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Bringing Nordic mathematics education into the future

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Preface

This volume presents Nordic mathematics education research, which will be presented at the Ninth Nordic Conference on Mathematics Education, NORMA 20, in Oslo, Norway, in June 2021. The theme of NORMA 20 regards what it takes or means to bring Nordic mathematics education into the future, highlighting that mathematics education is continuous and represents stability just as much as change.

NORMA conferences are always organized in collaboration with the Nordic Society for Research in Mathematics Education (NoRME). NoRME is open to membership from national societies for research in mathematics education in the Nordic and Baltic countries.

Inclusive classrooms and "mathematics education for all" have traditionally been at the core of Nordic mathematics education. Currently, the digital development and possibilities for individualized learning activities widen the understanding of adaption in compulsory education. This push and pull between inclusion and adaption bring the possibility of renewing mathematics education, including pre-school and tertiary levels, while still maintaining the principle of student-centred mathematics education. Mathematics education is also changing at the level of teacher education, which is reflected in the conference papers included in this preceeding.

The International Programme Committee (IPC) of NORMA 20 represents all Nordic countries and includes one representative from the Baltic countries, with a mix of junior and senior researchers. The IPC has organized the submission and review process leading to this volume. The members of the IPC were:

- Guri A. Nortvedt University of Oslo (Chair), Norway
- Nils Buchholtz, University of Oslo, Norway, and University of Cologne, Germany
- Janne Fauskanger, University of Stavanger, Norway
- Freyja Hreinsdóttir, University of Iceland, Iceland
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- Jelena Radišić, University of Oslo, Norway
- Anna Wernberg, Malmö University, Sweden

The first NORMA conference on mathematics education, NORMA 94, was held in Lahti, Finland, in 1994. Four years later, the conference was held in Kristiansand, Norway; since then, it has taken place every third year. After each conference, selected papers are published in a proceeding. Due to the

COVID-19 pandemic, the NORMA 20 conference was postponed until 2021; however, many conference papers were in progress and authors were given the opportunity to continue working on them within the original planned timespan. Traditionally, papers are presented at the conference, allowing the authors to receive feedback that is valuable towards finalizing the paper. Instead, the authors have used two rounds of reviewer feedback to substantially improve their papers. In this process, the NORMA community established in 1995, together with external reviewers who are experts in the different fields studied and presented in the papers, have played an important role in producing the Preceeding.

We believe that the NORMA 20 Preceeding is the first conference preceeding to be published, containing 36 papers from authors representing six countries.

After the conference, a traditional Proceeding will be published, containing papers written by submitting authors who decided to wait until after the conference to finalise their papers, to take advantage of feedback from both conference participants and reviewers when they revise their papers.

The IPC would like to extend our thanks to all authors and reviewers for their efforts towards this volume.

Oslo, January 2021, on behalf of the IPC

Guri A. Nortvedt

Contents

Realization of the mathematical signifier 25 × 12

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This paper identifies how a mathematical signifier "25 × 12" is realized through the practice of multiple solution strategies when teaching multiplication. This case study is conducted in a fifth-grade classroom where the teacher employs a context-based explorative teaching approach. Analyses of data from teaching sessions show that the teacher prioritizes effective mathematical communication through representations, gestures and visual mediators. The result shows that the students attained a set of mathematical realizations of the signifier by examining the discursive equivalence established through mathematical communication, gestures, multiple connections and use of visual mediators in the discourse. Possible implications of these findings are discussed.

Keywords: Signifiers and realizations, mathematical communication, visual mediators, gestures, multiplication

Introduction

For some decades, researchers in mathematics education have worked extensively to develop concepts, frameworks and theories to enhance teaching-learning activities (Adler & Ronda, 2015; Ball, Thames, & Phelps, 2008; Radford, 2003; Sfard, 2007, 2008). The recent years have witnessed that a discourse perspective for making sense of classroom processes has evolved and earned major attention (Adler & Ronda, 2015; Adler & Sfard, 2016; Sfard, 2008). A discourse perspective, related to an interpretive framework for the study of learning is called commognitive, combining the terms cognitive and communicational (Sfard, 2007). In a commognitive framework, Sfard (2007) considers that a mathematical discourse consists of four interrelated discursive features: word use, visual mediators, routines and narratives.

Multiplication is one of the four fundamental operations in arithmetic and mastering it is important to build confidence for higher-level mathematics. For some decades, research has highlighted four models that influence students' understanding of multiplication: equal groups, (rectangular) arrays, rectangular area and multiplicative comparison (for more details, see Greer, 1992). In a Nordic context, a recent Swedish study has investigated two students' multiplicative understanding for multi-digits and decimals (Larsson, Pettersson, & Andrews, 2017). Individual interviews were conducted over five semesters during grades 5-7. The authors found that the two students revealed a robust conceptualization of multiplication as equal groups, but they had difficulties in connecting calculations to models for multiplication.

Developing inaccurate and inefficient counting methods and encountering problems in learning multiplication tables may promote mathematics difficulties (Wong & Evans, 2007). While different models have been adopted in teaching multiplication, these authors suggest that the "use of concrete materials, pictures, diagrams, and discussion increases students' familiarity with the process" (Wong & Evens, 2007, p. 89). This viewpoint agrees with Sfard's commognitive framework which considers visual mediators as "visible objects that are operated upon as a part of the process of communication" (Sfard, 2008, p. 133). Arithmetic deals basically with calculations involving numbers. Multiplication is one of such calculations that is governed by well-framed procedures. Sfard (2008) considers such calculations “as a discursive sequence built according to well-defined rules that, once uttered or written, counts as a confirmation of the discursive equivalence of two numerical expressions” (p. 52).

In the present study, we consider the two numerical expressions, twenty-five times twelve (or, in written symbolic form, 25 × 12) and three hundred (300). These two expressions are discursive equivalents. Discursive equivalence means that we can use these two numerical expressions interchangeably for any communicational purpose. While we try to objectify arithmetical discourse, thereby multiplication, we “need to account for the fact that two different symbolic or verbal strings, count as exchangeable” (Sfard, 2008, p. 52). 25 × 12 is a signifier, which is a primary object, used in communication. Signifiers are words or symbols that work as nouns in communication, and a single signifier may have several realizations. Sfard (2008, p. 301) defines a realization as "a procedure that pairs a signifier with another primary object or the product of such procedure". She suggests that the process of achieving the realization in question can be visual (that includes verbal, iconic, concrete and gestural) and vocal.

The major focus of the present study is to explore mathematics communication and paint generic pictures of processes and acts like visual mediators and gestures that arise from mathematical discourses in a classroom. To be precise, we will address the following research question: How is the mathematical signifier 25 × 12 realized in a discursive classroom when teaching multiplication in grade 5 in a Norwegian school? To seek an answer to this question, we will analyze two episodes that focus on the use of examples in teaching multiplication to identify students’ signs of discursive development. A particular focus will be made on the realizations of the mathematical signifier 25 × 12.

Methodology

This study is a part of a project conducted by the Mathematics Education Research Group (MERG)-2018 at the University of Stavanger. The project was conducted in two fifth-grade classrooms. The mathematics teacher of these classes employed a context-based explorative teaching method. The primary goal of the project was to focus on the mathematical discourse developed in the classroom during the teaching-learning process. The present study focuses on how a signifier 25 × 12 was manipulated to achieve its realizations. It has used the qualitative approach with observation, exploration, and interpretation as tools for understanding these discourses. Kieran, Forman, and Sfard (2001, p. 1) see a clear shift from the classical *background-method-sample-finding-discussion* structure to "a distinct and relatively new type of research" in mathematics education which they term as a discursive approach. As a part of this approach and using the commognitive perspective (Sfard 2007, 2008) as our analytical framework, we will focus on the student interactions in small groups, their two-way communication with the teacher and the shift in the discourses as they are challenged with the mathematical tasks.

The empirical material of the MERG project includes observations of 16 teaching sessions captured as video recordings in which all sessions were transcribed. For this study, we have selected one teaching session in which the students applied different solution methods to find the product of two numbers. The task, written on the whiteboard, was to compute 25 × 12. The teacher asked her pupils to compute this with different methods. From this teaching session, we have selected two different episodes that showed a clear shift in the discourse (Adler & Ronda, 2015). This shift could be a task change, a shift in focus from one student group to another one, or a shift in the teacher's presentation. Here, it implies a shift in the multiplication method. These two episodes were then thoroughly analyzed, using the commognitive framework with a focus on visual mediators, revealing the realization of the mathematical signifier embedded in those episodes. As we chose to focus on two episodes of a specific teaching session from 16 such sessions, owing to our methodological choice, we might have missed the opportunity to take into account the mathematical discourses that were developed in other teaching sessions. However, the selected episodes illustrate the specific realizations of the mathematical signifier 25 × 12.

Analyses and results

The first episode accounts for gestures, speech and multiple connections of the realizations, while the second episode focuses on the role of visual mediators in achieving the realizations of the signifier. The analyses will focus on the subjects of the discourse (teacher and/or student/s), but also on the signifier 25 × 12, searching for the signs of realizations of the signifier in the tasks and the utterances within or across the episodes. These realizations are achieved through different activities. Both the teacher and the students use symbolic representations across the episodes to perform the mathematical operations. Although the students uniformly use a dot (·) to represent multiplication, the teacher often uses a cross (×) while she uses a dot (·) sometimes. Despite this ambiguity, no signs of confusion were noticed with the students.

Episode 1. Gestures, speech and multiple connections

Sequence 1: The interplay between gestures and speech

The teacher is here visiting two boys, Tor and Peter who have formed a group. The dialogue shows a verbal conversation between the teacher and Peter since Tor was silent here. The teacher is looking at Peter's notebook, following Peter's strategy of solving the multiplication problem.

069	Teacher:	25 times 4 plus 25 times 4 plus 25 times 4 (Moves finger pointing to each term of 25 · 4 + 25 · 4 + 25 · 4 that the student has written)
070	Tor:	[Yeah]
	Peter:	[Yeah]
071	Teacher:	Yeah. (3s) Why should (unknown text) times 12 there? (Points to what the student has written from term to term)
072	Peter:	(Unknown text) (Shows something in the exercise book)
073	Teacher:	Umm (.) This is not multiplied like this here, 25 times 4 plus 25 times 4 plus 25 times 4. Whether you can think of a different way of writing this? (Moves index finger steadily from left to right and then from right to left pointing 25 · 4 + 25 · 4 + 25 · 4)
074	Peter:	25 times 4 times 3 (Looks at the teacher)
		In the conversation that followed, the teacher asks Peter to write what he said. He writes 25 · 4 · 3 and the teacher asks if this could be written differently.
080	Teacher:	Can this be written as a different multiplication than this? Only 2 factors. These here are 2 factors, and these here are 3 factors. (2s). Could it be thought that there is a third way to write the same multiplication? (Peter looks into his book)

The teacher's utterances (069, 073) are the examples of the use of mathematical objects in the discourse. Although the task is not a descriptive multiplication simplified with the use of the symbols, the correct interpretation of the operation (multiplication here) with the symbols gives the first impression of the use of mathematical objects. As the teacher explains the procedure, she moves her index finger from left to right, pointing each term Peter has written (see Figure 1). Peter looks attentively at what is pointed. The student has written 25 · 4 + 25 · 4 + 25 · 4.

The teacher (073) utters, *this is not multiplied like this here, 25 times 4 plus 25 times 4 plus 25 times 4* as she moves her index finger steadily from left to right and then from right to left again pointing to the expression 25 · 4 + 25 · 4 + 25 · 4, indicating that she was talking about the whole expression and not only the specific terms. Peter had written the expression correctly but had made a mistake in his calculation. After the explanation from the teacher, he makes correction in his calculation. The teacher (080) refers to factors of 300, expressing and pointing that *these here are two factors, and these here are three factors*. The pointing gestures the teacher used were coordinated with the speech.

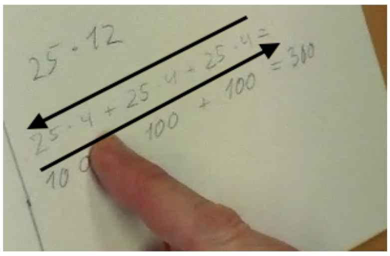

Figure 1: The teacher moves her index finger steadily from left to right and then from right to left again pointing the expression.

Sequence 2: Establishing discursive equivalence through multiple connections

The teacher goes around and interacts with her students about their methods. A student (Mia) has performed the multiplication as $25 \cdot 10 + 25 \cdot 2 = 300$. In the communication that followed, the teacher refers to this as Mia's method.

140	Per:	First, it is 10 times 12 (which) is 120. And then it is 10 times 12 again. It is 120. 120 plus 120, it becomes well 240. And then we have what: (.)5 again. Then we have 5 times 12 (.) which becomes well thus 60. 60 plus 240 becomes 300.
141	Teacher:	Was it more difficult than Mia's (method)?
142	Per:	Can (be) more difficult than Mia's (method).
143	Teacher:	How is it (the solution) different from Mia's and how is it similar to Mia's?
144	Per:	We split differently. For example, I split 25 while she split 12.

In this sequence, the teacher is moving a step ahead. She wants to compare the methods (141) employed by two students: *How is it different from Mia's and how is it similar to Mia's*? (143). Per answers that they split the rectangle differently (144). He wrote 25 (the length of the rectangle) as 10 + 10 + 5 and multiplied each number with 12 (the breadth of the rectangle). He, finally, summed up the products to get 300 (140). Mia, on the other hand, split 12 (the breadth of the rectangle) into 2 numbers as 10 + 2 and multiplied each number with 25 (the length of the rectangle). She then summed up the products to get 300.

The correct symbolic representations for multiplication of the factors are the manipulations of mathematical signifiers, i.e., symbols. The teacher talks about four different symbolic strings: 25×10, 25 × 2, 25 × 12 and 300 and asks if the students consider that the first two together are the same as the third and/or the fourth. Making use of students' solutions, the teacher tries to make multiple connections and account for the fact that 25 × 12 and 300 are discursive equivalents and thus, count as exchangeable.

Episode 2. Visual mediators

In episode 2, the teacher and a student (Mads) both use visual mediators to demonstrate the multiplication of 25 × 12.

246	Teacher:	So, you explain how you cut the grid, Mads!
247	Mads:	How I split? (Low voice)
248	Teacher:	You did it yesterday
249	Mads:	Yeah:: I thought 4 grids (Looks at the teacher)
250	Teacher:	(Unknown text) (Very low voice)
251	Mads:	Um: instead of 25 times 12, I took 50 times 6 (Draws the figure below the one drawn by the teacher (25 · 12) and plots 50 times 6)

The dialogue illustrates that the teacher invites Mads to present his solution (246, 248). Mads manipulates the 25 × 12 rectangle by doubling the length and halving the width while keeping the area intact to make a 50 × 6 rectangle (251). Though not included in the dialogue here, Mads goes one step further and makes a 100 × 3 rectangle (refer 248). He explains and performs the manipulation. The iconic visual mediator (drawing of the rectangle here), whose concrete equivalent could be a grid or other 3D objects that could be manipulated to fit into the operation, seems to be an important tool in order for Mads to come up with his solution (see Figure 2).

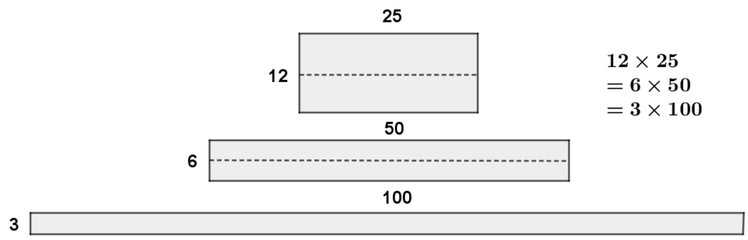

Figure 2: A 25 × 12 rectangle manipulated to 50 × 6 and then to 100 × 3.

Presenting some of the discourses developed in a mathematics teaching session across the two episodes, we have given an insight into a discursive classroom. Figure 3 shows different types of representations that the students used to multiply 25 by 12 (see for example: 069, 080, 251). They have used both symbolic and iconic representations. These different representations are realizations of a single mathematical signifier 25 × 12. Acquiring an abstract mathematical realization of a signifier is not an easy task. But realizing the principal signifier (here 25 × 12) can be made easier by examining the relation of developing an iconic representation using visual mediators which are the grids or the rectangles drawn in students' notebook or on the board in our case here (Heyd-Metzuyanim & Sfard, 2012).

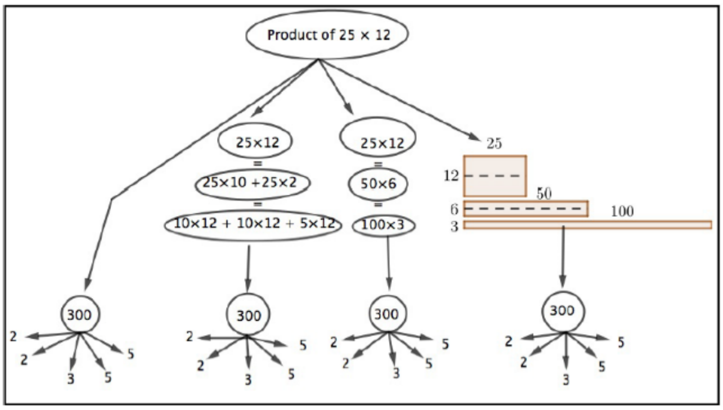

Figure 3: A realization tree of the signifier 25 × 12. The concept of the sketch is derived from Sfard (2008, p. 165)

When we organize these realizations as shown in Figure 3, we tend to construct a tree of realizations. With effective employment of verbal, gestural, iconic and concrete communication, we achieve a series of mathematical realizations. The realization tree that we have achieved by using the solution procedures (each realization) of the students can be extended further. For example, finding the factors of 300 (= 2 × 2 × 3 × 5 × 5) in the third row, we have extended the tree. In other words, "any mathematical realization may be used as a signifier and then realized even further" (Sfard, 2008, p. 165).

Concluding discussion

The teacher provided her students the opportunity to compute 25 × 12 in different ways in this teaching session. Supplementing the pupils' example models with explanation, she showed how different methods could be used to multiply 25 by 12. Unlike as noted by Zodik and Zaslavsky (2008), the teacher did not use the example models just for the sake of using them. Except for some direct calculations, she adopted explorative teaching and invited her students to the mathematical discourse. Adler and Sfard (2016) argue that only explorative mathematics is a tool for life. It is only in such explorative and interactive classrooms the students can establish generalizations, objectify mathematics and attain mathematical realizations as assumed to have been achieved by the students of this teacher. Using example models and performing the operations by splitting 25 and 12 appropriately, the students got the opportunity to compare the procedures. They also got the opportunity of using procedural knowledge, apply it and make and multiple connections. This formed a means to achieve realizations of the signifier 25 × 12.

By providing some perspectives of a discursive mathematics classroom of a Norwegian primary school, the results from this study contribute to the literature about teaching practices in mathematics (Erath, Prediger, Quasthoff, & Heller, 2018; Shinno, 2018; Williams, 2016). The primary focus of the present study was to analyze how the use of the examples (and example solutions) could be used for acquiring the realizations of a signifier in a discursive mathematics lesson. The results show that the practical use of example solutions is instrumental in developing an understanding of mathematical processes in a general sense. More specifically, it can be pointed out that the students got an opportunity to manipulate the symbols in the operations which acted as a signifier. The students also got the opportunity of using symbolic and iconic visual mediators, thus developing a visual realization. As seen in the discourse, the interplay between gestures and speech (effective communication) played an important role in supporting students' understanding. The vocal mathematical communication signified the importance of mathematical language for formal mathematical discourse. Summing up, by combining the realizations of the signifier evident from the students' task, a realization tree of signifier could be drawn.

Since, students tend to depend on superficial mathematical reasoning, discursive practice in mathematics teaching is often challenging. Supported by the presented result and the discussion that followed, we consider that the teacher can help the students attain realizations of a mathematical signifier by: 1) converting the abstract mathematical concepts (of multiplication) to concrete numerical strings, 2) using visual mediators like drawings, grids, gestures, and 3) establishing generalization through exemplification and associated explanation. It would be worthwhile to see how the use of colloquial language might enhance the process of attaining realizations of a mathematical signifier. It will be meaningful to consider this question for future research. Not standing totally against the colloquial language, we stand in line with Sfard (2008) and argue that it is important to develop mathematical concepts whenever it is possible as these, which are the part of the formal discourse, signify learning mathematics.

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