the leader agent despite the nonlinear input. As future work, the proposed approach will be tested considering a fleet of UAVs subject to cyber-attacks on an experimental platform.

References

[1] T. Han, Z.-H. Guan, Y. Wu, D.-F. Zheng, X.-H. Zhang, and J.-W. Xiao, "Three-dimensional containment control for multiple unmanned aerial vehicles," *Journal of the Franklin Institute*, vol. 353, no. 13, pp. 2929–2942, 2016.

[2] J. Wang, "Time-varying formation of second-order discrete-time multi-agent systems under non-uniform communication delays and switching topology with application to UAV formation flying," *IET Control Theory & Applications*, vol. 14, pp. 1947–1956(9), September 2020.

[3] J. F. Guerrero-Castellanos, A. Vega-Alonzo, N. Marchand, S. Durand, J. Linares-Flores, and G. Mino-Aguilar, "Real-time event-based formation control of a group of VTOL-UAVs," in *2017 3rd International Conference on Event-Based Control, Communication and Signal Processing (EBCCSP)*, May 2017, pp. 1–8.

[4] D. G. Lui, A. Petrillo, and S. Santini, "An optimal distributed PID-like control for the output containment and leader-following of heterogeneous high-order multi-agent systems," *Information Sciences*, vol. 541, pp. 166–184, 2020.

[5] A. Amini, A. Asif, and A. Mohammadi, "Dynamic event-triggered formation control for multi-agent systems: A co-design optimization approach," in *2020 American Control Conference (ACC)*, 2020, pp. 707–712.

[6] Z. Xu, H. Liu, and Y. Liu, "Fixed-time leader-following flocking for nonlinear second-order multi-agent systems," *IEEE Access*, vol. 8, pp. 86 262–86 271, 2020.

[7] Z. Li and Z. Duan, *Cooperative control of multi-agent systems: a consensus region approach*. CRC Press, 2014.

[8] Y. Dong and J. Huang, "Leader-following rendezvous with connectivity preservation of single-integrator multi-agent systems," *IEEE/CAA Journal of Automatica Sinica*, vol. 1, no. 1, pp. 19–23, Jan 2014.

[9] Y. Quan, W. Chen, Z. Wu, and L. Peng, "Distributed fault detection for second-order delayed multi-agent systems with adversaries," *IEEE Access*, vol. 5, pp. 16 478–16 483, 2017.

[10] Y. Hua, X. Dong, Q. Li, and Z. Ren, "Distributed fault-tolerant time-varying formation control for high-order linear multi-agent systems with actuator failures," *ISA Transactions*, vol. 71, no. Part 1, pp. 40 – 50, 2017, special issue on Distributed Coordination Control for Multi-Agent Systems in Engineering Applications.

[11] Z. Zhang, F. Hao, L. Zhang, and L. Wang, "Consensus of linear multi-agent systems via event-triggered control," *International Journal of Control*, vol. 87, no. 6, pp. 1243–1251, 2014.

[12] K. Li, C. Hua, X. You, and X. Guan, "Finite-time observer-based leader-following consensus for nonlinear multiagent systems with input delays," *IEEE Transactions on Cybernetics*, pp. 1–9, 2020.

[13] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, Jan 2007.

[14] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, Sept 2004.

[15] M. He, J. Mu, and X. Mu, "$\mathcal{H}_\infty$ leader-following consensus of nonlinear multi-agent systems under semi-Markovian switching topologies with partially unknown transition rates," *Information Sciences*, vol. 513, pp. 168 – 179, 2020.

[16] J. A. Vazquez Trejo, D. Rotondo, M. Adam Medina, and D. Theilliol, "Observer-based event-triggered model reference control for multi-agent systems," in *2020 International Conference on Unmanned Aircraft Systems (ICUAS)*, 2020, pp. 421–428.

[17] A. Belkadi, L. Ciarletta, and D. Theilliol, "Particle swarm optimization method for the control of a fleet of unmanned aerial vehicles," *Journal of Physics: Conference Series*, vol. 659, p. 012015, nov 2015.

[18] M. Ille and T. Namerikawa, "Collision avoidance between multi-UAV-systems considering formation control using MPC," in *2017 IEEE International Conference on Advanced Intelligent Mechatronics (AIM)*, July 2017, pp. 651–656.

[19] T. Xiong, Z. Pu, and J. Yi, "Time-varying formation finite-time tracking control for multi-UAV systems under jointly connected topologies," *International Journal of Intelligent Computing and Cybernetics*, 2017.

[20] W. Zou and Z. Xiang, "Event-triggered leader–following consensus of non-linear multi-agent systems with switched dynamics," *IET Control Theory & Applications*, vol. 13, pp. 1222–1228(6), 2019.

[21] M.-C. Fan and Y. Wu, "Global leader-following consensus of nonlinear multi-agent systems with unknown control directions and unknown external disturbances," *Applied Mathematics and Computation*, vol. 331, pp. 274 – 286, 2018.

[22] X. You, C. Hua, and X. Guan, "Event-triggered leader-following consensus for nonlinear multiagent systems subject to actuator saturation using dynamic output feedback method," *IEEE Transactions on Automatic Control*, vol. 63, no. 12, pp. 4391–4396, 2018.

[23] L. Ding and G. Guo, "Sampled-data leader-following consensus for nonlinear multi-agent systems with Markovian switching topologies and communication delay," *Journal of the Franklin Institute*, vol. 352, no. 1, pp. 369 – 383, 2015.

[24] A. N. Langville and W. J. Stewart, "The Kronecker product and stochastic automata networks," *Journal of Computational and Applied Mathematics*, vol. 167, no. 2, pp. 429 – 447, 2004.

[25] K. Zhou and J. C. Doyle, *Essentials of robust control*. Prentice hall Upper Saddle River, NJ, 1998, vol. 104.

[26] H. Kheloufi, A. Zemouche, F. Bedouhene, and M. Boutayeb, "On LMI conditions to design observer-based controllers for linear systems with parameter uncertainties," *Automatica*, vol. 49, no. 12, pp. 3700 – 3704, 2013.

[27] A. Mustafa and H. Modares, "Attack analysis and resilient control design for discrete-time distributed multi-agent systems," *IEEE Robotics and Automation Letters*, vol. 5, no. 2, pp. 369–376, 2020.

[28] C. Chen, F. L. Lewis, S. Xie, H. Modares, Z. Liu, S. Zuo, and A. Davoudi, "Resilient adaptive and $\mathcal{H}_\infty$ controls of multi-agent systems under sensor and actuator faults," *Automatica*, vol. 102, pp. 19 – 26, 2019.

[29] W. Ni and D. Cheng, "Leader-following consensus of multi-agent systems under fixed and switching topologies," *Systems & Control Letters*, vol. 59, no. 3, pp. 209 – 217, 2010.

[30] J. Chen, W. Zhang, Y. Y. Cao, and H. Chu, "Observer-based consensus control against actuator faults for linear parameter-varying multiagent systems," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 47, no. 7, pp. 1336–1347, July 2017.

[31] A. Zecevic and D. D. Siljak, *Control of complex systems: Structural constraints and uncertainty*. Springer, 2010.

[32] D. Rotondo, H. Sánchez, F. Nejjari, and V. Puig, "Análisis y diseño de sistemas lineales con parámetros variantes utilizando LMIs," *Revista Iberoamericana de Automática e Informática industrial*, vol. 16, no. 1, pp. 1–14, 2019.

[33] G. Lafferriere, A. Williams, J. Caughman, and J. Veerman, "Decentralized control of vehicle formations," *Systems & Control Letters*, vol. 54, no. 9, pp. 899 – 910, 2005.

[34] J. Rodrigues, D. Figueira, C. Neves, and I. Ribeiro, "Leader-following graph-based distributed formation control," in *Proc. of Robotica 2008-8th Conference on Autonomous Robot Systems and Competitions*, vol. 95, 2008.

[35] Z. Li, "Dynamic consensus of linear multi-agent systems," *IET Control Theory & Applications*, vol. 5, pp. 19–28(9), January 2011.

[36] C. Ma and J. Zhang, "On formability of linear continuous-time multi-agent systems," *Journal of Systems Science and Complexity*, vol. 25, no. 1, pp. 13–29, 2012.