Properties of Paley–Wiener spaces: sampling sets, contractions, and geometry of the unit ball

by

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Thesis submitted in fulfilment of
the requirements for the degree of
PHILOSOPHIAE DOCTOR
(PhD)

University
of Stavanger

Faculty of Science and Technology
Department of Mathematics and Physics
2023

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ISBN: 978-82-8439-153-3
ISSN: 1890-1387
PhD: Thesis UiS No. 689

Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of Philosophiae Doctor (Ph.D.) at the University of Stavanger, Faculty of Science and Technology, Norway. The research has been carried out at the University of Stavanger from August 2019 to December 2022.

I am very much grateful to my supervisor Alexander Ulanovskii for his guidance and continuous support. During the three years I spent in Norway, I was constantly surrounded by a very safe and friendly atmosphere created by Alexander. He was very generous sharing his ideas and made an invaluable contribution to my research, particularly in four papers in this Thesis, and the development of my scientific career. Alexander has greatly influenced my views both in mathematics and in life.

I am also grateful to my co-advisor Sergei V. Kislyakov, who, in particular, lectured me on Mathematical Analysis in Saint-Petersburg and introduced to me the world of Harmonic Analysis.

I am deeply grateful to Aleksei Kulikov, with whom we worked together on two papers contained in this Thesis. In particular, Aleksei came up with an elegant proof of the theorem on uniqueness sets of lacunary polynomials which made it possible to obtain the main results in Paper V in the final form. Many thanks to Kristian Seip, who organized my research visit to NTNU and provided me with an opportunity to work with Aleksei.

I would like to thank Alexander Rashkovskii for his significant contribution to the development of the paper on the mobile sampling problem.

I would like to thank the University of Stavanger, especially, the Department of Mathematics and Physics for a warm and friendly environment.

Finally, I would like to thank my parents, sister, and my girlfriend for their love, continued support, and patience.

Ilia Zlotnikov
Stavanger, December 2022

Abstract

This Thesis is based on five papers, four of which are published and one has been submitted for publication. For convenience of the reader, we also include one chapter that contains a brief overview of the results, some relevant background material, and comments on methods used in the proofs.

The Thesis studies several problems from modern Harmonic analysis. More precisely, we focus on studying various properties of Paley-Wiener spaces: sampling sets, extreme and exposed points of the unit ball, and contraction operators.

List of papers

Paper I

Alexander Rashkovskii, Alexander Ulanovskii, Ilya Zlotnikov, (2023).
On 2-dimensional mobile sampling.
Applied and Computational Harmonic Analysis 62, 1-23,
DOI: 10.1016/j.acha.2022.08.001, preprint: arxiv.org/abs/2005.11193

Paper II

Alexander Ulanovskii, Ilya Zlotnikov, (2021).
Reconstruction of bandlimited functions from space-time samples.
Journal of Functional Analysis 280 (9), 108962, 1-14,
DOI: 10.1016/j.jfa.2021.108962, preprint: arxiv.org/abs/2007.11366

Paper III

Alexander Ulanovskii, Ilya Zlotnikov, (2022).
On geometry of the unit ball of Paley-Wiener space over two symmetric intervals.
In press in International Mathematics Research Notices,
DOI: 10.1093/imrn/rnac043, preprint: arxiv.org/abs/2108.08093

Paper IV

Aleksei Kulikov, Ilya Zlotnikov, (2022).
Contractive projections in Paley-Wiener spaces.
In press in Proceedings of the American Mathematical Society,
DOI: 10.1090/proc/16336, preprint: https://arxiv.org/abs/2207.09278

Paper V

Aleksei Kulikov, Alexander Ulanovskii, Ilya Zlotnikov, (2022).
Completeness of Certain Exponential Systems and Zeros of Lacunary Polynomials.
Submitted to journal,
preprint: https://arxiv.org/abs/2210.00504

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Appendix

1 Introduction

1.1 Paley-Wiener spaces

The spaces of bandlimited functions will play the central role in this Thesis. Given a compact set $\Omega, \Omega \subset \mathbb{R}^d$, the Bernstein space B_Ω consists of all continuous bounded functions in $\mathbb{R}^d$, which are the inverse Fourier transform[1] of tempered distributions supported by Ω. Equipped with uniform norm $\|\cdot\|_\infty$, B_Ω is a Banach space. We introduce the Paley–Wiener space PW^p_Ω, $1 \le p < \infty$, by

$$PW^p_\Omega = B_\Omega \cap L^p(\mathbb{R}^d)$$

and endow it with L^p-norm $\|\cdot\|_p$. In case $p = 2$ the space PW^2_Ω is a Hilbert one.

The Paley–Wiener spaces turned out to be very useful in mathematical analysis and applications, particularly, in signal theory and theory of entire functions, see e.g. [25], [19], [16]. For example, the classical Whittaker–Kotelnikov–Shannon theorem states that every function f belonging to the $PW^2_{[-\pi,\pi]}$ admits a unique representation

$$f(x) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin(\pi(x-n))}{\pi(x-n)}.$$

We can interpret this result as follows. Every function from the space $PW^2_{[-\pi,\pi]}$ can be reconstructed from its values measured at integer points. It is very natural to generalize this result and ask when it is possible to recover every function from PW^p_Ω or B_Ω (or some other space) from certain samples (measurements)?

1.2 Sampling problems

The classical sampling problem deals with the stable reconstruction of bandlimited functions from uniformly discrete samples. Let Ω be a

[1]We warn the reader that the normalization of Fourier transform varies in Papers I-V. The author apologizes for this inconvenience.

compact subset of $\mathbb{R}^d$ and $1 \le p < \infty$. Assume that the signal f belongs to the Paley–Wiener space PW_Ω^p or Bernstein space B_Ω. It is well known that to ensure the stable recovery of the signal f from the samples $\{f(\lambda)\}_{\lambda\in\Lambda}$ it suffices to provide the inequalities

$$A\|f\|_p^p \le \sum_{\lambda\in\Lambda} |f(\lambda)|^p \le B\|f\|_p^p, \quad 1 \le p < \infty; \tag{1.1}$$

$$\|f\|_\infty \le K \sup_{\lambda\in\Lambda} |f(\lambda)|, \quad p = \infty, \tag{1.2}$$

with constants A, B, K independent of f.

In the one-dimensional setting and $\Omega = [-\sigma, \sigma]$, the sampling problem was completely solved by Beurling [5] for the Bernstein space and by Ortega-Cerda and Seip [20] for the Paley-Wiener ones. Beurling's theorem relates the certain density of the set of measurements with the size (the length) of the spectrum Ω.

In the multi-dimensional setting, a full description of the sampling sets is unknown. There is a gap between the necessary condition given by Landau in [14] (see also [18]) and the sufficient condition established by Beurling in [4]. The main obstacle in higher dimensions is that the zero sets of analytic functions are not discrete.

In applications, different sampling methods appear. Below, we will focus on mobile and dynamical (space-time) sampling.

1.3 Mobile sampling (Paper I)

The mobile sampling problem deals with the recovery of the signal from the measurements taken along some trajectories, i.e. we know the values of the function on a union of continuous paths. In this setting, the inequalities (1.1) and (1.2) are replaced by the estimates

$$A\|f\|_p^p \le \int_P |f(\mathbf{u})|^p \, ds \le B\|f\|_p^p, \quad \text{for every } f \in PW_\Omega^p, \tag{1.3}$$

$$\|f\|_\infty \le K \sup_{\mathbf{x}\in P} |f(\mathbf{x})|, \quad \text{for every } f \in B_\Omega, \tag{1.4}$$

where P is a union of locally rectifiable curves and we integrate with respect to arc length. Of course, as usual, the constants A, B, K do not depend on the function f. We will say that if for the trajectory P the conditions (1.3) and (1.4) are satisfied for some space PW^p_Ω or B_Ω then P is a mobile sampling set for the corresponding space.

In what follows, we will focus on the dimension 2 and consider the planar trajectories. The mobile sampling problem was previously investigated in a number of papers, see [3], [12], [26], [27] and references therein. We also note that mobile sampling theory has nice applications, in particular, in magnetic resonance imaging, where the anatomy of a person is captured by moving sensors, see e.g. [13] and [12].

We mention some relevant results obtained in the papers [3] and [13]. Benedetto and Wu studied the mobile sampling problems on the spiral curves. In particular, by applying Beurling covering theorem, they have established a sufficient condition for the Archimedean spiral

$$A^\eta = \{(\eta\theta\cos(2\pi\theta), \eta\theta\sin(2\pi\theta)) : \quad \theta \geq 0\}$$

to form a set of stable mobile sampling for PW^2_Ω space, i.e. the conditions (1.3) are satisfied. In the recent paper [13], Jaming, Negreira, and Romero introduced a generalization of Archimedean spiral trajectories and called them spiraling curves. For this large collection of trajectories, they provided necessary and sufficient conditions to form a set of mobile sampling. Among other requirements, the assumption of asymptotic equispacing was imposed.

In this Thesis, we contribute to the development of the mobile sampling theory for Paley-Wiener spaces. In Paper I, which is a joint work with A. Rashkovskii and A. Ulanovskii, we studied three types of possible trajectories:

- parallel lines;
- dilations of a convex curve;
- translation of a unit circle.

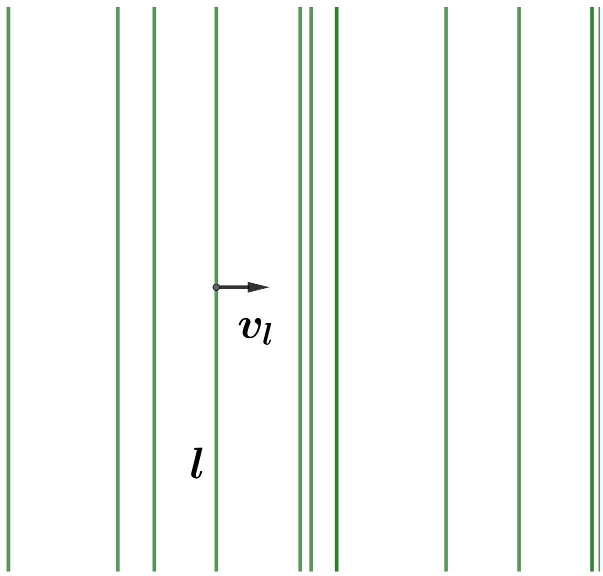
We mention two main features of our results. First, we do not require the trajectories to be uniformly distributed, i.e. we managed to get rid of the equispacing requirement from [13]. Second, we carefully studied the interaction between the spectrum set Ω and the set of trajectories P.

Before passing to the main results, we note that we formulate them for the Bernstein space B_Ω. The transition to the Paley-Wiener space is provided by Theorem 5 (Paper I), which states that solving the mobile sampling problem in these two spaces is almost equivalent (up to an infinitesimally small deformation of the spectra set Ω).

To formulate the main results we need some auxiliary definitions.

Definition 1 *Given any u.d. set $H = \{a_k\} \subset \mathbb{R}$ we denote by $D^-(H)$ its lower uniform density defined by*

$$D^-(H) = \lim_{r\to\infty} \inf_{x\in\mathbb{R}} \frac{\# H \cap (x-r, x+r)}{2r}.$$



For the collection of *parallel lines*, we described the mobile sampling properties in terms of the density of the set of distances between them.

Let $l \in \mathbb{R}^2$ be a straight line through the origin, and let v_l be a unit vector orthogonal to l. Assume that $H = \{a_k\}$ is a set of real numbers. Let P be the collection of translates of l by the values a_k in the given direction v_l, i.e.

$$P = l + Hv_l = \bigcup_{j \in \mathbb{Z}} (l + a_k v_l).$$

The necessary and sufficient for the trajectory P to be a set of mobile sampling is given by the following

Theorem 1 *Set*

$$\Gamma = \Omega - \Omega = \{x - y \,\big|\, x \in \Omega,\ y \in \Omega\}.$$

The set P is a mobile sampling set for B_Ω if and only if

$$D^-(H)v_l \notin \Omega - \Omega.$$

Combining this result and the classical Beurling technique, we managed to solve the mobile sampling problem for the *dilations of a convex curve*.

Let $D \subset \mathbb{R}^2$ be a closed convex set of finite positive measure such that $0 \in \operatorname{Int}(D)$. Denote by ∂D the boundary of D, by $\operatorname{Ext}(D)$ the closed set of extreme points of D and by

$$D^o := \{\mathbf{x} \in \mathbb{R}^2 : \langle \mathbf{x}, \mathbf{y} \rangle \le 1, \mathbf{y} \in D\}$$

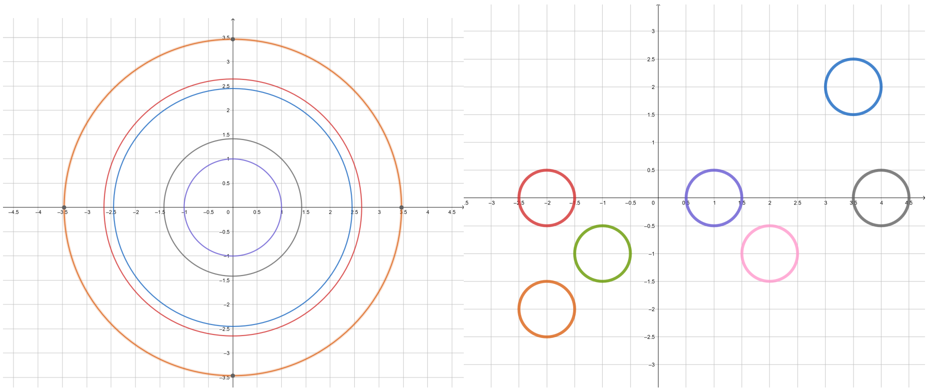
the polar set of D.

Given a u.d. set $Q = \{q_k\} \subset (0, \infty)$, consider the set

$$P = Q\partial D := \bigcup_{k=1}^{\infty} \bigcup_{w \in \partial D} \{q_k w\}. \tag{1.5}$$

Theorem 2 *The set P in* (1.5) *is a mobile sampling set for B_Ω if and only if*

$$D^-(Q \cup (-Q))v \notin \Omega - \Omega, \quad \textit{for every } v \in \operatorname{Ext}(D^o). \tag{1.6}$$



Finally, for the *translates of a circle* we provide the solution to mobile sampling in terms of (two-dimensional lower uniform) density of the sets of translates.

Take any circle $T := \{x \in \mathbb{R}^2 : |x| = r\}, r > 0$. Let $V = \{v_k\}_{k=1}^{\infty} \subset \mathbb{R}^2$ be a u.d. set. Set

$$P = V + T := \bigcup_{k=1}^{\infty} (v_k + T). \tag{1.7}$$

Theorem 3 *The set P in (1.7) is a mobile sampling set for B_Ω if and only if*

$$D^-(V) = \lim_{r\to\infty} \inf_{\mathbf{x}\in\mathbb{R}^2} \frac{\#V \cap B_r(x)}{|B_r(x)|} > 0.$$

1.4 Space-time sampling (Paper II)

Another way of acquisition of samples is to measure the signal in space and in time simultaneously, i.e. instead of the measurements $\{f(\lambda)\}_{\lambda\in\Lambda}$ taken on some uniformly discrete set Λ we consider the samples $\{f * \varphi_u(\lambda)\}_{\lambda\in\Lambda, u\in I}$, where I is some interval, $\{\varphi_u\}$ is a collection of kernels that satisfy some additional properties, and $f * \varphi_u$ is the convolution between the function f and kernel φ_u.

More precisely, the following problem was formulated by A. Aldroubi, K. Grochenig, and etc. in the paper [1]:

Main problem (space-time sampling) Assume $I \subset \mathbb{R}_+$. Let Λ be a uniformly discrete subset of $\mathbb{R}$ and let $\varphi_u(x)$ be a collection of functions parametrized by $u \in I$. What assumptions should be imposed on the spatial set Λ, the index set I, and functions φ_u to enable the recovery of every band-limited signal f from its space-time samples $\{f * \varphi_u(\lambda)\}_{\lambda\in\Lambda,\alpha\in I}$?

For the Paley-Wiener space PW_σ^p with $1 \le p < \infty$, the stable recovery is possible if the inequalities

$$A\|f\|_p^p \le \sum_{\lambda\in\Lambda} \int_I |f*\varphi_u(\lambda)|^p \, du \le B\|f\|_p^p, \quad \text{for every } f \in PW_\sigma^p \tag{1.8}$$

are true with some constants A and B independent of f.

We would like to provide an example of application from the paper [1] that motivates solving the main problem. Consider the initial value problem for the heat equation:

$$\frac{\partial}{\partial t} w(x,u) = k^2 \frac{\partial^2}{\partial x^2} w(x,u) \quad x \in \mathbb{R},\ u > 0 \tag{1.9}$$

with initial condition

$$w(x,0) = f(x). \tag{1.10}$$

It is well-known that the solution is given by $w(x,u) = f * \varphi_u(x)$, where

$$\varphi_u(x) = \frac{1}{\sqrt{4\pi uk}} e^{-\frac{x^2}{4uk}}.$$

Note that Main Problem applied to this initial value problem provides the reconstruction of the initial function f from the states $\{w(\lambda,u)\}_{\lambda\in\Lambda,u\in I}$. Here, I may be taken as $I = [a,b], 0 < a < b < \infty$.

We would like to mention some results obtained in [1].

- Unlike the classical sampling setting, the assumptions that should be imposed on the set Λ to solve the Main Problem *cannot be expressed in terms of some density* of Λ (see Example 4.1. in the mentioned paper). More precisely, one may construct a set with of arbitrarily small density that provides the stable reconstruction of the initial signal.

- For the solution of Main Problem we have to require Λ to be *relatively dense,* i.e. Λ can't have arbitrarily large gaps: there exists $R > 0$ such that for every $a \in \mathbb{R}$ we have

$$[a - R, a + R] \cap \Lambda \neq \emptyset.$$

Estimates of this value R were obtained in terms of constants A, B from the space-time sampling inequality (1.8).

In Paper II, which is a joint work with A. Ulanovskii, we propose a solution to the Main problem. To formulate the main result we need to make two remarks. First, we have to specify the assumptions that we impose on the kernels $\{\varphi_u\}$. However, they are quite bulky. For brevity of exposition, we omit the full list of requirements. But the following facts are worth mentioning.

- Among others, we require that φ_u decrease at the infinity, be real and even, and collection $\{\varphi_u\}_{u\in I}$ should satisfy a certain completeness property;

- One can easily verify that the Gaussian kernel $\varphi_u(x) = e^{-ux^2}$ satisfy all the assumptions and is the main example.

Second, we refer the reader to the definition of the set $W(\Lambda)$ of weak limits of all possible translates $\Lambda - t_n$ of the set Λ to the book [19].

It turns out that, unlike the classical sampling problem, the set Λ must be in some sense irregularly distributed, see condition (c) in the theorem below. Our Main Result is as follows:

Theorem 4 *Given a u.d. set $\Lambda \subset \mathbb{R}$ and a kernel $\{\varphi_u, u \in I\}$ that satisfy some certain assumptions (see conditions $(\beta) - (\theta)$ in Paper II). The following conditions are equivalent:*

(a) *The inequalities*

$$A\|f\|_2^2 \le \sum_{\lambda\in\Lambda} \int_I |f * \varphi_u(\lambda)|^2 \, d\alpha \le B\|f\|_2^2 \quad \textit{for every } f \in PW^2_\sigma$$

are true for every $\sigma > 0$ *and some* $A = A(\sigma)$ *and* $B = B(\sigma)$*;*

(b) *For every* $\sigma > 0$ *there is a constant* $K = K(\sigma)$ *such that*

$$\|f\|_\infty \le K \sup_{\lambda\in\Lambda, u\in I} |(f * \varphi_u)(\lambda)|, \quad \forall f \in B_\sigma; \tag{1.11}$$

(c) $W(\Lambda)$ *does not contain the empty set, and no element* $\Lambda^* \in W(\Lambda)$ *lies in an arithmetic progression.*

We conclude this section with the following examples.

- Set $\Lambda = \{\frac{1}{2^{|n|}} + n,\ n \in \mathbb{Z}\}$. Then $\mathbb{Z} \in W(\Lambda)$, since $x_k - \Lambda$ weakly converges to $\mathbb{Z}$ for $x_k = k$. Therefore, the space-time sampling is not possible, i.e. condition (1.8) is not satisfied.
- Set $\Lambda = n^2$. Then $\emptyset \in W(\Lambda)$, since in Λ there are arbitrarily large gaps. Therefore, the space-time sampling is not possible.
- Set $\Lambda = \mathbb{Z} \cup \sqrt{2}\mathbb{Z}$ solves the space-time sampling problem since no weak limit of translates of Λ lies in an arithmetic progression.

1.5 On geometry of the unit ball of Paley–Wiener spaces with a spectral gap (Paper III)

In Paper III, written together with A. Ulanovskii, we study the properties of Paley-Wiener spaces from a slightly different point of view that is closer to the Functional Analysis. The main focus is on the geometry of these spaces, more precisely, we study the extreme and exposed points of the unit ball of the Paley-Wiener spaces with one spectral gap.

Given a Banach space X, let $B \subset X$ be the closed unit ball of X. Recall that an element f from B is called extreme, if f is not a proper convex combination of two distinct points of B, i.e. there are no such f_-, f_+ belonging to B and $\alpha \in (0, 1)$ that

$$f = \alpha f_- + (1 - \alpha) f_+.$$

We note that due to some trivial reasons it is only interesting to study the geometry of the unit ball of the subspaces L^1 or L^∞ equipped, of course, with $\|\cdot\|_1$ and $\|\cdot\|_\infty$ norm respectively.

The geometry of the unit ball was studied for various function spaces. We mention the celebrated work by K. de-Leeuw and W. Rudin [15] in which the description of the extreme points of the unit ball was presented for the classical Hardy spaces. The exposed points of the unit ball of Hardy spaces emerged in various problems of Harmonic analysis. In particular, we mention the relations to Toeplitz operators, completely nondeterministic Gaussian processes, and theory of de Brange-Rovnyak spaces, see details in [21], [24], and [7].

The problems of a description of the set of exposed and extreme points for polynomials on the unit circle and classical Paley-Wiener spaces $PW^1_{[-\sigma,\sigma]}, \sigma > 0$, were investigated by K. Dyakonov in [10]. To formulate his characterization of extreme points for the Paley-Wiener space $PW^1_{[-\sigma,\sigma]}$, we need to define certain subsets of the complex plane:

Definition 2 *For the function $f, f \in PW^1_{[-\sigma,\sigma]}$, we introduce $\Lambda(f)$ and $\Omega(f)$ as follows.*

(i) Denote by $\Lambda(f) \subset \mathbb{C} \setminus \mathbb{R}$ the (possibly empty) set of all points $\lambda = a + ib, a, b \in \mathbb{R}, b \neq 0$, such that $f(\lambda) = f(\bar{\lambda}) = 0$, where $\bar{\lambda} = a - ib$.

(ii) Denote by $\Omega(f) \subset \mathbb{R}$ the (possibly empty) multi-set of all points $x \in \mathbb{R}$ such that $f(x) = f'(x) = \ldots = f^{(2n(x)-1)}(x) = 0$, where $n(x) \geq 1$ denotes the integer such that f has zero of multiplicity $2n(x)$ or $2n(x) + 1$ at x. Every point $x \in \Omega(f)$ is counted with multiplicity $2n(x)$.

For the functions with spectra supported on the single interval, Dyakonov presented the following characterization of extreme and exposed points.

Theorem 5 *The set of extreme points of the unit ball of* $PW^1_{[-\sigma,\sigma]}$, $\sigma > 0$, *consists precisely of the functions* f *satisfying the three conditions:*

$$\|f\|_1 = 1, \tag{1.12}$$

$$\text{at least one of the points } \{\pm\sigma\} \in \mathrm{Sp}(f), \tag{1.13}$$

and

$$\Lambda(f) = \emptyset. \tag{1.14}$$

The set of exposed points of the unit ball $PW^1_{[-\sigma,\sigma]}$, $\sigma > 0$, *consists precisely of the functions* f *that are extreme points and satisfy the conditions:*

$$\Omega(f) = \emptyset; \tag{1.15}$$

$$\int_{\mathbb{R}} |f(x)| w(x) dx = \infty, \forall w \in \mathrm{Hol}(\mathbb{C}), \mathrm{Type}(w) = 0, w|_{\mathbb{R}} \geq 0, w \neq const. \tag{1.16}$$

Here the conditions $w \in \mathrm{Hol}(\mathbb{C}), \mathrm{Type}(w) = 0$, *mean that w is an entire function of zero exponential type, i.e. for every* $\epsilon > 0$ *there is a constant* C_ϵ *such that*

$$|w(z)| \leq C_\epsilon e^{\epsilon |z|}, \quad z \in \mathbb{C}.$$

In his recent paper [11], Dyakonov put forward the following

Problem Assume S is a compact subset of $\mathbb{R}$. Describe the extreme and exposed points of the unit ball of PW^1_S.

We study this problem in the case of one spectral gap. More precisely, in Paper III, we consider the spectrum set $S = [-\sigma, -\rho] \cup [\rho, \sigma] = [-\sigma, \sigma] \setminus (-\rho, \rho)$. It turns out that the description of the extreme and exposed points strongly depends on the relative size of the gap, i.e. the ratio σ/ρ.

In Paper III, we completely describe the extreme and exposed points for the Paley-Wiener spaces with one large spectral gap $\sigma/\rho > 1/2$. To present our main result we fix some notation first.

Let $S = [-\sigma, \sigma] \setminus (-\rho, \rho)$, $0 < \rho < \sigma$. Observe that every function $f \in PW_S^1$ admits a unique representation

$$f(z) = f_-(z) + f_+(z), \quad \mathrm{Sp}(f_-) \subset [-\sigma, -\rho],\ \mathrm{Sp}(f_+) \subset [\rho, \sigma]. \quad (1.17)$$

Theorem 6 *Assume that $S = [-\sigma, \sigma] \setminus (-\rho, \rho)$, where $\sigma/2 < \rho < \sigma$. Then the following statements are true.*

1. *The set extreme points of PW_S^1 consists precisely of the functions f satisfying $\|f\|_1 = 1$ and the conditions*

$$\text{at least one of the points } \{\pm\sigma\}, \{\pm\rho\} \text{ belongs to } \mathrm{Sp}(f); \quad (1.18)$$

$$\Lambda(f_-) \cap \Lambda(f_+) = \emptyset. \quad (1.19)$$

2. *The set of exposed points consists precisely of the functions f that are extreme points and in addition satisfying* (1.16) *and*

$$\Omega(f_-) \cap \Omega(f_+) = \emptyset. \quad (1.20)$$

When the gap is small, i.e. $0 < \rho \leq \sigma/2$, the situation is more complicated. We show that the description obtained for the large gaps is not relevant anymore: there is a function that satisfies all the assumptions of the previous theorem, however, it is not extreme.

What is more surprising is that the extreme points can't be described in terms of symmetric (with respect to $\mathbb{R}$) zeroes.

Theorem 7 *Assume that $S = [-\sigma, \sigma] \setminus (-\rho, \rho)$ and $0 < \rho < \sigma/4$. Then there exist functions f_1 and f_2 from PW_Ω^1, $\|f_1\|_1 = \|f_2\|_1 = 1$, satisfying* (1.18) *and* (1.19) *such that*

$$\Lambda(f_1) = \Lambda(f_2),\ f_1 \in \mathrm{Ext}(PW_\Omega^1),\ f_2 \notin \mathrm{Ext}(PW_\Omega^1).$$

In addition, for the case of a large gap, we also show that the set of extreme points is dense on the unit sphere. Moreover, every function f from the unit sphere of Paley-Wiener space can be represented as

$f = (f_1 + f_2)/2$, where f_1 and f_2 are extreme points. We note that a similar result for the Hardy spaces was obtained in [15].

Our proof is based on two remarkable (and very complicated) results in Harmonic analysis. First, we use the Beurling-Malliavin theorem on the completeness radius of the exponential family, see [6]. Second, in our argument, we apply the recent result due to A. Poltoratski and M. Mitkovski [17] on the sign changes of real measures with a spectral gap at the origin.

1.6 Contractive projections (Paper IV)

In Paper IV, which is a joint work with A. Kulikov, we continue to study the Paley-Wiener spaces from the point of view of Functional analysis. More precisely, we consider the following

Main problem.

Let X, Y be spaces of functions. Assume that Y is a subspace of X and $P : X \to Y$ is a projection. What assumptions should be imposed on X, Y, and P to ensure that P is a contraction?

This problem was already studied for various function spaces X and Y. For example, the following statement can be deduced from [2] and [23], see also Theorem 1.1 in [8].

Theorem 8 *Let $X = L^p(\mathbb{T}^d)$, $Y = \{f \in L^p(\mathbb{T}^d) \mid \hat{f}(\lambda) = 0, \lambda \notin \Lambda\}$, and the projection P being an idempotent Fourier multiplier. Then projection P is a contraction if and only if either $p = 2$ or $\Lambda \subset \mathbb{Z}^d$ is a coset.*

Recently, O.F. Brevig, J. Ortega-Cerdà, and K. Seip [8] studied the contractivity of the similar idempotent Fourier multipliers in the case when X is a Hardy space, that is

$$X = H^p(\mathbb{T}^d) = \{f \in L^p(\mathbb{T}^d) \mid \hat{f}(n_1, n_2, \ldots, n_d) = 0 \text{ if } n_k < 0 \text{ for some } k\}.$$

They showed that if $p \notin 2\mathbb{N}$ then the only contractions are the same as in the Theorem 8, while for $p = 2k, k \in \mathbb{N}$, there exist non-trivial examples if $d \geq 3$. For the complete statement of their results, see [8], Theorem 1.2.

It is natural to consider the analog of this problem in the setting of Paley-Wiener spaces.

Assume that S_1 and S_2 are disjoint compact sets in $\mathbb{R}^d$. The *canonical projection* P acting from $PW^p_{S_1 \cup S_2}$ to $PW^p_{S_1}$ is defined by

$$P(f)(x) := \mathcal{F}^{-1}[\mathcal{F}(f) \cdot \chi_{S_1})](x),$$

where $\mathcal{F}$ stands for the Fourier transform.

For a set $S \subset \mathbb{R}^d$ and $k \in \mathbb{N}_0$ we define kS inductively as $0S = \{0\}$, $(k+1)S = kS + S$, where $+$ denotes the Minkowski sum.

The main result of Paper IV is the description of canonical contractive projections P acting from $PW^p_{S_1 \cup S_2}$ to $PW^p_{S_1}$.

Theorem 9 *Let S_1 and S_2 be disjoint finite unions of parallelepipeds in $\mathbb{R}^d$. Let P be a canonical projection from $PW^p_{S_1 \cup S_2}$ to $PW^p_{S_1}$, where $1 \leq p \leq \infty$. We have*

1. *If $p \in 2\mathbb{N}$ then P is a contraction if and only if*

$$\operatorname{mes}\left(\left(\frac{p}{2}S_1 + \left(\frac{p}{2} - 1\right)(-S_1)\right) \cap S_2\right) = 0. \qquad (1.21)$$

2. *If $p \notin 2\mathbb{N}$ then P is a contraction if and only if* $\operatorname{mes}(S_1) = 0$ *or* $\operatorname{mes}(S_2) = 0$.

1.7 Completeness of certain exponential families (Paper V)

In Paper V, which is a joint work with A. Kulikov and A. Ulanovskii, we return to studying Paley-Wiener spaces in the context of sampling theory. It is well known (e.g., see [19] and [9]) that the problems of reconstruction of the signal with spectrum supported on the bounded set S in a unique and stable way from the samples measured at the set

Λ are intimately connected to the problems of the completeness and frame properties of exponential families

$$E(\Lambda) = \{e^{i\lambda t}\}_{\lambda\in\Lambda}$$

in the space $L^2(S)$. Together with the samples $\{f(\lambda)\}_{\lambda\in\Lambda}$ one can measure the derivatives of the function f at the same points, i.e. $\{f^{(k)}(\lambda)\}_{\lambda\in\Lambda,k\in\Gamma}$ and $\Gamma \subset \mathbb{N}$. In particular, this approach allows for the reduction of the density of the set Λ or the number of sensors that measure the function. Therefore, the completeness or frame properties of the exponential system

$$E(\Lambda,\Gamma) = \{t^k e^{i\lambda t} \,:\, \lambda \in \Lambda, k \in \Gamma\}, \quad \Gamma \subset \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

are also worthy of investigation. Moreover, one can study these properties in various functions spaces. Below, we will focus on the system $E(\Lambda,\Gamma)$ and spaces of square-integrable functions and continuous functions.

The standard "measure of completeness" of the system $E(\Lambda,\Gamma)$ in L^2 is the so-called *completeness radius* that can be introduced as

$$CR(\Lambda,\Gamma) = \sup\{a \geq 0 : E(\Lambda,\Gamma) \text{ is complete in } L^2(-a,a)\}.$$

Similarly, one can introduce the *frame radius* as

$$FR(\Lambda,\Gamma) = \sup\{a \geq 0 : E(\Lambda,\Gamma) \text{ is a frame in } L^2(-a,a)\}.$$

The analog of the completeness radius in the space of continuous functions is straightforward:

$$CR_C(\Lambda,\Gamma) := \sup\{a \geq 0 : E(\Lambda,\Gamma) \text{ is complete in } C([-a,a])\}.$$

The completeness of exponential families is a classical problem in Harmonic analysis and a lot of remarkable papers were written in this field. We mention several relevant results.

- Of course, for $E(\mathbb{Z},\{0\}) = \{e^{2\pi int}\}_{n\in\mathbb{Z}}$ we have

$$CR(\mathbb{Z},\{0\}) = CR_C(\mathbb{Z},\{0\}) = \frac{1}{2}.$$

- From the celebrated Beurling–Malliavin theorem (see [6]) we know that

$$CR(\Lambda,\{0\}) = CR_C(\Lambda,\{0\}) = D^*(\Lambda),$$

  where D^* is the so-called upper Beurling–Malliavin density.

- It follows from the classical 'Beurling Sampling Theorem'(see [5] and [19]) that $FR(\Lambda) = D^-(\Lambda)$, where Λ is a uniformly discrete set and $D^-(\Lambda)$ is the lower uniform density of Λ.

- Let $\Gamma_N = \{0, 1, \dots, N-1\}$ then the system $E(\mathbb{Z},\Gamma_N) = \{t^k e^{2\pi int}\}_{n\in\mathbb{Z},k\in\Gamma_N}$ has radius of completeness

$$CR(\mathbb{Z},\Gamma_N) = CR_C(\mathbb{Z},\Gamma_N) = \frac{\#\Gamma_N}{2} = \frac{N}{2},$$

  see e.g. [22].

Note that in all the examples above the radii of completeness of the exponential families in the spaces L^2 and C coincide. It is natural to ask: is it always the case? More precisely,

Question. Does the equality

$$CR(\Lambda,\Gamma) = CR_C(\Lambda,\Gamma)$$

hold true for any exponential family $E(\mathbb{Z},\Gamma)$?

Surprisingly, the answer is generally negative, when $\Gamma \subset \mathbb{N}_0$ has "gaps". In Paper V, we managed to compute the completeness and frame radii for the exponential system $E(\Lambda,\Gamma)$ in the particular case $\Lambda = \mathbb{Z}$. It turns out that the radius of completeness in space of continuous functions depends on the cardinalities of the sets

$$\Gamma_{even} = \Gamma \cap 2\mathbb{Z} \quad \text{and} \quad \Gamma_{odd} = \Gamma \cap 2\mathbb{Z}+1.$$

Let us introduce the value

$$r(\Gamma) := \begin{cases} \#\Gamma_{odd} + \frac{1}{2}, \text{ if } \#\Gamma_{odd} < \#\Gamma_{even}, \\ \#\Gamma_{even}, \text{ if } \#\Gamma_{odd} \geq \#\Gamma_{even}. \end{cases}$$

Clearly, $r(\Gamma) \leq \#\Gamma/2$, and the inequality is strict whenever

$$\#\Gamma_{even} \neq \#\Gamma_{odd} \quad \text{and} \quad \#\Gamma_{even} \neq \#\Gamma_{odd} + 1.$$

The main result of Paper V is the following

Theorem 10 *Given any finite or infinite set* $\Gamma \subset \mathbb{N}_0$ *satisfying* $0 \in \Gamma$. *Then*

(i) $CR(\mathbb{Z}, \Gamma) = \#\Gamma/2$*;*

(ii) $CR_C(\mathbb{Z}, \Gamma) = FR(\mathbb{Z}, \Gamma) = r(\Gamma)$.

The proof is based on the properties of totally positive matrices, generalized Vandermonde matrices, but the key ingredient is a new result on the certain uniqueness sets of lacunary polynomials. We conclude this section by formulating this theorem.

Given any finite set $M \subset \mathbb{N}_0$, let $P(M)$ denote the set of real polynomials with exponents in M:

$$P(M) := \{p(x) = \sum_{m_j \in M} c_j x^{m_j} : c_j \in \mathbb{R}\}.$$

Given N distinct real numbers $t_1, \ldots, t_N$, set

$$S(t_1, \ldots, t_N) := \{(-1)^k t_k\}_{k=1}^N. \tag{1.22}$$

Theorem 11 *Assume* $0 < t_1 < t_2 < \cdots < t_N$. *Then both sets*

$$\pm S(t_1, \ldots, t_N)$$

are uniqueness sets for every space $P(M)$*, where* $M \subset \mathbb{N}_0, \#M = N$.

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Appendix

Paper I

On 2-dimensional mobile sampling

Appl. Comput. Harmon. Anal. 62 (2023) 1–23

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

On 2-dimensional mobile sampling

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ARTICLE INFO

Article history:
Received 13 August 2020
Accepted 3 August 2022
Available online 10 August 2022
Communicated by Yang Wang

Keywords:
Mobile sampling
Paley–Wiener spaces
Bernstein spaces
Beurling density
Uniqueness set

ABSTRACT

Necessary and sufficient conditions are presented for several families of planar curves to form a set of stable sampling for the Bernstein space $\mathcal{B}_\Omega$ over a convex set $\Omega \subset \mathbb{R}^2$. These conditions 'essentially' describe the mobile sampling property of these families for the Paley-Wiener spaces $\mathcal{PW}^p_\Omega, 1 \le p < \infty$.

1. Mobile sampling problem

The *classical sampling problem* is to determine when every continuous signal (function) f from a certain function space can be reconstructed from its discrete samples $f(\lambda), \lambda \in \Lambda$. The classical signal spaces are the Paley–Wiener spaces $\mathcal{PW}^p_\Omega$ of L^p-functions in $\mathbb{R}^d$ whose spectrum lies in a fixed set $\Omega \subset \mathbb{R}^d$. When $p < \infty$, the sampling problem asks for which discrete sets $\Lambda \subset \mathbb{R}^d$ there exist positive constants A, B such that

$$A\|f\|_p^p \le \sum_{\lambda \in \Lambda} |f(\lambda)|^p \le B\|f\|_p^p, \qquad \text{for every } f \in \mathcal{PW}^p_\Omega. \tag{1}$$

A different method for the acquisition of samples is when the samples of a multi-dimensional signal f are taken by a mobile sensor that moves along a continuous path γ. The *mobile sampling problem* is then to reconstruct the signal from its samples on a continuous path or a union P of continuous paths. In this case one needs to establish a 'continuous variant' of the inequalities above:

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https://doi.org/10.1016/j.acha.2022.08.001
1063-5203/

$$A\|f\|_p^p \leq \int\limits_P |f(u)|^p\, ds \leq B\|f\|_p^p, \quad \text{for every } f \in \mathcal{PW}_\Omega^p, \tag{2}$$

where we assume that P is locally rectifiable and integrate with respect to arc length.

The mobile sampling problem has recently attracted much attention. We refer the reader to [4–6,18,19] for motivation and recent results. The sampling property in Paley–Wiener spaces of several families has been considered:

(i) Parallel straight lines in $\mathbb{R}^d$ (see i.e. [4,18,19] and references therein).

(ii) In [1], a sufficient condition for the Archimedes spiral is presented to form a set of stable sampling. In [6], a wide family of *spiraling curves* in $\mathbb{R}^2$ is introduced and necessary and sufficient conditions for sampling in Paley–Wiener spaces with *convex symmetric spectrum* on these trajectories obtained.

In this paper, we consider the mobile sampling problem for three families of trajectories in $\mathbb{R}^2$. For each trajectory P from one of these families, we present a necessary and sufficient condition for sampling in the Bernstein space $B_\Omega := \mathcal{PW}_\Omega^\infty$ with *arbitrary convex spectrum* Ω. This condition 'essentially' describes the sampling property of P for the Paley-Wiener space $\mathcal{PW}_\Omega^p$. We therefore bypass the requirement of a uniform (or asymptotically uniform) distribution of trajectories, which has been essential in some previous approaches to the subject.

The rest of the paper is organized as follows. First, we give definitions of the classical Paley-Wiener and Bernstein spaces, then a short list of notations. In Section 4, we present the classical Beurling's sampling theorem. Our main results are formulated in Section 5 and proved in Sections 7-10. In Section 6, we discuss the connection between the sampling in Bernstein spaces and mobile sampling in Paley-Wiener spaces. In Section 11, we prove some uniqueness theorems which may have independent interest. Finally, in Section 12, we present some higher-dimensional results.

2. Bernstein and Paley–Wiener spaces

In what follows we will use the standard form of the Fourier transform:

$$\hat{f}(\mathbf{y}) = \int\limits_{\mathbb{R}^d} e^{-2\pi i\langle \mathbf{y},\mathbf{x}\rangle} f(\mathbf{x})d\mathbf{x}, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^d, \tag{3}$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{R}^d$.

We will consider the following classical spaces of signals (functions):

Definition 1. Let $\Omega \subset \mathbb{R}^d, d \geq 1$, be a compact set.

1. The Bernstein space $\mathcal{B}_\Omega$ consists of all continuous bounded functions in $\mathbb{R}^d$, which are the inverse Fourier transforms of tempered distributions supported by Ω. Equipped with uniform norm $\|\cdot\|_\infty$, $\mathcal{B}_\Omega$ is a Banach space.

2. Assume $\Omega \subset \mathbb{R}^d$ has positive measure. The Paley-Wiener spaces $\mathcal{PW}_\Omega^p, 1 \leq p < \infty$, are defined as

$$\mathcal{PW}_\Omega^p := \mathcal{B}_\Omega \cap L^p(\mathbb{R}^d).$$

Equipped with L^p-norm $\|\cdot\|_p$, $\mathcal{PW}_\Omega^p$ is a Banach space.

When $p = 2$, the space $\mathcal{PW}_\Omega^2$ is a Hilbert space consisting of all L^2-functions whose Fourier transform vanishes a.e. outside Ω.

Observe also that when $\Omega \subset \mathbb{R}^d$ is a compact convex set, the space $\mathcal{B}_\Omega$ admits an analytic description: It consists of all entire functions f satisfying

$$|f(\mathbf{z})| \leq C\exp\{-2\pi \max_{\mathbf{u}\in\Omega}\langle \mathbf{u},\mathbf{y}\rangle\}, \quad \mathbf{z} = \mathbf{x} + i\mathbf{y},\ \mathbf{x},\mathbf{y} \in \mathbb{R}^d.$$

3. Notations

Given $\mathbf{v} \in \mathbb{R}^d, d \geq 1$, and $r > 0$, we denote by $B_r(\mathbf{v}) := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x} - \mathbf{v}| \leq r\}$ the closed ball in $\mathbb{R}^d$ of radius r centered at $\mathbf{v}$. By $|E|$ we denote the (d-dimensional Lebesgue) measure of a set $E \subset \mathbb{R}^d$ and $\#E$ means the number of elements in E.

Set $\mathbb{R}^2_+ := \{\mathbf{x} = (x_1, x_2) : x_2 \geq 0\}$, $B^+_r(\mathbf{x}) := B_r(\mathbf{x}) \cap \mathbb{R}^2_+$ and $|\mathbf{x}| := \sqrt{x_1^2 + x_2^2}$.

Given sets $E, S \subset \mathbb{R}^2$, $Q \subset \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, we write

$$E + S = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in E, \mathbf{y} \in S\}, \ E - S = \{\mathbf{x} - \mathbf{y} : \mathbf{x} \in E, \mathbf{y} \in S\},$$
$$QE = \{q\mathbf{x} : \mathbf{x} \in E, q \in Q\}, \quad \operatorname{dist}(\mathbf{x}, S) := \inf_{\mathbf{y} \in S} |\mathbf{x} - \mathbf{y}|.$$

We say that the Hausdorff distance between E and S is $\leq \epsilon$ if $E \subset S + B_\epsilon(0)$ and $S \subset E + B_\epsilon(0)$.

4. Sampling in $\mathcal{B}_\Omega$

Definition 2. We say that a set $P \subset \mathbb{R}^d, d \geq 1$, is a sampling set (SS) for the Bernstein space B_Ω, where Ω is a compact in $\mathbb{R}^d$, if there is a constant $C > 0$ such that

$$\|f\|_\infty \leq C\|f|_P\|_\infty, \quad \text{for every } f \in \mathcal{B}_\Omega, \tag{4}$$

where

$$\|f|_P\|_\infty := \sup_{\mathbf{x} \in P} |f(\mathbf{x})|.$$

Definition 3. 1. A set $\Lambda \subset \mathbb{R}^d, d \geq 1$, is called uniformly discrete (u.d.) if

$$\delta(\Lambda) := \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda} |\lambda - \lambda'| > 0. \tag{5}$$

The constant $\delta(\Lambda)$ is called the separation constant for Λ.

2. The lower uniform density of a set $\Lambda \subset \mathbb{R}^d$ is defined as

$$D^-(\Lambda) = \lim_{r \to \infty} \inf_{\mathbf{x} \in \mathbb{R}^d} \frac{\#\Lambda \cap B_r(\mathbf{x})}{|B_r(\mathbf{x})|}.$$

In the classical situation where $d = 1$ and Ω is an interval in $\mathbb{R}$, the sampling problem for $\mathcal{B}_\Omega$ was completely solved by Beurling:

Theorem 1. ([3]) *Let $\Omega \subset \mathbb{R}$ be a compact interval. A set $P \subset \mathbb{R}$ is an SS for $\mathcal{B}_\Omega$ if and only if it contains a u.d. set Λ satisfying $D^-(\Lambda) > |\Omega|$.*

Observe that if $P \subset \mathbb{R}^d$ is an SS for $\mathcal{B}_\Omega$, then P contains a discrete subset which is also an SS for $\mathcal{B}_\Omega$:

Proposition 1. *Assume $P \subset \mathbb{R}^d, d \geq 1$, is an SS for $\mathcal{B}_\Omega$. Then there exists $\eta > 0$ such that every subset $\Lambda \subset P$ satisfying*

$$P \subset \Lambda + B_\eta(0) \tag{6}$$

is also an SS for B_Ω.

We omit the proof, as it easily follows from *Bernstein's inequality*, see [10], p. 21.

In particular, the set Λ in this result can be chosen to be u.d.

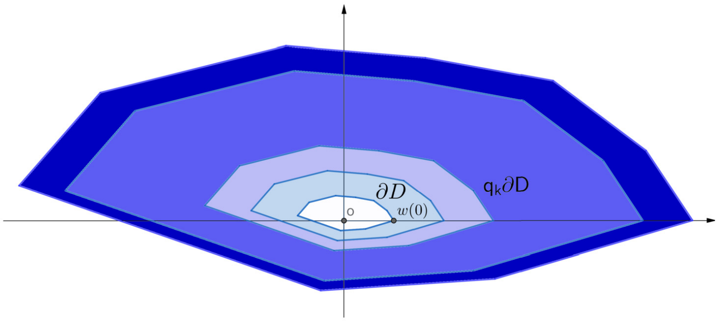

Fig. 1. Dilation of a convex curve.

5. Results

In what follows, we assume that $\Omega \subset \mathbb{R}^2$ is a convex set of positive measure.

We will consider the sampling problem for three families of curves in $\mathbb{R}^2$: parallel lines, dilations of a convex closed curve around the origin and translations of a circle. For every family we present a sufficient and necessary condition for sampling in $\mathcal{B}_\Omega$.

5.1. Parallel lines

Let $l \in \mathbb{R}^2$ be a straight line through the origin, and let $\mathbf{v}_l$ be a unit vector orthogonal to l. Given any u.d. set

$$H := \{a_k\}_{k\in\mathbb{Z}} \subset \mathbb{R},$$

consider the set of parallel lines

$$P = l + H\mathbf{v}_l := \bigcup_{k\in\mathbb{Z}} (l + a_k\mathbf{v}_l). \tag{7}$$

Theorem 2. *The set P in (7) is an SS for $\mathcal{B}_\Omega$ if and only if*

$$D^-(H)\mathbf{v}_l \notin \Omega - \Omega.$$

5.2. Dilations of a convex curve

Let $D \subset \mathbb{R}^2$ be a closed convex set of finite positive measure such that $0 \in \mathrm{Int}(D)$. Denote by ∂D the boundary of D, by $\mathrm{Ext}(D)$ the closed set of extreme points of D and by

$$D^o := \{\mathbf{x} \in \mathbb{R}^2 : \langle \mathbf{x}, \mathbf{y}\rangle \le 1, \mathbf{y} \in D\}$$

the polar set of D.

Given a u.d. set $Q = \{q_k\} \subset (0, \infty)$, consider the set

$$P = Q\partial D := \bigcup_{k=1}^{\infty} \bigcup_{\mathbf{w}\in\partial D} \{q_k\mathbf{w}\}, \tag{8}$$

see Fig. 1. Set $d^-(Q) := D^-(Q \cup (-Q))$.

Theorem 3. *The set P in (8) is an SS for $\mathcal{B}_\Omega$ if and only if*

$$d^-(Q)\mathbf{v} \notin \Omega - \Omega, \quad \textit{for every } \mathbf{v} \in \mathrm{Ext}(D^o). \tag{9}$$

The following is a simple corollary of this result and Remark 1 in Section 7.

Corollary 1. (i) *Assume* $\mathrm{Ext}(D) = D$. *The set P in (8) is an SS for $\mathcal{B}_\Omega$ if and only if*

$$d^-(Q)\partial D^o \cap (\Omega - \Omega) = \emptyset.$$

In particular, if D is the unit circle then P is an SS for $\mathcal{B}_\Omega$ if and only if $\mathrm{Diam}(\Omega) < d^-(Q)$.

(ii) *Let D be the square* $[-1,1]^2$. *The set P in (8) is an SS for $\mathcal{B}_\Omega$ if and only if the vectors* $(d^-(Q), 0)$ *and* $(0, d^-(Q))$ *do not lie in* $\Omega - \Omega$.

5.3. Translations of a circle

Take any circle $T := \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| = r\}, r > 0$. Let $V = \{\mathbf{v}_k\}_{k=1}^\infty \subset \mathbb{R}^2$ be a u.d. set. Set

$$P = V + T := \bigcup_{k=1}^\infty (\mathbf{v}_k + T). \tag{10}$$

Theorem 4. *The set P in (10) is an SS for $\mathcal{B}_\Omega$ if and only if $D^-(V) > 0$.*

5.4. Remarks

Let $\Omega \subset \mathbb{R}$ be a compact interval. Beurling's Theorem 1 solves the sampling problem for $\mathcal{B}_\Omega$ in terms of the lower uniform density $D^-(\Lambda)$ of sampling set Λ. The sampling property of Λ in the Paley-Wiener space $\mathcal{PW}^2_\Omega$ can be 'essentially' described in terms of $D^-(\Lambda)$, see i.e. [10]. See [17] for necessary and sufficient conditions for sampling in $\mathcal{PW}^2_\Omega$.

If $\Omega \subset \mathbb{R}$ is a disconnected set, already when it is a union of two intervals, the sampling property of u.d. sets Λ cannot be described in terms of any density of Λ. This is also the case for the spectra $\Omega \subset \mathbb{R}^d, d > 1$, see [10,11]. However, the necessary density condition for sampling remains valid for general spectra $\Omega \subset \mathbb{R}^d$: Landau [7] proved that if a u.d. set Λ is an SS for $\mathcal{PW}^2_\Omega$, then it satisfies $D^-(\Lambda) \geq |\Omega|$ (see [9] for different simpler proof, which in particular extends Landau's result to unbounded spectra).

Observe that for the mobile sampling (see definition in the next section) there is no analogue of Landau's result. One can define a path density of trajectory P as the 'average length' covered by a curve. An example is constructed in [4] showing that trajectories P of arbitrarily small path density may nevertheless provide mobile sampling for PW^p_Ω. Theorem 5 below reduces the mobile sampling problem for the Paley-Wiener spaces to the sampling problem for the Bernstein spaces. Hence, our Theorem 4 presents another example in this direction. Observe also that it easily follows from Corollary 1 (ii) and Theorem 5, that a union of equidistant squares

$$P := \bigcup_{n \in \mathbb{N}} \{\mathbf{x} = (x_1, x_2) : \max\{|x_1|, |x_2|\} = n\}$$

provides mobile sampling for $\mathcal{PW}^p_\Omega, 1 \leq p < \infty$, for certain convex sets Ω of arbitrarily large measure. However, by Corollary 1 (i) and Theorem 5, the union of equidistant circles

$$P := \bigcup_{n \in \mathbb{N}} \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| = n\}$$

is not an SS for $\mathcal{PW}^p_\Omega$, whenever Ω is a convex compact set of measure strictly greater than $\pi/4$.

6. Sampling in $\mathcal{B}_\Omega$ versus mobile sampling in PW^p_Ω

Following [4], we call $P \subset \mathbb{R}^d$ a trajectory, if P is a countable union of locally rectifiable (continuous) curves.

Definition 4. A trajectory $P \subset \mathbb{R}^d$ is called a stable sampling trajectory (ST) for $\mathcal{PW}^p_\Omega, 1 \le p < \infty$, if condition (2) holds with some positive constants A, B.

When the sampling set $\Lambda \subset \mathbb{R}^d$ is discrete, it is well-known that the right inequality in (1) is equivalent to the condition that there exist $r, C > 0$ such that $\#(\Lambda \cap B_r(\mathbf{x})) < C$, for every $\mathbf{x} \in \mathbb{R}^d$ (such sets Λ are called relatively uniformly discrete).

In the case of mobile sampling, one has

Proposition 2. *Let $1 \le p < \infty$, $\Omega \subset \mathbb{R}^d$ be a compact set of positive measure and P be a trajectory. The following conditions are equivalent:*

(i) *There is a constant C such that*

$$\int_P |f(\mathbf{u})|^p ds \le C\|f\|_p^p, \quad \textit{for every } f \in PW^p_\Omega;$$

(ii) *There are constants $r > 0$ and $C > 0$ such that*

$$\sup_{\mathbf{u}\in\mathbb{R}^d} \int_{P\cap B_r(\mathbf{u})} ds \le C. \tag{11}$$

We skip the proof which is similar to the corresponding proof for sampling on u.d. sets, see [20].

We will need a condition which prohibits P to contain separated curves of arbitrarily small length:

$$\textit{There exists } r > 0 \textit{ such that } \inf_{\mathbf{x}\in P} \int_{P\cap B_\delta(\mathbf{x})} ds \ge \delta, \quad \textit{for every } \delta \le r. \tag{12}$$

The following result establishes a connection between sampling in Bernstein and mobile sampling in Paley–Wiener spaces:

Theorem 5. *Let $1 \le p < \infty$ and $0 < \epsilon < 1$. Let $\Omega \subset \mathbb{R}^d$ be a compact convex set of positive measure and P a trajectory satisfying (11).*

(i) *If P is an SS for $\mathcal{B}_\Omega$ and satisfies (12), then it is an ST for $\mathcal{PW}^p_{(1-\epsilon)\Omega}$.*

(ii) *If P is not an SS for $\mathcal{B}_\Omega$, then it is not an ST for $\mathcal{PW}^p_{(1+\epsilon)\Omega}$.*

This is an analogue of the corresponding result for the discrete sampling sets, see Theorem 5.30 in [10]. We omit the proof of Theorem 5 since it is rather similar to the proof of the mentioned result from [10].

One can check that part (i) of the theorem ceases to be true if condition (12) is dropped.

7. Proof of Theorem 2

Before passing to the proof, we recall several well-known facts.

A sequence of u.d. sets $Q_k \subset \mathbb{R}, k \in \mathbb{N}$, is said to converge weakly to a u.d. set Q' if for every $R > 0$ satisfying $\pm R \notin Q'$, the Hausdorff distance between $Q_k \cap (-R, R)$ and $Q' \cap (-R, R)$ tends to zero as $k \to \infty$.

The following lemma is well-known:

Lemma 1. *Assume u.d. sets $Q_k \subset \mathbb{R}$ satisfy $\inf_k \delta(Q_k) > 0$, where $\delta(Q_k)$ is the separation constant defined in (5). Then there is a subsequence Q_{k_n} which converges weakly to some u.d. set Q' satisfying $D^-(Q') \geq \limsup_{n\to\infty} D^-(Q_{k_n})$.*

One can also define the weak convergence of trajectories. However, in what follows we will not use this definition.

The next statement is obvious.

Remark 1. Assume Ω is a convex set and $\mathbf{v}_\theta$ is a non-zero vector with argument θ, i.e. $\mathbf{v}_\theta = |\mathbf{v}_\theta|(\cos\theta, \sin\theta)$. Then Ω does not contain any segment of length $|\mathbf{v}_\theta|$ parallel with $\mathbf{v}_\theta$ if and only if $\mathbf{v}_\theta \notin \Omega - \Omega$.

Now, we pass to the proof of Theorem 2. Without loss of generality we may assume that l is a vertical line and $D^-(H) = 1$. To prove Theorem 2 we have to show that the set P in (7) is an SS for $\mathcal{B}_\Omega$ if and only if

$$(1, 0) \notin \Omega - \Omega. \tag{13}$$

(i) Assume P is not an SS for B_Ω. We have to prove that (13) is not true.

Since P is not an SS for B_Ω, there is a sequence of functions $f_k \in \mathcal{B}_\Omega, k \in \mathbb{N}$, such that

$$\|f_k|_P\|_\infty \leq \frac{1}{k}, \quad \|f_k\|_\infty = 1 \text{ and } \quad f_k(\mathbf{x}_k) \geq 1 - \frac{1}{k}, \tag{14}$$

for some points $\mathbf{x}_k = (u_k, w_k) \in \mathbb{R}^2$.

Set

$$g_k(\mathbf{x}) := f_k(\mathbf{x} + \mathbf{x}_k).$$

Clearly, we have $g_k(0) \geq 1 - \frac{1}{k}$ and $\|g_k|_{\{l+(H-u_k)(1,0)\}}\|_\infty \leq \frac{1}{k}$. By Lemma 1, there is a subsequence k_n such that the translates $H - u_k$ converge weakly to some u.d. set H' satisfying $D^-(H') \geq D^-(H) = 1$. By the compactness property of $\mathcal{B}_\Omega$ (see [10]), (taking if necessary a subsequence of k_n) we may assume that g_{k_n} converge (uniformly on compacts in $\mathbb{C}^2$) to some non-trivial function $g \in \mathcal{B}_\Omega$. Clearly, $\|g\|_\infty = g(0) = 1$ and

$$g(q, x_2) = 0, \quad q \in H', \quad x_2 \in \mathbb{R}. \tag{15}$$

Fix any small $\varepsilon > 0$. Since $D^-(H') \geq 1$, by a result of Seip (see [16], Theorem 2.3), H' contains a subset $H'' \subset H'$ such that the exponential system $E(H'') := \{e^{2\pi i q t} : q \in H''\}$ forms a Riesz basis in $L^2(-(1-\varepsilon)/2, (1-\varepsilon)/2)$.

Next, we invoke Pavlov's characterization of Riesz bases from [13] to ensure that the generating function of the exponential system above

$$\varphi(z_1) := \lim_{R\to\infty} \prod_{q\in H'', |q|\leq R} \left(1 - \frac{z_1}{q}\right) \tag{16}$$

is well defined, of exponential type $\pi(1-\varepsilon)$, and

$$|\varphi(x_1 - i)|^2 \in A_2, \tag{17}$$

i.e. it belongs to the Muckenhoupt class A_2.

We briefly note that in [13] and [16], the zeroes of φ are supposed to lie strictly above the real line. However, one can overcome this obstacle by noting that the exponential system $E(H'' + i) := \{e^{2\pi i(q+i)}, q \in H''\}$ is also a Riesz basis for $L^2(-(1-\varepsilon)/2, (1-\varepsilon)/2)$, and its generating function $\tilde{\varphi}$ satisfies $\varphi(x_1 - i) = \varphi(-i)\tilde{\varphi}(x_1)$.

Claim 1. *There is a constant $\delta > 0$ such that*

$$|\varphi(x_1 - i)| \geq \frac{\delta}{1 + |x_1|^3}, \quad x_1 \in \mathbb{R}. \tag{18}$$

This claim can be easily deduced from (17) and Bernstein's inequality.

Recall that the generating function for a Riesz basis satisfies $|\varphi(x_1)|/(1 + |x_1|) \in L^2(\mathbb{R})$. Choose any $q_0 \in H''$ and set $\varphi_1(x_1) := \varphi(x_1)/(x_1 - q_0)$.

Then $\varphi_1 \in \mathcal{PW}_{[-(1-\varepsilon)/2,(1-\varepsilon)/2]}$, and the points $\pm(1-\varepsilon)/2$ belong to the spectrum of φ_1. Set

$$\psi(z_1, z_2) := \frac{g(z_1, z_2)\sin^4\left(\frac{\varepsilon}{4}z_1\right)}{\varphi(z_1){z_1}^4}. \tag{19}$$

By (15) and (16), ψ is holomorphic in $\mathbb{C}^2$. By (18), it belongs to L^2 on $(\mathbb{R} - i) \times \mathbb{R}$. Therefore, $\psi \in \mathcal{PW}_{\Omega'}$, where $\Omega' \subset \mathbb{R}^2$ is the spectrum of ψ.

Consider the equality

$$\varphi_1(z_1)\psi(z_1, z_2) = g(z_1, z_2)\psi_\varepsilon(z_1), \quad \psi_\varepsilon(z_1) := \frac{\sin^4\left(\frac{\varepsilon}{4}z_1\right)}{z_1^4}. \tag{20}$$

The spectrum of φ_1 contains the endpoints of the interval

$$I := [-(1-\varepsilon)/2, (1-\varepsilon)/2)]$$

on the x_1-axis. The spectrum of ψ_ε is the interval $I_\varepsilon := [-\varepsilon, \varepsilon]$ on the x_1-axis. One may now use an analogue of the Titchmarsh convolution theorem for higher dimensions:

$$\text{c.h.}(I + \Omega') = \text{c.h.}(\text{Sp}\,\varphi \cdot \psi) = \text{c.h.}(\text{Sp}\,g \cdot \psi_\varepsilon) \subset \text{c.h.}(I_\varepsilon + \Omega) \subset \Omega + B_\varepsilon(0),$$

where c.h. means the convex hull and Sp the spectrum. Clearly, the set c.h.$(I + \Omega')$ contains a horizontal interval of length $|I| = 1 - \varepsilon$. This is also true for $\Omega + B_\varepsilon(0)$. It follows that Ω contains a horizontal interval of length $1 - 3\varepsilon$. Using Remark 1, we see that the point $(1 - 3\varepsilon, 0) \in \Omega - \Omega$. Since ε can be chosen arbitrarily small, we conclude that (13) does not hold.

Note, that in this reasoning it is essential that Ω is a convex set.

(ii) Assume (13) does not hold. We have to show that P is not an SS for $\mathcal{B}_\Omega$. This is an easy consequence of Beurling's Theorem 1. Indeed, since translations of Ω do not change the sampling property of P, we may assume that $[-1/2, 1/2] \in \Omega$. Using Theorem 1, for every $\varepsilon > 0$ there is a function $f(x_1) \in \mathcal{B}_{[-1/2,1/2]}$ satisfying $\|f|_H\|_\infty \leq \varepsilon$ and $\|f\|_\infty = 1$. It follows that P is not an SS for $\mathcal{B}_{[-1/2,1/2]}$. Therefore, P is not an SS for $\mathcal{B}_\Omega$.

8. Auxiliary results for the proof of Theorem 3

In this section, $Q, D, \partial D$ have the same meaning as in Theorem 3.

Denote by $\arg \mathbf{w}$ the argument of vector $\mathbf{w}$, i.e. the angle θ such that $\mathbf{w} = |\mathbf{w}|(\cos\theta, \sin\theta)$. We also denote by $\mathbf{w}(\theta) \in \partial D$ the unique vector which lies on ∂D satisfying $\arg \mathbf{w}(\theta) = \theta$, $-\pi < \theta \le \pi$.

Recall that for every convex or concave function $f(x)$ defined on an interval $I \subset \mathbb{R}$, both one-sided derivatives of f exist at every interior point $w_0 \in I$. It follows that for every boundary point $\mathbf{w}(\theta) \in \partial D$, both semi-tangent lines $\mathbf{w}(\theta) + l_+(\theta)$ and $\mathbf{w}(\theta) + l_-(\theta)$ exist, where $l_\pm(\theta)$ are straight lines through the origin. In particular, if $\theta = 0$ then there exist two lines $l_+(0)$ and $l_-(0)$ such that

$$\operatorname{dist}(\mathbf{w}(\theta) - \mathbf{w}(0), l_+(0)) = o(\theta), \quad \theta \downarrow 0, \tag{21}$$

and

$$\operatorname{dist}(\mathbf{w}(\theta) - \mathbf{w}(0), l_-(0)) = o(|\theta|), \quad \theta \uparrow 0.$$

For the proof of Theorem 3, we need two lemmas:

Lemma 2. *Let $l_+(0)$ be the line satisfying (21). Assume a sequence of vectors $\mathbf{x}_k$ satisfies*

$$\arg(\mathbf{x}_k) > 0, \quad k \in \mathbb{N}, \quad |\mathbf{x}_k| \to \infty, \quad \arg(\mathbf{x}_k) \to 0, \quad k \to \infty. \tag{22}$$

Then there exists a subsequence $\mathbf{x}_{k_n}$ and a u.d. set $Q' \subset \mathbb{R}$ such that

$$|\mathbf{w}(\arg(\mathbf{x}_{k_n}))|Q - |\mathbf{x}_{k_n}| \textit{ converge weakly to } Q', \;\; D^-(Q') \ge \frac{d^-(Q)}{|\mathbf{w}(0)|}. \tag{23}$$

Condition (23) implies for every $R > 0$ that

$$\mathrm{d}_R^+(Q \cdot \partial D - \mathbf{x}_{k_n}, \mathbf{w}(0) + l_+(0) + Q' \cdot (1,0)) \to 0, \;\; n \to \infty. \tag{24}$$

Here $d_R^+(A, B)$ denotes the Hausdorff distance between $A \cap B_R^+(0)$ and $B \cap B_R^+(0)$.

Remark 2. (i) Note that if in (22) we assume that the arguments of $\mathbf{x}_k$ are negative and tend to zero, then a similar to (24) condition holds with the 'lower' semi-tangent line $l_-(0)$.

(ii) Assume additionally that $l_+(0) = l_-(0)$, i.e. there is a tangent line to D through $\mathbf{w}(0)$. Then one may check that for every $R > 0$ the Hausdorff distance between

$$(Q \cdot \partial D - \mathbf{x}_{k_n}) \cap B_R(0)$$

and

$$(\mathbf{w}(0) + l_+(0) + Q' \cdot (1,0)) \cap B_R(0)$$

tends to zero as $n \to \infty$.

Lemma 3. *Assume P is not an SS for $\mathcal{B}_\Omega$. Then for every $n \in \mathbb{N}$ and $\epsilon > 0$ there exist $f_n \in B_{(1+\epsilon)\Omega}$ and $\mathbf{x}_n \in \mathbb{R}^2$ such that*

$$|\mathbf{x}_n| > n, \quad \|f_n\|_\infty = 1, \quad \|f_n|_P\|_\infty < \frac{1}{n}, \quad |f_n(\mathbf{x}_n)| > 1 - \frac{1}{n}. \tag{25}$$

8.1. Proof of Lemma 2

Condition (23) follows easily from Lemma 1. In what follows, for simplicity we assume that

$$|\mathbf{w}(\theta_n)| \cdot Q - |\mathbf{x}_n| \text{ converge weakly to } Q', \quad n \to \infty. \tag{26}$$

We have to deduce (24) from (23). Before we proceed with the proof, observe that (24) is intuitively clear. Indeed, since $\mathbf{w}(0) + l_+(0)$ is a semi-tangent line to ∂D, when q_n and $|\mathbf{x}_n|$ tend to infinity, the set $(q_n \partial D - \mathbf{x}_n) \cap B_R^+(0)$ is either empty or 'looks more and more like' a segment as $n \to \infty$. However, the formal proof is somewhat technical.

Below we denote by C different positive constants.

Recall that D is a convex set of positive measure around the origin.

Given a straight line l through the origin, denote by $\varphi(l), 0 \le \varphi(l) < \pi$, the angle from the positive ray $\mathbb{R}_+(1,0)$ to l in the counterclockwise direction. Recall that $\mathbf{w}(\theta) + l_+(\theta)$ and $\mathbf{w}(\theta) + l_-(\theta)$ denote the semi-tangent lines to ∂D at the boundary point $\mathbf{w}(\theta) \in \partial D$, where we assume that $\varphi(l_+(\theta)) \ge \varphi(l_-(\theta))$ for small positive values of θ.

Clearly, for all small enough positive angles $\theta > \theta'$ we have

$$\varphi(l_+(\theta')) \le \arg(\mathbf{w}(\theta') - \mathbf{w}(\theta)) \le \varphi(l_-(\theta)), \quad 0 < \theta' < \theta, \tag{27}$$

and

$$|\mathbf{w}(\theta') - \mathbf{w}(\theta)| < C(\theta' - \theta), \quad 0 < \theta' < \theta, \tag{28}$$

where one may take $C = 2|\mathbf{w}(0)|$.

Clearly, $\varphi(l_\pm(\theta)) \downarrow \varphi(l_+(0))$, as $\theta \downarrow 0$. Therefore, for every $\epsilon > 0$ there is an angle $\theta(\epsilon) > 0$ such that

$$0 < \varphi(l_+(\theta)) - \varphi(l_+(0)) < \epsilon, \quad 0 < \theta < \theta(\epsilon). \tag{29}$$

Set $\theta_n := \arg \mathbf{x}_n$. By (22), $\theta_n > 0$ and $\theta_n \to 0$ as $n \to \infty$. We assume that n is large enough so that $\theta_n < \theta(\epsilon)$.

Assume $\theta_n < \theta < \theta(\epsilon)$, and denote by $\mathbf{w}_n(\theta)$ the point with argument θ lying on the semi-tangent line $\mathbf{w}(\theta_n) + l_+(\theta_n)$:

$$\mathbf{w}_n(\theta) \in \mathbf{w}(\theta_n) + l_+(\theta_n), \quad \arg \mathbf{w}_n(\theta) = \theta. \tag{30}$$

From (27), (28) and (29) we may deduce that

$$d = |\mathbf{w}(\theta) - \mathbf{w}_n(\theta)| < 2C\epsilon(\theta - \theta_n), \quad \theta_n < \theta < \theta(\epsilon), \tag{31}$$

provided ϵ is sufficiently small (see Fig. 2).

Fix any $R > 0$ satisfying $\pm R \notin Q'$. Then fix a positive number $\epsilon < 1/R^2$.

To prove (24) we show that the Hausdorff distance between

$$(Q \cdot \partial D - \mathbf{x}_n) \cap B_R^+(0) \tag{32}$$

and

$$(\mathbf{w}(0) + l_+(0) + Q' \cdot (0,1)) \cap B_R^+(0)$$

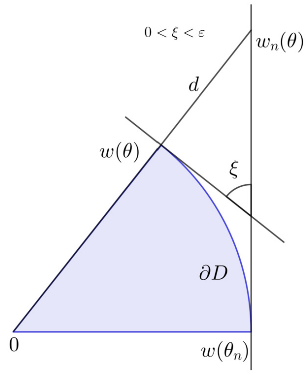

Fig. 2. Approximation of the curve by lines.

tends to zero as $n \to \infty$. Since $\theta_n \to 0$, it suffices to check this for the Hausdorff distance between the set in (32) and the set

$$(\mathbf{w}(\theta_n) + l_+(\theta_n) + Q' \cdot (\cos\theta_n, \sin\theta_n)) \cap B_R^+(0).$$

Write $Q' \cap (-R, R) = \{q'(1), ..., q'(m)\}$. By (26), for every large enough n,

$$Q \cap \left(\frac{|\mathbf{x}_n| - R}{|\mathbf{w}(\theta_n)|}, \frac{|\mathbf{x}_n| + R}{|\mathbf{w}(\theta_n)|} \right) = \{q_n(1), ..., q_n(m)\},$$

where

$$|\mathbf{w}(\theta_n)|q_n(j) - |\mathbf{x}_n| - q'(j) \to 0, \quad n \to \infty, \quad j = 1, ..., m.$$

Since $\arg \mathbf{w}(\theta_n) = \arg \mathbf{x}_n = \theta_n$, this yields

$$|\mathbf{w}(\theta_n)q_n(j) - \mathbf{x}_n - q'(j)(\cos\theta_n, \sin\theta_n)| \to 0, \;\; n \to \infty, \;\; j = 1, ..., m. \tag{33}$$

We see that it suffices to check that for every $j = 1, ..., m$, the Hausdorff distance between

$$\{q_n(j)\mathbf{w}(\theta) : \theta \geq \theta_n\} \cap (\mathbf{x}_n + B_R^+(0)) \tag{34}$$

and

$$(\mathbf{x}_n + q'(j)(\cos\theta_n, \sin\theta_n) + l_+(\theta_n)) \cap (\mathbf{x}_n + B_R^+(0)) \tag{35}$$

tends to zero as $n \to \infty$.

Observe that for every sufficiently large n, condition $q\mathbf{w}(\theta) \in \mathbf{x}_n + B_R(0)$ implies

$$\theta < \theta_n + 2R/|\mathbf{x}_n|, \quad q \in (C|\mathbf{x}_n|/2, 2C|\mathbf{x}_n|),$$

where we may take $C = 1/|\mathbf{w}(0)|$. Hence, from (31) one may easily check that the distance between the set

$$\{q_n(j)\mathbf{w}(\theta) : \theta_n \leq \theta \leq \theta_n + 2R/|\mathbf{x}_n|\}$$

and the set

$$\{q_n(j)\mathbf{w}_n(\theta) : \theta_n \le \theta \le \theta_n + 2R/|\mathbf{x}_n|\} \tag{36}$$

is less than $CR\epsilon < C\sqrt{\epsilon}$.

On the other hand, by (30), the point $q_n(j)\mathbf{w}_n(\theta)$ has argument θ and lies on the line $q_n(j)\mathbf{w}(\theta_n)+l_+(\theta_n)$. Let $\mathbf{u}_j(\theta)$ be the point on $\mathbf{x}_n + q'(j)(\cos\theta_n, \sin\theta_n) + l_+(\theta_n)$ satisfying $\arg(\mathbf{u}_j(\theta)) = \theta$. By (33), $|\mathbf{u}_j(\theta) - q_n(j)\mathbf{w}_n(\theta)| \to 0$ as $n \to \infty$, which implies that for sufficiently large n, the Hausdorff distance between the sets in (34) and (35) is less than $C\sqrt{\epsilon}$. Since ϵ can be chosen arbitrarily small, this proves (24).

8.2. Proof of Lemma 3

Since P is not an SS for $\mathcal{B}_\Omega$, there is a sequence of functions $g_k \in \mathcal{B}_\Omega$ satisfying

$$\|g_k\|_\infty = 1, \quad \|g_k|_P\|_\infty < \frac{1}{k}.$$

For every k choose a point $\mathbf{y}_k$ such that $|g_k(\mathbf{y}_k)| > 1 - 1/k$. If

$$\limsup_{k\to\infty} |\mathbf{y}_k| \to \infty,$$

then condition (25) holds for $f_n(\mathbf{x}) := g_{k_n}(\mathbf{x}) \in \mathcal{B}_\Omega$, for some suitable subsequence k_n.

Assume that the sequence $\mathbf{y}_k$ is bounded. We may assume that it converges to some point $\mathbf{y}_0 \in \mathbb{R}^2$. Using the compactness property of Bernstein spaces, see [10], we may also assume that g_n converge to some function $g_0 \in \mathcal{B}_\Omega$. Clearly, g_0 satisfies

$$\|g_0\|_\infty = |g_0(\mathbf{y}_0)| = 1, \quad g_0|_P = 0.$$

Consider two cases.

1. Assume g_0 tends to zero fast in the sense that for every $\mathbf{m} \in (\mathbb{N} \cup \{0\})^2$ we have $|\mathbf{x}^\mathbf{m} g_0(\mathbf{x})| \to 0$ as $|\mathbf{x}| \to \infty$, where $\mathbf{x}^\mathbf{m} = x_1^{m_1}x_2^{m_2}, \mathbf{x} = (x_1, x_2), \mathbf{m} = (m_1, m_2)$. Choose a point $\mathbf{y}_\mathbf{m}$ such that

$$\|\mathbf{x}^\mathbf{m} g_0(\mathbf{x})\|_\infty = \max_{\mathbf{x}\in\mathbb{R}^2} |\mathbf{x}^\mathbf{m} g_0(\mathbf{x})| = |\mathbf{y}_\mathbf{m}^\mathbf{m} g_0(\mathbf{y}_\mathbf{m})|.$$

Clearly,

$$|\mathbf{y}_\mathbf{m}| \to \infty, \quad \max_{\mathbf{x}\in\mathbb{R}^2} |\mathbf{x}^\mathbf{m} g_0(\mathbf{x})| \to \infty, \quad |\mathbf{m}| \to \infty.$$

Therefore, for a suitable subsequence $\mathbf{m}_n$, the functions

$$f_n(\mathbf{x}) := \mathbf{x}^{\mathbf{m}_n} g_0(\mathbf{x})/\|\mathbf{x}^{\mathbf{m}_n} g_0(\mathbf{x})\|_\infty$$

belong to $\mathcal{B}_\Omega$ and satisfy condition (25).

2. If g_0 does not satisfy the decrease condition above, then there exist $\mathbf{m} \in (\mathbb{N} \cup \{0\})^2$ and a sequence of points $\mathbf{y}_k$ such that

$$|\mathbf{y}_k| > k, \qquad |\mathbf{y}_k^\mathbf{m} g_0(\mathbf{y}_k)| \ge 1, \quad k \in \mathbb{N}.$$

Consider the functions

$$\varphi_k(\mathbf{x}) := \mathbf{x}^\mathbf{m} g_0(\mathbf{x}) \mathrm{sinc}^{2|\mathbf{m}|}(\epsilon(\mathbf{x} - \mathbf{y}_k)/2|\mathbf{m}|),$$

where

$$\operatorname{sinc}(\mathbf{x}) := \frac{\sin x_1}{x_1}\frac{\sin x_2}{x_2}, \quad \mathbf{x} = (x_1, x_2).$$

Clearly, φ_k belongs to $\mathcal{B}_{(1+\epsilon)\Omega}$, vanishes on P and $|\varphi_k(\mathbf{y}_k)| \geq 1$. Moreover, for every $\mathbf{x}$ in the disk $|\mathbf{x}| < |\mathbf{y}_k|/2$, we have

$$|\varphi_k(\mathbf{x})| \leq \frac{|\mathbf{x}^{\mathbf{m}}|}{\epsilon^{2|\mathbf{m}|}|\mathbf{x}-\mathbf{y}_k|^{2|\mathbf{m}|}} \leq \frac{2^{|\mathbf{m}|}}{\epsilon^{2|\mathbf{m}|}|\mathbf{y}_k|^{|\mathbf{m}|}} \to 0, \quad k \to \infty.$$

Therefore, the functions

$$f_n(\mathbf{x}) := \frac{\varphi_{k_n}(\mathbf{x})}{\|\varphi_{k_n}\|}$$

satisfy (25) for a suitable subsequence k_n.

9. Proof of Theorem 3

9.1. Proof of sufficiency

Assume P is not an SS for $\mathcal{B}_\Omega$. We have to show that

$$\text{There exists } \mathbf{v} \in \operatorname{Ext}(D^o) \text{ satisfying } d^-(Q)\mathbf{v} \in \Omega - \Omega. \tag{37}$$

Fix any $\epsilon > 0$. By Lemma 3, there exist $f_n \in \mathcal{B}_{(1+\epsilon)\Omega}$ and $\mathbf{x}_n \in \mathbb{R}^2$ satisfying (25). Without loss of generality, we may assume that the sequence of arguments $\arg(\mathbf{x}_n)$ converges to zero. For convenience, we may assume that it converges to zero 'from above', i.e. that it satisfies (22).

Using the compactness principle for Bernstein spaces, we may assume that

$$f_n(\mathbf{x}+\mathbf{x}_n) \text{ converge to some function } f \in \mathcal{B}_{(1+\epsilon)\Omega}. \tag{38}$$

By convergence we mean the uniform convergence on compacts in $\mathbb{C}^2$. Clearly, the limit function f satisfies $\|f\|_\infty = |f(0)| = 1$.

By (24), without loss of generality, we may assume that f vanishes on the set of segments $(l_+(0)+\mathbf{w}(0)+Q'(1,0)) \cap B_R^+(0)$. Since f is an entire function, it vanishes on the sets of lines $l_+(0)+\mathbf{w}(0)+Q'(1,0)$.

Denote by $\mathbf{v}_l := (\cos\varphi, \sin\varphi), |\varphi| < \pi/2$, be a unite vector orthogonal to $l_+(0)$. Denote by $A \subset \mathbb{R}$ the uniformly discrete set such that

$$l_+(0)+\mathbf{w}(0)+Q'(1,0) = l_+(0) + A\mathbf{v}_l.$$

It is easy to check that

$$D^-(A) = \frac{D^-(Q')}{\cos\varphi}.$$

Hence, by (23),

$$D^-(A) \geq \frac{d^-(Q)}{\cos\varphi|\mathbf{w}(0)|}.$$

Also, since f vanishes on $l_+(0) + A\mathbf{v}_l$, this set of lines is not an SS for $\mathcal{B}_{(1+\epsilon)\Omega}$. Then, by Theorem 2,

$$\frac{d^-(Q)}{\cos\varphi|\mathbf{w}(0)|}\mathbf{v}_l \in (1+\epsilon)\Omega - (1+\epsilon)\Omega. \tag{39}$$

We will consider two cases:

Case 1. There is a unique point $\mathbf{v}_0 \in \partial D^o$ such that $\langle \mathbf{w}(0), \mathbf{v}_0\rangle = 1$.

Claim 2. $\mathbf{v}_0 \in \mathrm{Ext}(D^o)$.

Indeed, assume $\mathbf{v}_0 \notin \mathrm{Ext}(D^o)$. Then $\mathbf{v}_0$ is an inner point of a segment $I \subset D^o$. If this segment is vertical, then there are infinitely many points $\mathbf{v} \in D^o$ satisfying $\langle \mathbf{w}(0), \mathbf{v}\rangle = 1$. If it is not vertical, we may find points $\mathbf{v} \in I$ for which $\langle \mathbf{w}(0), \mathbf{v}\rangle > 1$. None of the above is true. This shows that $\mathbf{v}_0 \in \mathrm{Ext}(D^o)$.

Claim 3. $\mathbf{v}_0$ *is orthogonal to* $l_+(0)$.

Indeed, if not, then there clearly exist infinitely many different vectors $\mathbf{v}$ such that $\langle \mathbf{v}, \mathbf{w}(0)\rangle = 1$ and $\langle \mathbf{v}, \mathbf{w}\rangle \le 1$ for all $\mathbf{w} \in D$. But then $\mathbf{v} \in D^o$, which contradicts the assumption above.

It follows that $\mathbf{v}_0 = |\mathbf{v}_0|\mathbf{v}_l$. Since $\langle \mathbf{v}_0, \mathbf{w}(0)\rangle = 1$, we conclude that

$$\mathbf{v}_0 = \frac{\mathbf{v}_l}{\cos\varphi|\mathbf{w}(0)|}.$$

Now, (37) follows from (39), since it holds for every $\epsilon > 0$.

Case 2. Assume there exist two points $\mathbf{v}_1, \mathbf{v}_2 \in \partial D^o$ such that $\langle \mathbf{w}(0), \mathbf{v}_j\rangle = 1,\ j = 1, 2$. Then, clearly, the above holds for every $\mathbf{v} \in \partial D^o$ on the *vertical* segment I between $\mathbf{v}_1$ and $\mathbf{v}_2$. We may assume that I is not a part of any larger segment which lies in ∂D^o. Then, clearly, $\mathbf{v}_j \in \mathrm{Ext}(D^o), j = 1, 2$. Moreover, clearly, there is no point $\mathbf{v} \in \partial D^o \setminus I$ such that $\langle \mathbf{w}(0), \mathbf{v}\rangle = 1$.

We may assume that $\mathbf{v}_1$ lies 'above' $\mathbf{v}_2$. Similarly to Claim 3, we show that $\mathbf{v}_2$ is orthogonal to $l_+(0)$.

The rest of the proof repeats the proof above.

9.2. Necessity

For convenience, below we write $\mathcal{B}_\sigma := \mathcal{B}_{[-\sigma,\sigma]}$.

We need a one-dimensional variant of Lemma 3.

Lemma 4. *Assume* $\Lambda \subset \mathbb{R}$ *is a uniformly discrete set, and let* $\sigma := D^-(\Lambda)/2$. *Then for every* $k \in \mathbb{N}$ *there is a function* $f_k \in \mathcal{B}_{\sigma+1/k}$ *and a point* $x(k)$ *satisfying*

$$\|f_k\|_\infty = 1,\ |f_k(x(k)| > 1 - 1/k,\ |x(k)| > k,\ \|f_k|_\Lambda\|_\infty < 1/k. \tag{40}$$

Observe that by Theorem 1, Λ is not an SS for B_σ.

The proof of Lemma 4 is similar to the proof of Lemma 3.

Lemma 5. *Assume a sequence of positive numbers* $\sigma(k)$ *converges to* $\sigma > 0$. *Let* $\Lambda(k) \subset \mathbb{R}$ *be u.d. sets satisfying* $\inf_k \delta(\Lambda_k) > 0$ *and such that each* $\Lambda(k)$ *is not an SS for* $B_{\sigma(k)}$. *Then there is a sequence* $x(k)$ *with* $|x(k)| \to \infty$ *such that* $\Lambda(k) - x(k)$ *converges weakly to some set* Λ *which is not an SS for* $\mathcal{B}_\sigma$. *If* $\Lambda(k)$ *are symmetric, then* $x(k)$ *can be chosen positive.*

Proof. Use Lemma 4 to find a sequence of functions $f_k \in \mathcal{B}_{\sigma(k)+1/k}$ satisfying (40) with $\Lambda = \Lambda(k)$. Clearly, if every $\Lambda(k)$ is symmetric, we may assume $x_k > 0$. Then the functions $f_k(x + x(k))$ converge to some non-trivial function $f \in \mathcal{B}_\sigma$.

By Lemma 1, we may assume that the translates $\Lambda_k - x(k)$ converge weakly to some set Λ. Clearly, $f|_\Lambda = 0$, which means that Λ is not an SS for $\mathcal{B}_\sigma$.

9.2.1. Proof of necessity

Assume $d^-(Q)\mathbf{v}_0 \in \Omega - \Omega$, for some $\mathbf{v}_0 \in \mathrm{Ext}(D^o)$. We have to show that P is not an SS for $\mathcal{B}_\Omega$, i.e. that for every small number $\eta > 0$ there is a function $f_\eta \in \mathcal{B}_\Omega$ satisfying

$$\|f_\eta\|_\infty = 1, \quad \|f_\eta|_P\|_\infty \le \eta. \tag{41}$$

Since $\mathbf{v}_0 \in \mathrm{Ext}(D^o)$, there is a point $\mathbf{w}_0 \in D$, satisfying $\langle \mathbf{v}_0, \mathbf{w}_0\rangle = 1$. We may assume that $\mathbf{w}_0 = \mathbf{w}(0)$, i.e. $\mathbf{w}_0 = (w, 0), w > 0$. Then the line $\{\mathbf{w} : \langle \mathbf{w}, \mathbf{v}_0\rangle = 1\}$ is a semi-tangent line to D at the point $\mathbf{w}(0)$. We may assume that it is the 'upper' semi-tangent line $\mathbf{w}(0) + l_+(0)$. It will be convenient to write it in the form

$$\mathbf{w}(0) + l_+(0) = \mathbf{w}(0) + t\mathbf{u}, \quad t \ge 0,$$

where $\mathbf{u}$ is a unite vector parallel to $l_+(0)$ (and so, orthogonal to $\mathbf{v}_0$).

Choose any positive sequence $\theta_k \to 0, k \to \infty$. Set

$$\sigma(k) := \frac{d^-(Q)}{2|\mathbf{w}(\theta_k)|}.$$

Then

$$\sigma(k) \to \sigma := \frac{d^-(Q)}{2|\mathbf{w}(0)|}.$$

By Theorem 1, the symmetric set

$$\frac{Q \cup (-Q)}{|\mathbf{w}(\theta_n)|}$$

is not an SS for $\mathcal{B}_{\sigma(n)}$. Hence, by Lemma 5, there exist $x(k) \to \infty$ such that the translates $Q|\mathbf{w}(\theta_k)| - x(k)$ converge weakly to some set Q', which is not an SS for $\mathcal{B}_\sigma$. By Lemma 2, condition (24) holds with $\mathbf{x}_k := x(k)(\cos\theta_k, \sin\theta_k)$.

Since Q' is not an SS for $\mathcal{B}_\sigma$, it follows from the compactness principle for Bernstein spaces, that for every $\epsilon > 0$ there exists $\delta > 0$ such that there is a function $g(x) \in \mathcal{B}_{\sigma-\delta}$ satisfying

$$\|g\|_\infty = 1, \quad \|g|_{Q'}\|_\infty \le \epsilon.$$

Consider the function $\varphi(\mathbf{x})$ defined as

$$\varphi(\mathbf{x}) := g\left(|\mathbf{w}(0)|\langle \mathbf{v}_0, \mathbf{x} - \mathbf{w}(0)\rangle\right).$$

It is easy to check that $\|\varphi\|_\infty = \|g\|_\infty = 1$ and

$$|\varphi(\mathbf{x})| \le \epsilon, \quad \mathbf{x} \in \mathbf{w}(0) + t\mathbf{u} + Q'(1,0), \quad t \in \mathbb{R}.$$

By Remark 1, condition $d^-(Q)\mathbf{v}_0 \in \Omega - \Omega$ means that Ω contains an interval I parallel to $\mathbf{v}_0$ and of length $d^-(Q)|\mathbf{v}_0|$. Since translations of Ω do not change the sampling property of P for $\mathcal{B}_\Omega$, we may assume that I is symmetric, $-I = I$. Observe that the spectrum of g lies on $[-\sigma + \delta, \sigma - \delta]$, where $2\sigma = d^-(Q)/|\mathbf{w}(0)|$. Then, clearly, the spectrum of φ lies on $(1-\delta)I$. Therefore, there is a small number $\delta' > 0$ such that $(1-\delta)I + B_{\delta'}(0) \subset \Omega$.

Now, choose a point y_0 such that $|g(y_0)| \geq 1/2$. Then we have $|\varphi(\mathbf{x})| \geq 1/2$ for all $\mathbf{x}$ on the line $\mathbf{w}(0) + (y_0, 0) + \mathbb{R}\mathbf{u}$.

Fix $R > 0$ and consider the function

$$\psi(\mathbf{x}) := \varphi(\mathbf{x})\mathrm{sinc}(\delta'(\mathbf{x} - \mathbf{x}_0)), \quad \mathbf{x}_0 := \mathbf{w}(0) - (y_0, 0) - 2R\mathbf{u}.$$

Then $\psi \in \mathcal{B}_\Omega$, $|\psi(\mathbf{x}_0)| \geq 1/2$ and

$$|\psi(\mathbf{x})| \leq \epsilon, \quad \mathbf{x} \in \mathbf{w}(0) + \mathbb{R}\mathbf{u} + Q'(1,0). \tag{42}$$

Since $|\mathrm{sinc}(\mathbf{x})| \to 0$ as $|\mathbf{x}| \to \infty$, we may assume that R is so large that

$$|\psi(\mathbf{x})| \leq \epsilon, \quad |\mathbf{x} - \mathbf{x}_0| \geq R. \tag{43}$$

By Lemma 3, we may assume that the Hausdorff distance between the set

$$(Q\partial D - \mathbf{x}_k) \cap B^+_{4R}(0)$$

and

$$(\mathbf{w}(0) + l_+(0) + Q'(1,0)) \cap B^+_{4R}(0)$$

tends to zero as $k \to \infty$. We may also assume that $B_R(\mathbf{x}_0) \subset B^+_{4R}(0)$. Then, the same is true for the sets

$$(Q\partial D - \mathbf{x}_k) \cap B_R(\mathbf{x}_0)$$

and

$$(\mathbf{w}(0) + \mathbb{R}\mathbf{u} + Q'(1,0)) \cap B_R(\mathbf{x}_0).$$

From (42), by Bernstein's inequality, for all large enough k we get

$$|\psi(\mathbf{x})| \leq C\epsilon, \quad \mathbf{x} \in (Q\partial D - \mathbf{x}_k) \cap B_R(\mathbf{x}_0),$$

where the constant C depends only on the diameter of Ω.

Finally, we see that the function

$$f(\mathbf{x}) := \frac{\psi(\mathbf{x} - \mathbf{x}_k)}{\|\psi\|_\infty}$$

belongs to $\mathcal{B}_\Omega$ and satisfies (41) with $\eta = C\epsilon$, where ϵ is any positive number.

10. Proof of Theorem 4

Recall that P is defined in (10).

(i) Assume that $D^-(V) = 0$. We have to check that P is not an SS for every space $\mathcal{B}_\Omega$, where Ω is a convex set of positive measure.

The proof is easy. Indeed, from $D^-(V) = 0$ it follows that there is a sequence of points $\mathbf{x}_n$ such that the discs $B_n(\mathbf{x}_n)$ do not intersect V. We may assume that $B_\delta(0) \subset \Omega$, for some $\delta > 0$. Then the functions

$$f_n(\mathbf{x}) := \operatorname{sinc}(\delta(\mathbf{x} - \mathbf{x}_n)), \quad n \in \mathbb{N},$$

belong to $\mathcal{B}_\Omega$ and satisfy $\|f_n\|_\infty = 1$. It is obvious that

$$\|f_n|_P\|_\infty \le \|f_n|_{\mathbb{R}^2 \setminus B_n(\mathbf{x}_n)}\|_\infty \to 0, \quad n \to \infty,$$

which proves that P is not an SS for $\mathcal{B}_\Omega$.

(ii) Assume that $D^-(V) > 0$. We have to check that P is an SS for every space $\mathcal{B}_\Omega$. The proof is a simple consequence of the uniqueness Theorem 8 below: Assume that P is not an SS for $\mathcal{B}_\Omega$, i.e. there is a sequence of functions $f_n \in \mathcal{B}_\Omega$ satisfying

$$\|f_n\|_\infty = 1, \quad \|f_n|_P\|_\infty \to 0, \quad n \to \infty.$$

Choose points $\mathbf{x}_n$ such that $|f_n(\mathbf{x}_n)| > 1 - 1/n$ and set $g_n(\mathbf{x}) := f_n(\mathbf{x} + \mathbf{x}_n)$. Then

$$g_n \in \mathcal{B}_\Omega, \ \|g_n\|_\infty = 1, \ |g_n(0)| > 1/n.$$

Then a subsequence g_{n_k} converges to some non-zero function $g \in \mathcal{B}_\Omega$.

We may assume that the translates $V - \mathbf{x}_{n_k}$ converge to some set $V' \subset \mathbb{R}^2$. We have $D^-(V') \ge D^-(V) > 0$. It is clear that $g|_{V'+T} = 0$. Theorem 8 yields $g = 0$. Contradiction.

11. Uniqueness sets

Uniqueness sets play an important role in the sampling theory. In particular, Beurling [3] proved that a u.d. set Λ is an SS for $\mathcal{B}_\sigma$ if and only if every weak limit of translates $\Lambda - x_n$ is a uniqueness set for $\mathcal{B}_\sigma$. A similar result holds in higher dimension.

Below we consider subsets of $\mathbb{R}^2$ that are uniqueness sets for some classes of entire functions of exponential type and, in particular, of the Bernstein spaces. We believe such results are of independent interest.

Given an entire function f in $\mathbb{C}^d$, let $Z_f = \{\mathbf{z} \in \mathbb{C}^d : \ f(\mathbf{z}) = 0\}$ denote its zero set. For a generic f, the set $Z_f \cap \mathbb{R}^d$ is discrete and, as far as we know, only discrete (actually, u.d.) uniqueness sets $P \subset \mathbb{R}^d$ for entire functions of exponential type, as well as for $\mathcal{B}_\Omega$ with $\Omega = I_1 \times \ldots \times I_d$ with $I_k = [-r_k, r_k]$, have been considered before (see, for example, [2], [14]). Here we will be interested in the case of non-discrete uniqueness sets $P \subset \mathbb{R}^2$.

Note that, for any entire function f, the set $Z_f \cap \mathbb{R}^2$ is represented by the equations $f_\Re(\mathbf{x}) = 0$ and $f_\Im(\mathbf{x}) = 0$ with the real analytic functions $f_\Re = \operatorname{Re} f$ and $f_\Im = \operatorname{Im} f$. Therefore, any set which is not a subset of a locally finite union of real analytic curves and discrete points in $\mathbb{R}^2$ is a uniqueness set for the whole class of entire functions in $\mathbb{C}^2$. In what follows, we work only with sets in $\mathbb{R}^2$ that are real analytic.

The idea of our considerations is, as in the case of discrete sets, getting control over the volume of the zero set Z_f in the balls $\mathbb{B}_t \subset \mathbb{C}^2$ in terms of a counting function, which in our case will be

$$\theta(t) = \#\{V \cap D_t\}, \quad t > 0, \tag{44}$$

where $V \subset \mathbb{R}^2$ is a discrete set related to P and $D_t = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < t\}$ is an open disk in $\mathbb{R}^2$.

Three main ingredients are as follows. First, if a set $E \subset \mathbb{R}^2$ is non-discrete, then there exists at most one irreducible analytic variety of complex dimension 1 (i.e., analytic curve) in $\mathbb{C}^2$ containing E. Indeed, if $E \subset \chi_j$ for two irreducible analytic curves χ_1 and χ_2, then $\dim_{\mathbb{C}} \chi_1 \cap \chi_2 = 1$ and thus $\chi_1 = \chi_2$.

Second, for any analytic curve χ and a point $\mathbf{a} \in \chi$, let $\sigma_{\chi,\mathbf{a}}(t)$ denote the volume of χ inside the ball

$$\mathbb{B}_t(\mathbf{a}) = \{z \in \mathbb{C}^2 : |\mathbf{z} - \mathbf{a}| < t\}$$

and $\sigma_\chi(t) = \sigma_{\chi,0}(t)$. Then, by Lelong's bound for volumes of analytic sets, see [8], Thm. 2.23,

$$\sigma_{\chi,\mathbf{a}}(t) \geq \pi t^2 \tag{45}$$

for any $t > 0$, with an equality if χ is a complex line.

Finally, we will use Jensen's formula for analytic functions in $\mathbb{C}^d$, see [15]:

$$m_{\log|f|}(r) := \frac{1}{V_{d-1}} \int_{S_1} \log|f(r\mathbf{z})|\, dS_1(\mathbf{z}) = \int_{r_0}^{r} \mu_f(t)\, t^{-2d+1}\, dt + C_{r_0,f},$$

where dS_1 is the normalized surface measure on $S_1 = \partial\mathbb{B}_1$, V_{2d-2} is the volume of the unit ball in $\mathbb{C}^{d-1}$, and $\mu_f(t)$ is the volume, computed with the multiplicities, of Z_f in $\mathbb{B}_t$. This gives us for $d = 2$

$$m_{\log|f|}(r) \geq \frac{1}{\pi} \int_{r_0}^{r} \sigma_{Z_f}(t)\, t^{-3}\, dt + C_{r_0,f}. \tag{46}$$

Another, and more classical, form of Jensen's formula uses the intersections of χ with complex lines. Assume $0 \notin \chi$ and, for any point $\mathbf{s}$ on the unit sphere S_1, let $n_{\mathbf{s}\chi}(t)$ be the number of intersection points of χ with the line $\mathbf{z} = \mathbf{s}\zeta$, $\zeta \in \mathbb{C}$. Let

$$n_\chi(t) = \int_{S_1} n_{\mathbf{s}\chi}(t)\, dS_1(\mathbf{s}), \tag{47}$$

then

$$m_{\log|f|}(r) \geq \int_{r_0}^{r} \frac{n_{Z_f}(t)}{t}\, dt + C_{r_0,f}. \tag{48}$$

11.1. Straight lines

We start by considering the case when $P = \{l_k\}$ is a collection of straight lines in $\mathbb{R}^2$. We will not assume, unlike in (7), that the lines are shifts of a single line l, the only condition being that none of the lines passes through the origin. Therefore, we can represent them as

$$l_k = \{\mathbf{x} \in \mathbb{R}^2 : \langle \mathbf{x}, \mathbf{e}_k \rangle = 1\} \tag{49}$$

for some vectors $\mathbf{e}_k \in \mathbb{R}^2$. Denote $\mathbf{v}_k = \mathbf{e}_k |\mathbf{e}_k|^{-2}$ and let $\theta(t)$ be defined by (44) for $V = \{\mathbf{v}_k\}$. Note that if all $\mathbf{e}_k = a_k^{-1}\mathbf{v}_l$ for $a_k \in \mathbb{R} \setminus \{0\}$, this gives us precisely the set P from (7).

Theorem 6. *In the above setting, P is the uniqueness set for entire functions of type A, provided*

$$\liminf_{t\to\infty}\frac{\theta(t)}{t}>\frac{3}{2}A.$$

Proof. Let $L_k=\{\mathbf{z}\in\mathbb{C}^2:\ \langle\mathbf{z},\mathbf{e}_k\rangle=1\}$ be the complex lines containing l_k. Then any entire function $f\not\equiv 0$ vanishing on all l_k also vanishes on all L_k, so

$$Z_f\supset Z:=\bigcup_k L_k.$$

By [12],

$$\sigma_{Z_f}(t)\ge\sigma_Z(t)=2\pi\int_0^t s\,n(s)\,ds, \tag{50}$$

where $n(s)$ is the amount of points $\mathbf{v}_k$ inside the ball $\mathbb{B}_s$. By the construction, all $\mathbf{v}_k$ are real, so $n(s)=\theta(s)$ and

$$\sigma_Z(t)=2\pi\int_0^t s\,\theta(s)\,ds.$$

There exist $A''>A'>A$ such that $\theta(s)\ge\frac{3}{2}A''s$ for all $s>r_0$ for some $r_0>0$. Therefore,

$$\sigma_Z(t)\ge 2\pi\int_{r_0}^t s\,\theta(s)\,ds\ge\pi A''(t^3-r_0^3),\quad t>r_0,$$

and

$$\frac{1}{\pi}\int_{r_0}^r\sigma_{Z_f}(t)\,t^{-3}\,dt\ge A''\,(r-\frac{3}{2}r_0).$$

If the function f is of type $A>0$, then $m_{\log|f|}(r)\le A'r$ for r sufficiently big, which, by (46), is impossible.

11.2. Dilations of circles

Next, we will be concerned with dilations of the unit circle $\mathbb{T}=\{\mathbf{x}\in\mathbb{R}^2:|\mathbf{x}|=1\}$. Let $Q=\{q_k\}\subset\mathbb{R}_+$ be a discrete set, $\theta(t)$ be the counting function of Q, and

$$P=Q\,\mathbb{T}=\bigcup_k q_k\mathbb{T}.$$

Theorem 7. *P is the uniqueness set for entire functions of type A if*

$$\liminf_{t\to\infty}\frac{\theta(t)}{t}>\frac{A}{2\alpha},$$

where

$$\alpha=\int_{S_1}|s_1^2+s_2^2|^{1/2}\,dS_1(\mathbf{s}).$$

Proof. Denote $\gamma = \{\mathbf{z} \in \mathbb{C}^2 : z_1^2 + z_2^2 = 1\}$, the unique irreducible analytic curve in $\mathbb{C}^2$ containing the circle $\mathbb{T}$. For any $\mathbf{s} \in S_1$ such that $s_1^2 + s_2^2 \neq 0$, the intersection $\mathbf{s}q\,\gamma$ of the quadric $q\,\gamma$, $q > 0$, with the line $\mathbf{z} = \mathbf{s}\,\mathbb{C}$ consists of two points given by $\mathbf{z} = \mathbf{s}\,\zeta$ with $(s_1^2 + s_2^2)\zeta^2 = q^2$. Therefore, (47) gives us

$$n_{Q\gamma}(t) = 2 \int_{S_1} \theta(|s_1^2 + s_2^2|^{1/2} t)\, dS_1(\mathbf{s}).$$

Take any small $\epsilon > 0$ and denote $E_\epsilon = \{\mathbf{z} \in \mathbb{C}^2 : |s_1^2 + s_2^2| < \epsilon^2\}$. If $\theta(t) \geq \frac{A'}{2\alpha} t$ for some $A' > A \geq 0$ and all $t > \epsilon\, r_0$, then

$$n_{Q\gamma}(t) \geq 2 \int_{S_1 \setminus E_\epsilon} \theta(|s_1^2 + s_2^2|^{1/2} t)\, dS_1(\mathbf{s}) \geq (1 - C_\epsilon) A' t$$

with $C_\epsilon \to 0$ as $\epsilon \to 0$, and

$$\int_{r_0}^{r} \frac{n_{Q\gamma}(t)}{t} dt \geq (1 - C_\epsilon) A' \,(r - r_0).$$

If an entire function $f \not\equiv 0$ vanishes on P, then $Z_f \supset Q\gamma$. By (48), it cannot have type $A < A'$.

11.3. Translations of circles

Finally, we consider translations of a circle $T = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| = r\}$.

Theorem 8. *Let $\theta(t)$ be defined by (44) for a discrete set $V \subset \mathbb{R}^2$. If*

$$\lim_{t \to \infty} \frac{\theta(t)}{t} = \infty, \tag{51}$$

then $P = T + V$ is a uniqueness set for entire functions of exponential type. More precisely, P is a uniqueness set for entire functions of type $A > 0$ if

$$B := \liminf_{t \to \infty} \frac{\theta(t)}{t} > \frac{3}{2\sqrt{2}} A. \tag{52}$$

Proof. Denote $\gamma_0 = \{\mathbf{z} \in \mathbb{C}^2 : \mathbf{z}_1^2 + \mathbf{z}_2^2 - r^2 = 0\}$, the unique irreducible analytic curve containing T.

Let $f \not\equiv 0$ be en entire function in $\mathbb{C}^2$ of exponential type, vanishing on P. Then its zero set Z_f contains $Z := \cup_k \gamma_k$, where $\gamma_k = \gamma_0 + \mathbf{v}_k$. Since $\gamma_k \cap \gamma_j$ for any $j \neq k$ is either empty or a finite set, we have

$$\sigma_{Z_f}(t) \geq \sigma_Z(t) = \sum_k \sigma_{\gamma_k}(t);$$

note that there is only finitely many γ_k intersecting $\mathbb{B}_t$.

Take any $t > 0$ sufficiently big and denote $K_t = \{k : \gamma_k \cap D_{t/2} \neq \emptyset\}$. Since $\mathbb{B}_{t/2}(a) \subset \mathbb{B}_t$ for any $a \in Z \cap D_{t/2}$, we get, by (45),

$$\sigma_Z(t) \geq \sum_{k \in K_t} \sigma_{\gamma_k}(t) \geq \theta(t/2)\, \pi (t/2)^2. \tag{53}$$

Assuming (51), take any $N > 0$ and let r_0 be such that $\theta(t) > Nt$ for all $t > r_0/2$. Then, by (53), we get

$$\frac{1}{\pi}\int_{r_0}^{r} \sigma_{Z_f}(t)\, t^{-3}\, dt \geq \frac{1}{4}\int_{r_0}^{r} \theta(t/2)\, t^{-1}\, dt > \frac{N}{8}\,(r - r_0),$$

which, in view of (46), means that f cannot have finite type. This proves the first statement and a weaker version of the second one, with $B > 8A$.

To prove the second statement in full, we need a sharper lower bound on the area of the variety Z than (53). Given $\epsilon \in (0, 1)$, let r_0 be such that

$$\theta(t)\, t^{-1} > (1 - \epsilon)B \tag{54}$$

for all $t > r_0$, and take $\mathbf{v}_k = (a_k, b_k)$ such that $r_0 < |\mathbf{v}_k| < t$. We have

$$\sigma_{\gamma_k}(t) = \int_{\gamma_k \cap \mathbb{B}_t} dm_2 = t^2 \int_{\gamma_{k,t} \cap \mathbb{B}_1} dm_2 = t^2 \sigma_{\gamma_k,t}(1), \tag{55}$$

where $\gamma_{k,t}$ is the analytic variety $\{(z_1 - a_k/t)^2 + (z_2 - b_k/t)^2 = t^{-2}\}$.

When $t \to \infty$, the varieties $\gamma_{k,t}$ converge to $\gamma_\infty := \{z_1 = \pm i z_2\}$. The convergence is not uniform in k, however we can choose r_0 such that, in addition to (54), we have

$$\sigma_{\gamma_{k,t}}(1) \geq (1 - \epsilon)\left(\sigma_{\Gamma^+_{k,t}}(1) + \sigma_{\Gamma^-_{k,t}}(1)\right) \tag{56}$$

for any $t \geq r_0$ and all k with $|\mathbf{v}_k| < t$, where $\Gamma^\pm_{k,t}$ are the complex lines

$$z_1 - a_k/t = \pm i(z_2 - b_k/t)$$

represent the families

$$\Gamma^\pm_t = \bigcup_k \Gamma^\pm_{k,t}$$

as in (49), by $\langle (z_1, z_2), \mathbf{e}^\pm_{k,t}\rangle = 1$ with

$$\mathbf{e}^\pm_{k,t} = \frac{(1, \pm i)}{c^\pm_k}\, t,$$

where $c^\pm_k = a_k \pm i b_k$. The corresponding reference vectors $\mathbf{v}_k$ are

$$\mathbf{v}^\pm_{k,t} = \mathbf{e}^\pm_{k,t} |\mathbf{e}^\pm_{k,t}|^{-2} \frac{(1, \pm i)\overline{c^\pm_k}}{2t},$$

so $n_{\Gamma^\pm_{k,t}}(s) = \theta(\sqrt{2}\, s\, t)$.

By (50), we have

$$\sigma_{\Gamma^\pm_t}(1) = 2\pi \int_0^1 s\, n_{\Gamma^\pm_t}(s)\, ds = 2\pi \int_0^1 s\, \theta(\sqrt{2}\, s\, t)\, ds = \pi t^{-2} \int_0^{\sqrt{2}\, t} s\, \theta(\sqrt{s})\, ds,$$

so (55), (56) give us

$$\sigma_Z(t) = t^2 \sum_k \sigma_{\gamma_{k,t}}(1) \geq (1-\epsilon)\, 2\pi \int\limits_0^{\sqrt{2}\,t} s\,\theta(s)\, ds.$$

Taking into account (54), we get

$$\frac{1}{\pi}\int\limits_{r_0}^{r} \sigma_{Z_f}(t)\, t^{-3}\, dt \geq (1-\epsilon) \int\limits_{r_0}^{r} \int\limits_0^{\sqrt{2}\,t} s\,\theta(s)\, ds\, t^{-3}\, dt$$

$$\geq (1-\epsilon) \int\limits_{\sqrt{2}\,r_0}^{\sqrt{2}\,r} \frac{\theta(s)}{2s}\left(2 - \frac{s^2}{r^2}\right) ds \geq (1-\epsilon)^2 \frac{2\sqrt{2}}{3} B(r - 3r_0),$$

which contradicts (46) if f has type $A < \frac{2\sqrt{2}}{3} B$. The proof is complete.

Remarks. 1. As was mentioned before, functions f from the Bernstein class $\mathcal{B}_\Omega$ have the bound

$$|f(\mathbf{x} + i\mathbf{y})| \leq C \exp\{2\pi H(\mathbf{y})\},$$

where $H(\mathbf{y}) = \max_{\mathbf{u}\in\Omega}\langle \mathbf{u}, \mathbf{y}\rangle$ is the support function of Ω, and so are of exponential type

$$A \leq 2\pi \max_{|\mathbf{y}|=1} H(\mathbf{y}).$$

Actually, as follows from the Jensen's inequality, the constant A in the both theorems can be chosen as

$$A = 2\pi \int\limits_{S_1} H(\mathbf{y})\, dS_1(\mathbf{z}).$$

2. The circle T can be replaced with a trace of arbitrary irreducible entire analytic curve. Moreover, the whole collection P can be formed by uniformly bounded arcs $T_k = \gamma_k \cap \mathbb{R}^2$ for irreducible analytic curves γ_k satisfying $\dim \gamma_k \cap \gamma_j = 0$, $j \neq k$, the set V in the definition of the counting function $\theta(t)$ being formed by arbitrary points $\mathbf{v}_k \in T_k$.

12. Multi-dimensional extensions

Below we assume that the dimension $d \geq 3$ and that Ω is a compact convex set of positive measure in $\mathbb{R}^d$.

1. The following extension of Theorem 2 holds true: *Let $l \subset \mathbb{R}^d$ be a hyperplane through the origin, $\mathbf{u}_l$ be a unit vector orthogonal to l and $H \subset \mathbb{R}$ be a u.d. set. Then $P = l + H\mathbf{u}_l$ is an SS for $\mathcal{B}_\Omega$ if and only if $D^-(H)\mathbf{u}_l \notin \Omega - \Omega$.* The proof is similar to the proof of Theorem 2.

2. One may check that an analogue of Theorem 4 holds in higher dimensions for $P = V + T$, where $V \subset \mathbb{R}^d$ is a u.d. set and $T \subset \mathbb{R}^d$ is a $(d-1)$-dimensional sphere.

3. We guess that a multi-dimensional analogue of Theorem 3 is also true. In any case, one may prove the following multi-dimensional variant of Corollary 1:

(i) Let $Q \subset (0, \infty)$ be a u.d. set and $B_1(0)$ the unit ball in $\mathbb{R}^d$. The set $Q\partial B_1(0)$ is an SS for $\mathcal{B}_\Omega$ if and only if $\mathrm{Diam}(\Omega) < d^-(Q)$.

(ii) Let D be a convex polytope around the origin. Then $Q\partial D$ is an SS for $\mathcal{B}_\Omega$ if and only if $d^-(Q)\mathbf{v} \notin \Omega - \Omega$, for every vertex in the polar set (polytope) $\mathbf{v} \in D^o$.

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Paper II

Reconstruction of bandlimited functions from space–time samples

Journal of Functional Analysis 280 (2021) 108962

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Reconstruction of bandlimited functions from space–time samples

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ARTICLE INFO

Article history:
Received 24 August 2020
Accepted 9 February 2021
Available online 19 February 2021
Communicated by K. Seip

Keywords:
Dynamical sampling
Paley–Wiener spaces
Bernstein spaces

ABSTRACT

For a wide family of even kernels $\{\varphi_u, u \in I\}$, we describe discrete sets Λ such that every bandlimited signal f can be reconstructed from the space-time samples $\{(f * \varphi_u)(\lambda), \lambda \in \Lambda, u \in I\}$.

1. Introduction

The *classical sampling problem* asks when a continuous signal (function) f can be reconstructed from its discrete samples $f(\lambda), \lambda \in \Lambda$. In the *dynamical sampling problem*, the set of space samples is replaced by a set of space-time samples (see e.g. [1], [2], [3], [5] and references therein). An interesting case is the problem of reconstruction of a bandlimited signal f from the space-time samples of its states $f * \varphi_u$ resulting from the convolution with a kernel φ_u. An important example (see [3] and [4]) is the Gaussian kernel $\varphi_u = \exp(-ut^2)$, which arises from the diffusion process.

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https://doi.org/10.1016/j.jfa.2021.108962
0022-1236/

Denote by PW_σ the Paley–Wiener space

$$PW_\sigma := \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subseteq [-\sigma, \sigma]\},$$

where $\hat{f}$ denotes the Fourier transform

$$\hat{f}(t) = \int_{\mathbb{R}} e^{-itx} f(x)\, dx.$$

A set $\Lambda \subset \mathbb{R}$ is called uniformly discrete (u.d.) if

$$\delta(\Lambda) := \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| > 0. \tag{1}$$

The following problem is considered in [3]: Given a u.d. set $\Lambda \subset \mathbb{R}$ and a kernel $\{\varphi_u, u \in I\}$, where I is an interval. What are the conditions that allow one to recover a function $f \in PW_\sigma$ in a stable way from the data set

$$\{(f * \varphi_u)(\lambda) : \lambda \in \Lambda, u \in I\}? \tag{2}$$

In what follows, we denote by Φ_u the Fourier transform of φ_u and assume that the functions $\varphi_u(x)$ and $\Phi_u(t)$ are continuous functions of (x, u) and (t, u), respectively.

It is remarked in [3], that the property of stable recovery formulated above is equivalent to the existence of two constants A, B such that

$$A\|f\|_2^2 \leq \int_I \sum_{\lambda \in \Lambda} |(f * \varphi_u)(\lambda)|^2\, du \leq B\|f\|_2^2, \quad \forall f \in PW_\sigma. \tag{3}$$

It often happens in the sampling theory that inequalities similar to the one in the right-hand side of (3) are not difficult to check. It is also the case here, it suffices to assume the uniform boundedness of the $L^1(\mathbb{R})$-norms $\|\varphi_u\|_1$:

Proposition 1. *Assume*

$$\sup_{u \in I} \|\varphi_u\|_1 < \infty. \tag{4}$$

Then for every $\sigma > 0$ and every u.d. set Λ there is a constant B such that

$$\int_I \sum_{\lambda \in \Lambda} |(f * \varphi_u)(\lambda)|^2\, du \leq B\|f\|_2^2, \quad \forall f \in PW_\sigma.$$

We present a simple proof in Section 3.

Hence, the main difficulty lies in proving the left-hand side inequality.

Recall that the classical Shannon sampling theorem states that every $f \in PW_\sigma$ admits a stable recovery from the uniform space samples $f(k/a), k \in \mathbb{Z}$, if and only if $a \geq \sigma/\pi$. The critical value $a = \sigma/\pi$ is called the Nyquist rate. Since the space-time samples (2) produce "more information" compared to the space samples, one may expect that every $f \in PW_\sigma$ can be recovered from the space-time uniform samples at sub-Nyquist spatial density. However, it is not the case, as shown in [4] for the convolution with the Gaussian kernel. On the other hand, it is proved in [3] that uniform dynamical samples at sub-Nyquist spatial rate allow one to stably reconstruct the Fourier transform $\hat{f}$ away from certain, explicitly described blind spots.

It is well-known that the nonuniform sampling is sometimes more efficient than the uniform one. For example, this is so for the universal sampling, see e.g. [6], Lecture 6. It is also the case for the problem above: For a wide class of even kernels, we show that data (2) always allows stable reconstruction, provided Λ is any relatively dense set "different" from an arithmetic progression.

To state precisely our main result, we need the following definition: Given a u.d. set Λ, the collection of sets $W(\Lambda)$ is defined as all weak limits of the translates $\Lambda - x_k$, where x_k is any bounded or unbounded sequence of real numbers (for the definition of weak limit see e.g. Lecture 3.4.1 in [6]).

Consider the following condition:

(α) $W(\Lambda)$ does not contain the empty set, and no element $\Lambda^* \in W(\Lambda)$ lies in an arithmetic progression.

The first property in (α) means that Λ is relatively dense, i.e. there exists $r > 0$ such that every interval $(x, x+r)$ contains at least one point of Λ. It follows that every element $\Lambda^* \in W(\Lambda)$ is also a relatively dense set.

The second condition in (α) means that no $\Lambda^* \in W(\Lambda)$ is a subset of $b + (1/a)\mathbb{Z}$, for any $a > 0$ and $b \in \mathbb{R}$.

Let us now define a collection of kernels $\mathcal{C}$: A kernel $\{\varphi_u, u \in I\}$, where I is an interval, belongs to $\mathcal{C}$ if it satisfies the following five conditions:

(β) There is a constant C such that

$$\sup_{u \in I} |\varphi_u(x)| \leq \frac{C}{1 + x^4}, \quad x \in \mathbb{R}; \tag{5}$$

(γ) There is a constant C such that

$$\|\varphi_{u'} - \varphi_u\|_1 \leq C|u - u'|, \quad u, u' \in I; \tag{6}$$

(ζ) Every φ_u is real and even: $\varphi_u(x) \in \mathbb{R}, \varphi_u(-x) = \varphi_u(x), x \in \mathbb{R}, u \in I$;

(η) $\sup_{u \in I} |\Phi_u(t)| > 0$ for every $t \in \mathbb{R}$;

(θ) For every $w \in \mathbb{C}$ and every $\sigma > 0$, the family $\{\Phi_u''(t) + w\Phi_u(t), u \in I\}$ forms a complete set in $L^2(0, \sigma)$.

Clearly, condition (5) implies that the derivatives $\Phi_u''(t), u \in I$, are continuous and uniformly bounded. Condition (ζ) implies that the functions Φ_u are real and even.

It is easy to check that $\mathcal{C}$ contains the Gaussian kernel, where $I = (a, b)$ is any interval such that $0 < a < b < \infty$. Indeed, it is trivial that conditions (β) - (η) are fulfilled. By using the Fourier transform, condition (θ) follows from the easy fact that there is no non-trivial function $g \in L^2(\mathbb{R}), \hat{g} \in L^2(0, \sigma)$, such that $g(x)$ is orthogonal to every function $(x^2 + w)\exp(-tx^2), t \in J$, where $J \subset (0, \infty)$, is any non-empty interval.

Our main result is as follows:

Theorem 1. *Given a u.d. set $\Lambda \subset \mathbb{R}$ and a kernel $\{\varphi_u, u \in I\} \in \mathcal{C}$. The following conditions are equivalent:*

(a) *The left inequality in (3) is true for every $\sigma > 0$ and some $A = A(\sigma)$;*
(b) *Λ satisfies condition (α).*

2. Space–time sampling in Bernstein spaces

The aim of this section is to prove a variant of Theorem 1 for the Bernstein space B_σ.

It is well-known that every function $f \in PW_\sigma$ admits an analytic continuation to the complex plane and satisfies

$$|f(x + iy)| \le Ce^{\sigma|y|}, \quad x, y \in \mathbb{R}, \tag{7}$$

where C depends only on f.

The Bernstein space B_σ is defined as the set of entire functions f satisfying (7) with some C depending only on f. An equivalent definition is that B_σ consists of the bounded continuous functions that are the inverse Fourier transforms of tempered distributions supported by $[-\sigma, \sigma]$.

Denote by $\mathcal{C}_0$ the collection of kernels $\{\varphi_u, u \in I\}$ satisfying the properties (β)-(θ) in the definition of $\mathcal{C}$ above. However, we do not require I to be an interval. In particular, it can be a countable set.

Theorem 2. *Given a u.d. set $\Lambda \subset \mathbb{R}$ and a kernel $\{\varphi_u, u \in I\} \in \mathcal{C}_0$. The following conditions are equivalent:*

(a) *For every $\sigma > 0$ there is a constant $K = K(\sigma)$ such that*

$$\|f\|_\infty \le K \sup_{\lambda \in \Lambda, u \in I} |(f * \varphi_u)(\lambda)|, \quad \forall f \in B_\sigma; \tag{8}$$

(b) *Λ satisfies condition (α).*

To prove this theorem we need a lemma:

Lemma 1. *Assume $f \in B_\sigma$ and $\{\varphi_u, u \in I\} \in \mathcal{C}_0$. If $(f * \varphi_u)(0) = 0, u \in I$, then f is odd, $f(-x) = -f(x), x \in \mathbb{R}$.*

Proof. 1. Given a function $f \in B_\sigma$, set

$$f_r(z) := \frac{f(z) + \overline{f(\bar{z})}}{2}, \quad f_i(z) := \frac{f(z) - \overline{f(\bar{z})}}{2i}.$$

Then f_r, f_i are real (on $\mathbb{R}$) entire functions satisfying $f = f_r + if_i$. It is clear that both f_r and f_i satisfy (7), so that they both lie in B_σ. Hence, since every φ_u is real, it suffices to prove the lemma for the real functions $f \in B_\sigma$.

2. Let us assume that $f \in B_\sigma$ is real. Write

$$f_e(x) := \frac{f(z) + f(-z)}{2}, \quad f_o(x) := \frac{f(z) - f(-z)}{2}.$$

Clearly, $f_e \in B_\sigma$ is even, $f_o \in B_\sigma$ is odd and $f = f_e + f_o$. Since φ_u is even, we have $(f_o * \varphi_u)(0) = 0, u \in I$. Hence, to prove Lemma 1, it suffices to check that if a real even function $f \in B_\sigma$ satisfies $(f * \varphi_u)(0) = 0, u \in I$, then $f = 0$.

3. Let us assume that $f \in B_\sigma$ is real, even and satisfies $(f * \varphi_u)(0) = 0, u \in I$. If f does not vanish in $\mathbb{C}$ then $f(z) = e^{iaz}$ for some $-\sigma \le a \le \sigma$, which implies $a = 0, f(z) \equiv 1$. Then $(f * \varphi_u)(0) = \Phi_u(0) = 0, u \in I$, which contradicts condition (η).

Hence, $f(w) = 0$ for some $w \in \mathbb{C}$. It follows that $f(-w) = 0$. Set

$$g(z) := \frac{f(z)}{z^2 - w^2}.$$

Denote by G the Fourier transform of g. Then G is continuous, even and vanishes outside $(-\sigma, \sigma)$. Now, condition $(f * \varphi_u) = 0, u \in I$, implies:

$$0 = \int_{\mathbb{R}} \varphi_u(s) f(s)\, ds = \int_{\mathbb{R}} (s^2 - w^2)\varphi_u(s) g(s)\, ds =$$

$$-\int_{-\sigma}^{\sigma} (\Phi_u''(t) + w^2\Phi_u(t))G(t)\, dt = -2\int_{0}^{\sigma} (\Phi_u''(t) + w^2\Phi_u(t))G(t)\, dt.$$

Using property (θ), we conclude that $G = 0$ and so $f = 0$. □

2.1. Proof of Theorem 2

We denote by C different positive constants.

1. Suppose $W(\Lambda)$ contains an empty set. It means that Λ contains arbitrarily long gaps: For every $\rho > 0$ there exists x_ρ such that $\Lambda \cap (x_\rho - 2\rho, x_\rho + 2\rho) = \emptyset$. Set

$$f_\rho(x) := \frac{\sin(\sigma(x - x_\rho))}{\sigma(x - x_\rho)} \in B_\sigma. \tag{9}$$

Then $\|f_\rho\|_\infty = 1$. Using (5), for all x such that $|x - x_\rho| \geq 2\rho$, we have

$$|(f_\rho * \varphi_u)(x)| \leq \int\limits_{|s| < \frac{|x - x_\rho|}{2}} \frac{2}{\sigma |x - x_\rho|} |\varphi_u(s)|\, ds + \int\limits_{|s| > \frac{|x - x_\rho|}{2}} |\varphi_u(s)|\, ds \leq \frac{C}{|x - x_\rho|}. \tag{10}$$

It readily follows that (8) is not true.

2. Suppose $\Lambda^* \subset b + (1/a)\mathbb{Z}$ for some $\Lambda^* \in W(\Lambda), b \in \mathbb{R}$ and $a > 0$. Since $\Lambda^* - b \in W(\Lambda)$, we may assume that $b = 0$.

Consider two cases: First, let us assume that $\Lambda \subset (1/a)\mathbb{Z}$. Set $\sigma = \pi a$. Clearly, the function $f(z) := \sin(\pi a z) \in B_\sigma$. Since every function φ_u is even while f is odd, one may easily check that $(f * \varphi_u)(k/a) = 0, k \in \mathbb{Z}$, so that (8) is not true.

Now, assume that $\Lambda^* \subset (1/a)\mathbb{Z}$, for some $\Lambda^* \in W(\Lambda)$. This means that for every small $\epsilon > 0$ and large $R > 0$ there is a point $v = v(\epsilon, R) \in \mathbb{R}$ such that $(\Lambda - v) \cap (-R, R)$ is close to a subset of $(1/a)\mathbb{Z}$ in the sense that for every $\lambda \in \Lambda \cap (v - R, v + R)$ there exists $k(\lambda) \in \mathbb{Z}$ with

$$|\lambda - v - k(\lambda)/a| \leq \epsilon, \quad \lambda \in \Lambda \cap (v - R, v + R).$$

For simplicity of presentation, we assume that $v = 0, a = 1$, and that

$$\Lambda \cap (-R, R) = \{\lambda_k : |k| \leq m\}, \quad |\lambda_k - k| \leq \epsilon, \quad m = [R], \quad |k| \leq m. \tag{11}$$

The proof of the general case is similar.

Fix $\epsilon := 1/\sqrt{R}$. Set

$$f(x) := \sin(\pi x) \frac{\sin(\epsilon x)}{\epsilon x} \in B_{\pi + \epsilon} \tag{12}$$

and

$$f_k(x) := \sin(\pi x) \frac{\sin(\epsilon \lambda_k)}{\epsilon \lambda_k}.$$

Then

$$\left|f(\lambda_k - s) - (-1)^{k+1} f_k(s)\right| \le \left|[\sin(\pi(\lambda_k - s)) - \sin(\pi(k-s))]\, \frac{\sin \epsilon(\lambda_k - s)}{\epsilon(\lambda_k - s)}\right| + \left|\sin(\pi s)\left(\frac{\sin\epsilon(\lambda_k - s)}{\epsilon(\lambda_k - s)} - \frac{\sin \epsilon\lambda_k}{\epsilon\lambda_k}\right)\right|. \tag{13}$$

By (11),

$$|\sin(\pi(\lambda_k - s)) - \sin(\pi(k-s))| \le \pi\epsilon, \quad s \in \mathbb{R},$$

and so the first term in the right-hand side of (13) is less than $\pi\epsilon$ for every $s \in \mathbb{R}$. To estimate the second term in (13), we use the classical Bernstein's inequality (see e.g. [6], Lecture 2.10):

$$\left|\frac{\sin\epsilon(\lambda_k - s)}{\epsilon(\lambda_k - s)} - \frac{\sin\epsilon\lambda_k}{\epsilon\lambda_k}\right| = \left|\int_0^s \left(\frac{\sin\epsilon(\lambda_k - u)}{\epsilon(\lambda_k - u)}\right)' du\right| \le |s| \left\|\left(\frac{\sin(\epsilon s)}{\epsilon s}\right)'\right\|_\infty \le \epsilon|s|.$$

Therefore,

$$|f(\lambda_k - s) - (-1)^{k+1} f_k(s)| \le \pi\epsilon(1 + |s|), \quad s \in \mathbb{R}.$$

Observe that

$$(f * \varphi_u)(\lambda_k) = \int_{\mathbb{R}} (f(\lambda_k - s) - (-1)^{k+1} f_k(s))\varphi_u(s)\, ds + (-1)^{k+1} \int_{\mathbb{R}} f_k(s)\varphi_u(s)\, ds.$$

Since f_k is odd, the last integral is equal to zero. It follows that for every $|k| \le m$ we have

$$|(f * \varphi_u)(\lambda_k)| \le \pi\epsilon \int_{\mathbb{R}} (1 + |s|)|\varphi_u(s)|\, ds, \quad u \in I.$$

Hence, using (5) we conclude that

$$|(f * \varphi_u)(\lambda)| \le C\epsilon, \quad \lambda \in \Lambda \cap (-R, R), \quad u \in I.$$

On the other hand, for all $\lambda \in \Lambda, |\lambda| \ge R$ and $|s| < 1/\epsilon = \sqrt{R}$, we get

$$|f(\lambda - s)| \le \frac{1}{\epsilon|\lambda - s|} \le \frac{\sqrt{R}}{R - \sqrt{R}} < 2\epsilon,$$

provided R is sufficiently large. This and (5) imply

$$|(f * \varphi_u)(\lambda)| \le 2\epsilon \int_{|s| < \sqrt{R}} |\varphi_u(s)|\, ds + \int_{|s| > \sqrt{R}} |\varphi_u(s)|\, ds \le C\epsilon, \quad \lambda \in \Lambda, |\lambda| \ge R, u \in I.$$

Since ϵ can be chosen arbitrarily small, we conclude that (8) is not true.

3. Assume condition (α) holds. We have to show that for every $\sigma > 0$ there is a constant $K = K(\sigma)$ such that (8) is true.

Assume this is not so. It means that there exist $\sigma > 0$ and a sequence of functions $f_n \in B_\sigma$ satisfying

$$\|f_n\|_\infty = 1, \qquad \sup_{u\in I, \lambda\in\Lambda} |(f_n * \varphi_u)(\lambda)| \leq 1/n.$$

Choose points $x_n \in \mathbb{R}$ such that $|f_n(x_n)| > 1-1/n$, and set $g_n(x) := f_n(x+x_n)$. It follows from the compactness property of Bernstein spaces (see e.g. [6], Lecture 2.8.3), that there is a subsequence n_k such that g_{n_k} converge (uniformly on compacts in $\mathbb{C}$) to some non-zero function $g \in B_\sigma$. We may also assume (by taking if necessary a subsequence of n_k) that the translates $\Lambda - x_{n_k}$ converge weakly to some $\Gamma \in W(\Lambda)$. By property (α), Γ is an *infinite set* which is not a subset of any arithmetic progression.

By Lemma 1, we see that every function $g(x-\gamma), \gamma \in \Gamma$, is odd, $g(x-\gamma) = -g(-x-\gamma)$. This gives $g(x) = -g(-x-2\gamma), x \in \mathbb{R}$. Hence, for every $\gamma, \mu \in \Gamma$ we have $g(x-2\gamma) = g(x-2\mu), x \in \mathbb{R}$. Clearly, this implies that g is a periodic function and Γ is a subset of an arithmetic progression whose difference is a half-integer multiple of the period of g. Contradiction.

3. Space–time sampling in Paley–Wiener spaces

In what follows we assume that I is an interval. Throughout this section we denote by C different positive constants.

The next statement easily follows from (6) and (8).

Corollary 1. *Assume condition (8) holds for some kernel $\{\varphi_u\}$ satisfying (6), a u.d. set Λ and $\sigma > 0$. Then there is a constant $K' = K'(\sigma)$ such that*

$$\|f\|_\infty^2 \leq K' \int_I \sup_{\lambda\in\Lambda} |(f * \varphi_u)(\lambda)|^2 \, du, \quad \forall f \in B_\sigma.$$

We skip the simple proof.

3.1. Proof of Proposition 1

Take any function $f \in PW_\sigma$ and denote by F its Fourier transform. It follows from (4) that $\|\Phi_u\|_\infty \leq C, u \in I$. Hence, the functions $F \cdot \Phi_u \in L^2(-\sigma, \sigma)$ and

$$\|f * \varphi_u\|_2 = \|F \cdot \Phi_u\|_2 \leq \|\Phi_u\|_\infty \|F\|_2 \leq C\|f\|_2.$$

Clearly, $f * \varphi_u \in PW_\sigma$, for every u. Using Bessel's inequality (see e.g. Proposition 2.7 in [6]), we get

$$\sum_{\lambda\in\Lambda} |(f*\varphi_u)(\lambda)|^2 \le C\|f*\varphi_u\|_2^2 \le C\|f\|_2^2, \quad u\in I,$$

which proves Proposition 1.

3.2. Connection between space–time sampling in B_σ and PW_σ

Observe that if Λ is a sampling set (in the 'classical sense') for the Paley–Wiener space $PW_{\sigma'}$, then it is a sampling set for the Bernstein spaces B_σ with a 'smaller' spectrum $\sigma<\sigma'$, and vice versa (see Theorem 3.32 in [6]). We provide a corresponding statement for the space-time sampling problem.

For the reader's convenience, we recall the main inequalities:

$$\|f\|_2^2 \le D\int_I \sum_{\lambda\in\Lambda} |(f*\varphi_u)(\lambda)|^2\,du, \tag{14}$$

$$\|f\|_\infty \le K \sup_{\lambda\in\Lambda, u\in I} |(f*\varphi_u)(\lambda)|. \tag{15}$$

Theorem 3. *Let Λ be a u.d. set, a kernel $\{\varphi_u\}$ satisfy (5) and (6) and $\sigma'>\sigma>0$.*

(i) *Assume that (15) holds with some constant K for all $f\in B_{\sigma'}$. Then there is a constant D such that (14) is true for every $f\in PW_\sigma$.*

(ii) *Assume that (14) holds with some constant D for all $f\in PW_{\sigma'}$. Then there is a constant K such that (15) is true for every $f\in B_\sigma$.*

Proof. The proof is somewhat similar to the proof of Theorem 3.32 in [6], but is more technical.

(i) Assume that (15) holds for every $f\in B_{\sigma'}$. Fix any positive number ε satisfying

$$\sigma+\varepsilon\le\sigma'. \tag{16}$$

Set

$$h_\varepsilon(x) := \frac{\sin\varepsilon x}{\epsilon x}, \quad \varepsilon>0. \tag{17}$$

It is easy to check that

$$h_\varepsilon(0)=1, \quad \|h_\varepsilon\|_2^2=\frac{C}{\varepsilon}, \quad \|h_\varepsilon'\|_2^2=C\varepsilon. \tag{18}$$

For every $f\in PW_\sigma$, we have

$$\|f\|_2^2=\int_{\mathbb{R}} |f(x)|^2\,dx \le \int_{\mathbb{R}} \sup_{s\in\mathbb{R}} |h_\varepsilon(x-s)f(s)|^2\,dx.$$

Note that $h_\varepsilon(x-s)f(s) \in PW_{\sigma+\varepsilon} \subset B_{\sigma'}$. By Corollary 1, for every x and s,

$$|h_\varepsilon(x-s)f(s)|^2 \leq C \int_I \sup_{\lambda\in\Lambda} \left| \int_{\mathbb{R}} \varphi_u(\lambda-s)h_\varepsilon(x-s)f(s)\,ds \right|^2 du \leq$$

$$C \int_I \sum_{\lambda\in\Lambda} \left| \int_{\mathbb{R}} \varphi_u(\lambda-s)h_\varepsilon(x-s)f(s)\,ds \right|^2 du.$$

Write

$$J = J_u(x,\lambda) := \left| \int_{\mathbb{R}} \varphi_u(\lambda-s)h_\varepsilon(x-s)f(s)\,ds \right|^2.$$

Then

$$\|f\|_2^2 \leq C \int_{\mathbb{R}} \sum_{\lambda\in\Lambda} \int_I J\,dudx. \tag{19}$$

Clearly,

$$J \leq 2(J_1 + J_2),$$

where

$$J_1 := \left| \int_{\mathbb{R}} \varphi_u(\lambda-s)h_\varepsilon(x-\lambda)f(s)\,ds \right|^2 = |h_\varepsilon(x-\lambda)|^2 \left|(f*\varphi_u)(\lambda)\right|^2,$$

$$J_2 := \left| \int_{\mathbb{R}} \varphi_u(\lambda-s)(h_\varepsilon(x-s)-h_\varepsilon(x-\lambda))f(s)\,ds \right|^2.$$

Using property (5) and the Cauchy–Schwarz inequality, we have

$$J_2 \leq \int_{\mathbb{R}} |\varphi_u(\lambda-s)|\,ds \int_{\mathbb{R}} |\varphi_u(\lambda-s)||h_\varepsilon(x-s)-h_\varepsilon(x-\lambda)|^2|f(s)|^2\,ds \leq$$

$$C \int_{\mathbb{R}} |\varphi_u(\lambda-s)||h_\varepsilon(x-s)-h_\varepsilon(x-\lambda)|^2|f(s)|^2\,ds.$$

Observe that

$$|h_\varepsilon(x-s)-h_\varepsilon(x-\lambda)|^2 = \left| \int_s^\lambda h'_\varepsilon(x-v)\,dv \right|^2 \leq |s-\lambda| \int_s^\lambda |h'_\varepsilon(x-v)|^2\,dv.$$

Hence,

$$J_2 \le C \int_{\mathbb{R}} |\varphi_u(\lambda - s)||s-\lambda| \left(\int_s^\lambda |h'_\varepsilon(x-v))|^2 dv \right) |f(s)|^2\, ds.$$

Using (18), we have

$$\int_{\mathbb{R}} \sum_{\lambda\in\Lambda} \int_I J_1\, dudx = \int_{\mathbb{R}} |h_\varepsilon(\lambda - x)|^2\, dx \sum_{\lambda\in\Lambda} \int_I |(f*\varphi_u)(\lambda)|^2 du \le$$
$$\frac{C}{\epsilon} \sum_{\lambda\in\Lambda} \int_I |(f*\varphi_u)(\lambda)|^2 du.$$

To estimate the second sum we switch the order of integration and apply (18):

$$\int_{\mathbb{R}} \sum_{\lambda\in\Lambda} \int_I J_2\, dudx \le$$
$$\int_{\mathbb{R}} \sum_{\lambda\in\Lambda} \int_I |\varphi_u(\lambda - s)||s-\lambda||f(s)|^2 \left(\int_{\mathbb{R}} \int_s^\lambda |h'_\varepsilon(x-v)|^2 dv\, dx \right) duds \le$$
$$C\varepsilon \int_{\mathbb{R}} \int_I \sum_{\lambda\in\Lambda} |\varphi_u(\lambda - s)||s-\lambda|^2 |f(s)|^2 duds.$$

Now, by (5) we get

$$\sum_{\lambda\in\Lambda} |\varphi_u(\lambda - s)||s-\lambda|^2 \le C \sum_{\lambda\in\Lambda} \frac{(\lambda - s)^2}{1+(\lambda - s)^4} < C, \quad u \in I, s \in \mathbb{R},$$

where the second inequality holds since Λ is a u.d. set (see definition in (1)). Hence,

$$\int_{\mathbb{R}} \sum_{\lambda\in\Lambda} \int_I J_2\, dudx \le C\varepsilon |I| \|f\|_2^2,$$

where $|I|$ is the length of I.

Combining this with the estimate for J_1 and using (19), we conclude that

$$\|f\|_2^2 \le \frac{C}{\varepsilon} \sum_{\lambda\in\Lambda} \int_I |(f*\varphi_u)(\lambda)|^2\, du + C\varepsilon |I| \|f\|_2^2.$$

Choosing ε small enough, we obtain (14).

(ii) Assume (14) holds with some constant D for all $f \in PW_{\sigma'}$.

We will argue by contradiction. Assume that there is no constant K such that (15) holds for every $f \in B_\sigma$. This means that there exist $g_j \in B_\sigma$ such that $\|g_j\|_\infty = 1$,

$$\sup_{u\in I,\lambda\in\Lambda} |(g_j * \varphi_u)(\lambda)| < \frac{1}{j}, \tag{20}$$

and for some points x_j we have $|g_j(x_j)| \geq 1/2$.

Assume $\varepsilon > 0$ satisfies (16) and let h_ε be defined by formula (17). Set

$$f_j(x) := g_j(x)h_\varepsilon(x - x_j).$$

It is clear that for every j we have $f_j \in PW_{\sigma'}$, $\|f_j\|_\infty \leq 1$, and that $|f_j(x_j)| \geq 1/2$. The last two inequalities and the Bernstein's inequality imply that there is a constant $K' > 0$ such that

$$\|f_j\|_2 \geq K', \quad j \in \mathbb{N}. \tag{21}$$

By (14), we get

$$\|f_j\|_2^2 \leq C \int_I \sum_{\lambda\in\Lambda} |(f_j * \varphi_u)(\lambda)|^2 du = C \int_I \sum_{\lambda\in\Lambda} \left| \int_{\mathbb{R}} g_j(x)\varphi_u(\lambda - x)h_\varepsilon(x - x_j)dx \right|^2 du.$$

This gives

$$\|f_j\|_2^2 \leq C(\tilde{J}_1 + \tilde{J}_2), \tag{22}$$

where $\tilde{J}_1$ and $\tilde{J}_2$ are defined as follows:

$$\tilde{J}_1 := \int_I \sum_{\lambda\in\Lambda} \left| \int_{\mathbb{R}} g_j(x)\varphi_u(\lambda - x)(h_\varepsilon(x - x_j) - h_\varepsilon(\lambda - x_j))dx \right|^2 du,$$

$$\tilde{J}_2 := \int_I \sum_{\lambda\in\Lambda} \left| \int_{\mathbb{R}} g_j(x)\varphi_u(\lambda - x)h_\varepsilon(\lambda - x_j)dx \right|^2 du.$$

By Bessel's inequality (see, e.g. [6], Proposition 2.7) and (18),

$$\sum_{\lambda\in\Lambda} |h_\varepsilon(\lambda - s)|^2 \leq C\|h_\varepsilon\|_2^2 \leq \frac{C}{\varepsilon}, \quad \forall s \in \mathbb{R}.$$

Therefore, using (20) we arrive at

$$\tilde{J}_2 \leq \frac{C}{\varepsilon j^2}|I|.$$

Let us now estimate $\tilde{J}_1$. Recall that $\|g_j\|_\infty = 1$. Using the change of variables $x = t+\lambda$, we get

$$\tilde{J}_1 \leq \int_I \sum_{\lambda\in\Lambda} \left(\int_{\mathbb{R}} \left| \varphi_u(-t) \int_0^t h'_\varepsilon(s+\lambda-x_j)\, ds \right| dt \right)^2 du.$$

Now, use the Cauchy–Schwarz inequality:

$$\tilde{J}_1 \leq \int_I \sum_{\lambda\in\Lambda} \int_{\mathbb{R}} |\varphi_u(-t)|^2 (1+t^2)^2\, dt \int_{\mathbb{R}} \frac{1}{(1+t^2)^2} \left| \int_0^t h'_\varepsilon(s+\lambda-x_j)\, ds \right|^2 dt\, du.$$

Using again the Cauchy–Schwarz inequality and condition (5), we arrive at

$$\tilde{J}_1 \leq C \int_I \sum_{\lambda\in\Lambda} \int_{\mathbb{R}} \frac{|t|}{(1+t^2)^2} \left| \int_0^t |h'_\varepsilon(s+\lambda-x_j)|^2\, ds \right| dt\, du.$$

Finally, Bessel's inequality yields

$$\sum_{\lambda\in\Lambda} |h'_\varepsilon(s+\lambda-x_j)|^2 \leq C\|h_\varepsilon\|_2^2 \leq C\varepsilon,$$

and we conclude that

$$\tilde{J}_1 \leq C|I|\varepsilon.$$

We now insert the estimates for $\tilde{J}_1, \tilde{J}_2$ in (22) and use (21) to get the estimate

$$(K')^2 \leq \frac{C}{\varepsilon j^2} + C|I|\varepsilon.$$

Choosing ε sufficiently small, we arrive at contradiction for all large enough j. □

3.3. Proof of Theorem 1

The proof easily follows from Theorems 2 and 3.

Assume that the assumptions of Theorem 1 hold.

(i) Assume that Λ satisfies condition (α). Then by Theorem 2, for every $\sigma > 0$ there exists $K = K(\sigma)$ such that inequality (8) is true. Applying Theorem 3, we see that there exists $A = A(\sigma) > 0$ the left-hand side inequality in (3) is also true for every $\sigma > 0$.

(ii) Assume that Λ does not satisfy condition (α). Then by Theorem 2, there exists $\sigma > 0$ such that there is no constant K for which condition (8) is true. Applying Theorem 3, we see that for every positive $\sigma' > \sigma$ there is no constant D such that inequality (14) holds for every $f \in PW_{\sigma'}$.

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Paper III

On geometry of the unit ball of Paley-Wiener space over two symmetric intervals

A. Ulanovskii and I. Zlotnikov (2022) "On Geometry of the Unit Ball of Paley–Wiener Space Over Two Symmetric Intervals,"
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–35
https://doi.org/10.1093/imrn/rnac043

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Paper IV

Contractions in Paley-Wiener spaces

Contractive projections in Paley-Wiener spaces

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July 19, 2022

Abstract

Let S_1 and S_2 be disjoint finite unions of parallelepipeds. We describe necessary and sufficient conditions on the sets S_1, S_2 and exponents p such that the canonical projection P from $PW^p_{S_1 \cup S_2}$ to $PW^p_{S_1}$ is a contraction.

1 Introduction

In the present short paper, we study the particular case of the following

Main problem.

Let X, Y be spaces of functions. Assume that Y is a subspace of X and $P : X \to Y$ is a projection. What assumptions should be imposed on X, Y, and P to ensure that P is a contraction?

The case when X is an L^p space and Y is an arbitrarily closed subspace of L^p was completely solved by T. Andô [1]. He showed that if $p \neq 2$ and P leaves constants intact then P is a conditional expectation with respect to some σ-algebra. He also obtained a complete characterization even without this assumption, see [1], Theorem 2.

One prominent example is the case $X = L^p(\mathbb{T}^d)$, $Y = \{f \in L^p(\mathbb{T}^d) \mid \hat{f}(\lambda) = 0, \lambda \notin \Lambda\}$ and the projection P being an idempotent Fourier multiplier. In this case the result of Andô has the following simple formulation.

Theorem 1. *The projection P is a contraction if and only if either $p = 2$ or $\Lambda \subset \mathbb{Z}^d$ is a coset.*

Recently, O.F. Brevig, J. Ortega-Cerdà, and K. Seip [2] studied the contractivity of the similar idempotent Fourier multipliers in the case when X is a Hardy space, that is

$$X = H^p(\mathbb{T}^d) = \{f \in L^p(\mathbb{T}^d) \mid \hat{f}(n_1, n_2, \dots, n_d) = 0 \text{ if } n_k < 0 \text{ for some } k\}.$$

They showed that if $p \notin 2\mathbb{N}$ then the only contractions are the same as in the result of Andô, while for $p = 2k, k \in \mathbb{N}$ there exist non-trivial examples if $d \geq 3$. For the complete statement of their results, see [2], Theorem 1.2.

Since the Hardy spaces are the subsets of the Lebesgue spaces on the torus with some restrictions on the Fourier transform, it is natural to consider the analogue of the problem solved in [2] in the setting of Euclidean spaces. Specifically, in this note we consider the operators acting on the Paley-Wiener spaces. For the Fourier transform of the function f from $L^p, 1 \le p \le \infty$, we fix the notation

$$\mathcal{F}(f)(t) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x,t\rangle} dx, \quad t \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product and the integral is understood in the sense of distributions.

We recall that for a compact set $S \subset \mathbb{R}^d$ and $1 \le p \le \infty$ the Paley-Wiener space PW_S^p is defined by

$$PW_S^p = \left\{ f \in L^p(\mathbb{R}^d) : \mathrm{Sp}(f) := \operatorname{supp} \mathcal{F}(f) \subset S \right\}.$$

Sometimes, the spaces PW_S^∞ are called Bernstein spaces in the literature. We refer the reader to books [3] and [4] for more details regarding properties of Paley-Wiener spaces.

Assume that S_1 and S_2 are disjoint compact sets. In what follows we deal with the *canonical projection* P acting from $PW^p_{S_1 \cup S_2}$ to $PW^p_{S_1}$ defined by

$$P(f)(x) := \mathcal{F}^{-1}[\mathcal{F}(f) \cdot \chi_{S_1}](x).$$

For a set $S \subset \mathbb{R}^d$ and $k \in \mathbb{N}_0$ we define kS inductively as $0S = \{0\}$, $(k+1)S = kS + S$, where $+$ denotes the Minkowski sum.

Theorem 2. *Let S_1 and S_2 be finite unions of parallelepipeds in $\mathbb{R}^d$ such that $\mathrm{mes}(S_1 \cap S_2) = 0$ and let P be a canonical projection from $PW^p_{S_1 \cup S_2}$ to $PW^p_{S_1}$. We have*

1. *If $p \in 2\mathbb{N}$ then P is a contraction if and only if*

$$\mathrm{mes}\left(\left(\frac{p}{2} S_1 + \left(\frac{p}{2} - 1 \right) (-S_1) \right) \cap S_2 \right) = 0. \tag{1}$$

2. *If $p \notin 2\mathbb{N}$ then P is a contraction if and only if $\mathrm{mes}(S_1) = 0$ or $\mathrm{mes}(S_2) = 0$.*

Remark 1. *Note that from this theorem it follows that if P is a contraction for $p = 2(n+k)$, $n, k \in \mathbb{N}$ then it is a contraction for $p = 2n$ as well. On the other hand, for each $n \in \mathbb{N}$ there are sets S_1, S_2 such that P is a contraction if and only if $p = 2m, m \in \mathbb{N}, m \le n$. For example, we could take $S_1 = [0,1], S_2 = [n, n+1]$. Note also that we can not have projections for all $p \in 2\mathbb{N}$ (unless $\mathrm{mes}(S_1) = 0$ or $\mathrm{mes}(S_2) = 0$) since for large enough n we have*

$$\mathrm{mes}\left((nS_1 + (n-1)(-S_1)) \cap S_2 \right) > 0.$$

Remark 2. *We also note that, in contrast with the Hardy space setting, our argument does not depend on the dimension under the assumption that S_1 and S_2 are finite unions of parallelepipeds (in fact, the proof is still valid under even more general assumptions, see Remark 4).*

2 Proofs

For simplicity, we consider the case $d = 1$, i.e. the sets involved are disjoint unions of intervals. The proof of the general case is done exactly in the same way and therefore is omitted.

We follow the argument from [2] (see Lemma 3.1 there) and invoke the criterion due to H.S. Shapiro from [5].

Lemma 1. *Assume that S_1 and S_2 are disjoint compact sets. Let $f \in PW^p_{S_1}$. The following statements are equivalent:*

(a) *The inequality*

$$\|f\|_{L^p} \leq \|f + g\|_{L^p}$$

is true for every $g \in PW^p_{S_2}$.

(b) *The equality*

$$\int_{\mathbb{R}} |f(x)|^{p-2} f(x)\overline{g(x)}\, dx = 0 \tag{2}$$

holds for every $g \in PW^p_{S_2}$.

For $1 < p < \infty$ this lemma immediately follows from [5], Theorem 4.2.1. In the case $p = 1$, the statement follows from [5], Theorem 4.2.2, since the set $\{x : f(x) = 0\}$ has measure zero for any non-trivial entire function f. Informally, this lemma corresponds to taking the derivative of $||f + \varepsilon g||^p_{L^p}$ at $\varepsilon = 0$.

2.1 Proof of Theorem 2, Part 1.

Now, we consider $p \in 2\mathbb{N}$, i.e. $p = 2k, k \in \mathbb{N}$. First, we note that the sufficiency of condition (1) follows from Lemma 1. Indeed, take any $q \in PW^p_S$ and denote by $f = P(q)$ and $g = q - f$. Note that $f \in PW^p_{S_1}$ and $g \in PW^p_{S_2}$. Since Schwartz functions are dense in PW^p_S if S is a finite union of intervals, we will assume that f, g are Schwartz functions so that all Fourier transforms are genuine functions. By Titchmarsh's theorem, we have

$$\mathrm{Sp}(|f(x)|^{p-2} f(x)) = \mathrm{Sp}(f^k(x)(\overline{f(x)})^{k-1}) \subset kS_1 + (k-1)(-S_1).$$

Using condition (1) and Plancherel's theorem, we get

$$\int_{\mathbb{R}} |f(x)|^{p-2} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}} \mathcal{F}(|f|^{p-2}\bar{f})(t)\overline{\mathcal{F}(g)(t)}\, dt = 0.$$

Applying Lemma 1, we finish the proof of the sufficiency:

$$\|q\|_{L^p} = \|f + g\|_{L^p} \geq \|f\|_{L^p}.$$

Second, we assume that for the sets S_1 and S_2 equation (1) does not hold and prove that the projection $P : PW^p_{S_1 \cup S_2} \to PW^p_{S_1}$, is not a contraction. We

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use again Lemma 1 and reduce the problem to the construction of functions f and g that violate condition (2). Recall that $p = 2k, k \in \mathbb{N}$, and set

$$f(x) = \mathcal{F}(\chi_{S_1})(x), \quad g(x) = \mathcal{F}(\chi_{S_2})(x),$$

where χ_S stands for the indicator function of the set S. Note that $f, g \in L^p(\mathbb{R})$ since $p \geq 2$. Denote by

$$\Phi(t) = \mathcal{F}^{-1}(|f|^{p-2}f)(t).$$

Since S_1 is a finite union of intervals, $\mathrm{Sp}(\Phi) = kS_1 + (k-1)S_1$ and, moreover, $\Phi(t) > 0$ on the interior of $kS_1 + (k-1)S_1$. We have

$$\int_{\mathbb{R}} |f(x)|^{p-2} f(x)\overline{g(x)}\, dx = \int_{S_2} \Phi(t)dt > 0.$$

Thus, we arrive at a contradiction, since by Lemma 1 we have

$$\|f\|_{L^p} > \|f + g\|_{L^p}.$$

2.2 Proof of Theorem 2, Part 2.

First, we prove that the canonical projection $P : PW^p_{S_1 \cup S_2} \to PW^p_{S_1}$ is not a contraction if p is not an even integer and $1 \leq p < \infty$.

We have to find functions f from $PW^p_{S_1}$ and g from $PW^p_{S_2}$ such that (2) does not hold. Clearly, it suffices to prove the following

Lemma 2. *Let $1 \leq p < \infty$ and $p \notin 2\mathbb{N}$. Assume that I and J are non-empty disjoint intervals. There is a function f from the Paley-Wiener space PW^p_I such that* $\mathrm{mes}(\mathrm{Sp}(|f|^{p-2}f) \cap J) > 0$.

Proof. Without loss of generality we can assume that $I = [-1, 1]$. First, we consider the case $p > 1$. Set

$$h(x) = (x^2 - 1)\frac{\cos(2\pi x) - \cosh(2\pi)}{\prod\limits_{k=1}^{N} ((x + k)^2 + 1)}, \tag{3}$$

where $N > \frac{1}{2(p-1)} + 2$ so that $|h(x)|^{p-1} \in L^1(\mathbb{R})$. Therefore, the function $g(x) = |h(x)|^{p-2}h(x)$ belongs to $L^1(\mathbb{R})$ as well. Note that by the Paley-Wiener Theorem $h \in PW^p_I$. On the other hand, since h changes sign at the points ± 1, the function g is not analytic. Therefore, its Fourier transform $G = \mathcal{F}(g)$ is a continuous function which is not compactly supported. Also, note that since h is even, G is even as well.

Let $J = [a, b]$ and assume without loss of generality that $a > 1$. Since G is not compactly supported, there exists $x_0 \in \mathbb{R}$ such that $x_0 > a$ and $x_0 \in \mathrm{supp}\, G$ (here we used that G is even so its support must extend both to $+\infty$ and $-\infty$).

Since G is continuous and $G(x_0) \neq 0$, we have $G(x) \neq 0, x_0 - \varepsilon < x < x_0 + \varepsilon$ for small enough ε.

Consider $f(x) = h(\frac{a}{x_0}x)$. Since $\frac{a}{x_0} < 1$, f also belongs to PW_I^p. On the other hand, we have

$$\mathcal{F}(|f|^{p-2}f)(\xi) = \frac{x_0}{a}G\left(\frac{x_0}{a}\xi\right).$$

Thus, $\mathcal{F}(|f|^{p-2}f)(a) = \frac{x_0}{a}G(x_0) \neq 0$ and $\operatorname{mes}(\operatorname{supp}\mathcal{F}(|f|^{p-2}f) \cap J) > 0$ as required.

Remark 3. *Although it seems plausible that* $\operatorname{Sp} G = \mathbb{R}$ *we only managed to prove that* $\operatorname{Sp} G$ *is unbounded.*

Next, we deal with $p = 1$. Consider the same function $h(x)$ with $N = 2$. Note that $Q(x) = |f(x)|^{p-2}f(x) = 2\chi_{[-1,1]}(x) - 1$, whence

$$\mathcal{F}(Q)(x) = \frac{2\sin(2\pi x)}{\pi x} - \delta_0,$$

where δ_0 is a Dirac delta measure at 0. Clearly, the support of the distribution $\mathcal{F}(Q)$ is $\mathbb{R}$. This finishes the proof of the lemma. □

It remains to consider $p = \infty$. In this case we will show that the projection is not contractive directly. Again, we assume that $S_1 \supset [-1, 1]$. Set

$$f(x) = \frac{\sin(2\pi x)}{2\pi x}, \quad f \in PW_{S_1}^\infty.$$

Put $g(x) = \mathcal{F}(\chi_{S_2})(x)$. Clearly, $g(0) > 0$ and $g \in PW_{S_2}^\infty$. Consider $f_\varepsilon(x) = f(x) - \varepsilon g(x)$. We are going to show that for sufficiently small positive ε we have $\|f_\varepsilon\|_{L^\infty} < 1$. This will contradict contractivity and finish the proof of the theorem.

Note that $f(x), g(x) \to 0$ as $|x| \to \infty$. Since $f(0) = 1$ and $|f(x)| < 1$ when $x \neq 0$, for every $\delta > 0$ there exists $\varepsilon > 0$ such that $|f(x) - \varepsilon g(x)| < 1$ if $|x| > \delta$. Thus, it remains to consider $|x| < \delta$.

We have $1 - x^2 \leq f(x) \leq 1$ if $|x| < \frac{1}{100}$. On the other hand, $g(x) = c_0 + O(x)$ for some $c_0 > 0$ if x is close enough to 0. Thus,

$$0 < \operatorname{Re}(f(x) - \varepsilon g(x)) \leq 1 - \varepsilon c_0 + C\varepsilon|x|$$

and

$$|\operatorname{Im}(f(x) - \varepsilon g(x))| \leq C\varepsilon|x|$$

for some constant C and small enough x. Therefore,

$$|f(x) - \varepsilon g(x)| \leq 1 - \varepsilon c_0 + 2C\varepsilon|x|.$$

Choosing δ so that $\delta < \frac{1}{100}$ and $2C\delta < c_0$, we get that

$$|f(x) - \varepsilon g(x)| < 1$$

for $|x| \leq \delta$. Thus, $||f - \varepsilon g||_{L^\infty(\mathbb{R})} < 1$ and the projection is not contractive.

Remark 4. *In fact, our results hold more generally when the sets S_1, S_2 are closures of open sets with boundaries of measure* 0 *and results for $p \notin 2\mathbb{N}$ require only that S_1, S_2 contain some open balls. On the other hand, it would be interesting to study the case of arbitrary closed sets S_1, S_2 of positive measure with empty interior.*

Acknowledgement

The authors are grateful to Ole Fredrik Brevig, Joaquim Ortega-Cerdà, and Kristian Seip for valuable discussions and constructive remarks.

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Paper V

Completeness of Certain Exponential Systems and Zeros of Lacunary Polynomials

Completeness of Certain Exponential Systems and Zeros of Lacunary Polynomials

Aleksei Kulikov Alexander Ulanovskii Ilya Zlotnikov

October 2, 2022

Abstract

Let Γ be a subset of $\{0, 1, 2, ...\}$. We show that if Γ has 'gaps' then the completeness and frame properties of the system $\{t^k e^{2\pi i n t} : n \in \mathbb{Z}, k \in \Gamma\}$ differ from those of the classical exponential systems. This phenomenon is closely connected with the existence of certain uniqueness sets for lacunary polynomials.

Keywords: completeness, frame, totally positive matrix, generalized Vandermonde matrix, uniqueness set, lacunary polynomials

1 Introduction

Let Λ be a *separated* set of real numbers. Denote by

$$E(\Lambda) := \{e^{2\pi i \lambda t}, \lambda \in \Lambda\}$$

the corresponding exponential system.

Approximation and representation properties of exponential systems in different function spaces is a classical subject of investigation. In particular, the completeness and the frame problems of $E(\Lambda)$ for the space $L^2(a, b)$ can be stated as follows: Determine if

(a) (*Completeness property of* $E(\Lambda)$) every function F in $L^2(a, b)$ can be approximated arbitrarily well in L^2-norm by finite linear combinations of exponential functions from $E(\Lambda)$;

(b) (*Frame property of* $E(\Lambda)$) there exist two positive constants A and B such that for every $F \in L^2(a, b)$ we have

$$A\|F\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle F, e^{2\pi i \lambda t} \rangle|^2 \leq B\|F\|_2^2,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $L^2(a, b)$.

Note that the notion of frame is very important and can be defined in similar manner for an arbitrary system of elements $E = \{e_\lambda\}$ in a Hilbert space H. If E is a frame in H, then every element f from H admits a (maybe, non-unique) representation

$$f = \sum_{e_\lambda \in E} c_\lambda e_\lambda,$$

for some l^2-sequence of complex numbers c_λ (see e.g. [3]).

It is easy to check that the completeness property of $E(\Lambda)$ is translation-invariant: If $E(\Lambda)$ is complete in $L^2(a,b)$, then it is complete in $L^2(a+c, b+c)$, for every $c \in \mathbb{R}$. As a 'measure of completeness', one may introduce the so-called *completeness radius* of $E(\Lambda)$:

$$CR(\Lambda) = \sup\{a \geq 0 : E(\Lambda) \text{ is complete in } L^2(-a,a)\}.$$

Similarly, the frame property of $E(\Lambda)$ is also translation-invariant, and one may introduce the *frame radius* as

$$FR(\Lambda) = \sup\{a \geq 0 : E(\Lambda) \text{ is a frame in } L^2(-a,a)\}.$$

Both radii above can be expressed in terms of certain densities:

(A) The celebrated Beurling–Malliavin theorem [1] states that $CR(\Lambda) = D^*(\Lambda)$. Here D^* is the so-called upper (or external) Beurling–Malliavin density.

(B) It follows from the classical 'Beurling Sampling Theorem' [2] (see also a detailed discussion in [7]) that $FR(\Lambda) = D^-(\Lambda)$, where Λ is a separated (also called uniformly discrete) set and $D^-(\Lambda)$ is the lower uniform density of Λ.

We refer the reader to [8] or [11] for a complete description of exponential frames for the space $L^2(a,b)$. It is not given in terms of a density of Λ.

Observe that the proofs of (A) and (B) use techniques from the complex analysis.

The density D^* can be defined and the Beurling–Malliavin formula for the completeness radius remains valid for the *multisets* $(\Lambda, \Gamma(\lambda))$, where $\Lambda \subset \mathbb{R}$ and $\Gamma(\lambda) = \{0, ..., n(\lambda) - 1\}$, i.e. for the systems

$$E(\Lambda, \Gamma(\lambda)) := \{t^k e^{2\pi i \lambda t} : \lambda \in \Lambda, t = 0, ..., n(\lambda) - 1\}. \tag{1}$$

Here $n(\lambda)$ is the multiplicity (number of occurrences) of the element $\lambda \in \Lambda$. The same is true for the frame radius, see [4]. In particular, if $\Lambda = \mathbb{Z}$ and $\Gamma(\lambda) = \Gamma_N := \{0, ..., N-1\}$, $\lambda \in \Lambda$, then one has

$$CR(\mathbb{Z}, \Gamma_N) = FR(\mathbb{Z}, \Gamma_N) = N/2 = \#\Gamma_N/2, \tag{2}$$

where $\#\Gamma$ is the number of elements of Γ, $CR(\mathbb{Z}, \Gamma_N)$ and $FR(\mathbb{Z}, \Gamma_N)$ are the completeness and frame radius of $E(\mathbb{Z}, \Gamma_N)$, respectively.

One may consider the completeness property of systems in (1) in $L^p(a,b)$ and $C([a,b])$. For each of these spaces, the completeness property is translation-invariant. Clearly, the completeness in $C([-a,a])$ implies the completeness in

$L^p(-a,a)$ for every $1 \le p < \infty$. Observe that if $E(\Lambda, \Gamma(\Lambda))$ is not complete in $C([-a,a])$, its *deficiency* in $C([-a,a])$ is at most 1, i.e. by adding to the system an exponential function $e^{2\pi iat}, a \notin \Lambda$, the new lager system becomes complete in $C([-a,a])$ (see e.g. discussion in [10]). It easily follows that every system in (1) has the same completeness radius for every space considered above.

2 Statement of Problem and Results

Let us now introduce somewhat more general systems. Assume that $\Lambda \subset \mathbb{R}$ is a discrete set and that to every $\lambda \in \Lambda$ there corresponds a finite or infinite set $\Gamma(\lambda) \subset \mathbb{N}_0 := \{0, 1, 2, 3, ...\}$. Set

$$E(\Lambda, \Gamma(\lambda)) = \{t^\gamma e^{2\pi i\lambda t} : \lambda \in \Lambda, \gamma \in \Gamma(\lambda)\}.$$

Inspired by a recent work of H. Hedenmalm [5], we ask: What are the completeness and frame properties of $E(\Lambda, \Gamma(\lambda))$? In this note we restrict ourselves to the case $\Lambda = \mathbb{Z}$ and $\Gamma(n) = \Gamma \subset \mathbb{N}_0, n \in \mathbb{Z}$, is a fixed set. That is, we will consider the completeness and frame properties of the system

$$E(\mathbb{Z}, \Gamma) := \{t^\gamma e^{2\pi int} : n \in \mathbb{Z}, \gamma \in \Gamma\}, \quad \Gamma \subset \mathbb{N}_0.$$

Let us now introduce the formal analogues of the completeness and frame radius:

$$CR(\mathbb{Z}, \Gamma) := \sup\{a \ge 0 : E(\mathbb{Z}, \Gamma) \text{ is complete in } L^2(-a,a)\},$$

$$FR(\mathbb{Z}, \Gamma) := \sup\{a \ge 0 : E(\mathbb{Z}, \Gamma) \text{ is a frame in } L^2(-a,a)\}.$$

We also define the completeness radius $CR_C(\mathbb{Z}, \Gamma)$ in the spaces of continuous functions:

$$CR_C(\mathbb{Z}, \Gamma) := \sup\{a \ge 0 : E(\mathbb{Z}, \Gamma) \text{ is complete in } C([-a,a])\}.$$

In what follows, to exclude trivial remarks, we will always assume that $0 \in \Gamma$.

Set

$$\Gamma_{even} = \Gamma \cap 2\mathbb{Z} \quad \text{and} \quad \Gamma_{odd} = \Gamma \cap (2\mathbb{Z}+1),$$

and introduce the following number

$$r(\Gamma) := \begin{cases} \#\Gamma_{odd} + \frac{1}{2}, & \text{if } \#\Gamma_{odd} < \#\Gamma_{even}, \\ \#\Gamma_{even}, & \text{if } \#\Gamma_{odd} \ge \#\Gamma_{even}. \end{cases}$$

Observe that $r(\Gamma) < \#\Gamma/2$ unless $\#\Gamma_{even} = \#\Gamma_{odd}$ or $\#\Gamma_{even} = \#\Gamma_{odd} + 1$.

It turns out that the completeness and frame properties of $E(\mathbb{Z}, \Gamma)$ may differ from the ones for the systems considered above. In particular, we have

Theorem 1. *Given any finite or infinite set $\Gamma \subset \mathbb{N}_0$ satisfying $0 \in \Gamma$. Then*
(i) $CR(\mathbb{Z}, \Gamma) = \#\Gamma/2$;
(ii) $CR_C(\mathbb{Z}, \Gamma) = FR(\mathbb{Z}, \Gamma) = r(\Gamma)$.

Below we prove more precise results.

Theorem 1 shows that property (2) is no longer true for the systems $E(\mathbb{Z},\Gamma)$.

The proof of part (i) uses mainly basic linear algebra. We will see that the completeness property of $E(\mathbb{Z},\Gamma)$ in $L^2(a,b)$ is translation invariant, and so $CR(\mathbb{Z},\Gamma)$ still can be viewed as a 'measure of completeness' of $E(\mathbb{Z},\Gamma)$.

On the other hand, neither the frame property in $L^2(a,b)$ nor the completeness property in $C([a,b])$ is translation invariant in the sense that both of them depend on the length of the interval (a,b) and also on its position. This phenomenon is intimately connected with the solvability of certain systems of linear equations and also with the existence of certain uniqueness sets for lacunary polynomials, see Theorem 2 below.

Given any finite set $M \subset \mathbb{N}_0$, let $P(M)$ denote the set of real polynomials with exponents in M:

$$P(M) := \{P(x) = \sum_{m_j \in M} c_j x^{m_j} : c_j \in \mathbb{R}\}.$$

If $M \subset \mathbb{N}_0$ consists of n elements (shortly, $\#M = n$), then clearly no set $X \subset \mathbb{R}$ satisfying $\#X \le n-1$ is a uniqueness set for $P(M)$, i.e. there is a non-trivial polynomial $P \in P(M)$ which vanishes on X. This is no longer true if $\#X = n$. Moreover, there exist real uniqueness sets X, $\#X = n$, that are uniqueness sets for every space $P(M), \#M = n$. Indeed, by Descartes' rule of signs, each $P \in P(M)$ may have at most $n-1$ distinct positive zeros, and so every set of n positive points is a uniqueness set for $P(M)$. Here we present a less trivial example of such sets. Given N distinct real numbers $t_1,\dots,t_N$, set

$$S(t_1,\dots,t_N) := \{(-1)^k t_k\}_{k=1}^N. \tag{3}$$

Theorem 2. *Assume $0 < t_1 < t_2 < \dots < t_N$. Then both sets $\pm S(t_1,\dots,t_N)$ are uniqueness sets for every space $P(M), M \subset \mathbb{N}_0, \#M = N$.*

The rest of the paper is organized as follows: In Section 3 several auxiliary results are proved. Theorem 2 is proved in Section 4. We consider the completeness property of $E(\mathbb{Z},\Gamma)$ in $L^2(a,b)$ and in $C([a,b])$ in Sections 5 and 6, respectively. Finally, in Section 7 we consider the frame property of $E(\mathbb{Z},\Gamma)$ and also present some remarks.

3 Auxiliary Lemmas

Given $N \in \mathbb{N}, \mathbf{x} = \{x_0,\dots,x_{N-1}\} \subset \mathbb{R}$, and $\Gamma = \{\gamma_0,\gamma_1,\dots,\gamma_{N-1}\} \subset \mathbb{N}$ we denote by $V(\mathbf{x},\Gamma)$ a *generalized $N \times N$ Vandermonde matrix*,

$$V(\mathbf{x};\Gamma) := \begin{pmatrix} x_0^{\gamma_0} & x_1^{\gamma_0} & x_2^{\gamma_0} & \dots & x_{N-1}^{\gamma_0} \\ x_0^{\gamma_1} & x_1^{\gamma_1} & x_2^{\gamma_1} & \dots & x_{N-1}^{\gamma_1} \\ \dots & \dots & \dots & \dots & \dots \\ x_0^{\gamma_{N-1}} & x_1^{\gamma_{N-1}} & x_2^{\gamma_{N-1}} & \dots & x_{N-1}^{\gamma_{N-1}} \end{pmatrix}. \tag{4}$$

We will usually assume that $0 \in \Gamma$. Note that if $\Gamma = \{0, 1, \dots, N-1\}$, then the matrix $V(\mathbf{x}; \Gamma)$ is a standard Vandermonde matrix, and it is easy to compute its determinant and establish whenever it is invertible or not. However, if Γ has gaps, the situation is more complicated. In the case when $x_i > 0$ for all $i = 0, \dots, n-1$, one may use the following result from the theory of *totally positive matrices*, see e.g. [6] and [9].

Proposition 1. (see [9], section 4.2) *If $0 < x_0 < x_1 < \dots < x_N$ and $\gamma_0 < \gamma_1 < \gamma_2 < \dots < \gamma_N$, then $V(\mathbf{x}; \Gamma)$ is a totally positive matrix. In particular, it is invertible.*

This statement is no longer true if $\mathbf{x}$ contains both positive and negative coordinates.

We will be interested in a particular case where $\mathbf{x} = (s, s+1, ..., s+N-1)$ for some $s \in \mathbb{R}$. Consider the problem: Describe the set of points $s \in \mathbb{R}$ such that the matrix $V((s, \dots, s+N-1); \Gamma)$ is invertible *for every* $\Gamma \subset \mathbb{N}_0, \#\Gamma = N$.

Lemma 1. *$V((x_0, x_1, \dots, x_{N-1}); \Gamma)$ is not invertible if and only if there exists a polynomial $P \in P(\Gamma)$ which vanishes on the set $\{x_0, x_1, ..., x_{N-1}\}$.*

Proof. Write $\Gamma = \{\gamma_0, \gamma_1, \dots, \gamma_{N-1}\}$. The matrix $V((x_0, x_1, \dots, x_{N-1}); \Gamma)$ is not invertible if and only if its transpose is not. The latter means that there is a non-zero vector $\mathbf{a} = (a_0, \dots, a_{N-1})$ satisfying $V((x_0, x_1, \dots, x_{N-1}); \Gamma)^T \mathbf{a}^T = 0$. This means that the polynomial $\sum_{j=0}^{N-1} a_j x^{\gamma_j}$ vanishes at the points $x_0, \dots, x_{N-1}$. $\square$

Lemma 2. *Given $N \geq 2$, the matrix $V((s, \dots, s+N-1); \Gamma)$ is invertible for every $\Gamma \subset \mathbb{N}_0, \#\Gamma = N$, $0 \in \Gamma$, if*

(i) $s \geq 0$;

(ii) $s \in (-N/2, -N/2+1) \setminus (1/2)\mathbb{Z}$.

Part (i) is a direct consequence of Proposition 1.

Part (ii) follows from Lemma 1, Theorem 2, and the observation that for every $s \in (-N/2, -N/2+1)$ such that s does not equal $k/2$ for some $k \in \mathbb{Z}$, the set $\{s, \dots, s+N-1\}$ can be written as $\pm S$, where S is defined in (3).

Clearly, by Lemma 2, the determinant of $V((s, \dots, s+N-1); \Gamma)$ is a non-trivial polynomial of s. Hence, for every fixed Γ, this matrix is invertible for every s outside a finite number of points.

In what follows, by measure we mean a finite, complex Borel measure on $\mathbb{R}$.

Given a measure μ, as usual we denote by $\hat{\mu}$ its Fourier-Stieltjes transform

$$\hat{\mu}(x) = \int_{\mathbb{R}} e^{-2\pi i x t} \, d\mu(t).$$

We also denote by δ_x the δ-measure concentrated at the point x.

Lemma 3. *Let μ be a measure supported by an interval $[\alpha, \alpha+1]$. The following are equivalent:*

(i) *$\hat{\mu}$ vanishes on $\mathbb{Z}$;*

(ii) *$\mu = A(\delta_\alpha - \delta_{\alpha+1})$, for some $A \in \mathbb{C}$.*

Proof. We present a proof of (i) $\Rightarrow$ (ii). The converse implication is trivial.

Since $\operatorname{supp}\mu \subset [\alpha, \alpha+1]$, it is easy to see that the entire function

$$f(z) := e^{2\pi i(\alpha+1/2)z}\hat{\mu}(z)$$

satisfies

$$|f(x+iy)| \le Ce^{\pi|y|}, \quad x, y \in \mathbb{R}, \tag{5}$$

with some constant C. Since f vanishes on $\mathbb{Z}$, the function $g(z) := f(z)/(\sin \pi z)$ is also entire. Clearly, there is a positive constant B such that

$$|\sin(\pi(x+iy))| \ge Be^{\pi|y|}, \quad \text{for all } x, y \in \mathbb{R}, \ \inf_{n\in\mathbb{Z}} |x+iy-n| \ge 1/4.$$

This, (5) and the maximum modulus principle imply that $g(z)$ is bounded in $\mathbb{C}$. hence, g is a constant function, from which the lemma follows. $\square$

Let us now consider measures μ that are "orthogonal" to $E(\mathbb{Z}, \Gamma)$:

$$\int_{\mathbb{R}} t^\gamma e^{-2\pi int}\, d\mu(t) = 0, \quad \text{for all } \gamma \in \Gamma, n \in \mathbb{Z}. \tag{6}$$

Lemma 4. *Assume that $\Gamma \subset \mathbb{N}_0, \#\Gamma = N, 0 \in \Gamma$, and that a measure μ is concentrated on $[\alpha, \alpha+N]$. If μ satisfies (6), then there is a finite set $S \subset (\alpha, \alpha+1)$ and measures $\mu_s, s \in S$, and ν such that*

(i) $\mu = \sum_{s\in S} \mu_s + \nu$;

(ii) ν *and* $\mu_s, s \in S$, *satisfy* (6);

(iii) *The representations are true:*

$$d\nu = \sum_{j=1}^{N+1} a_j \delta_{\alpha+j-1}, \quad d\mu_s = \sum_{j=1}^{N} c_{s,j}\delta_{s+j-1}, \quad s \in S,\ c_{s,j} \in \mathbb{R},\ a_j \in \mathbb{R}. \tag{7}$$

Note that μ_s satisfies (6) if and only if

$$\sum_{j=1}^{N} (s+j-1)^\gamma c_{s,j} = 0, \quad \text{for every } \gamma \in \Gamma,\ s \in S. \tag{8}$$

A similar observation is true for the measure ν.

Proof of Lemma 4. Clearly, μ admits a unique representation

$$d\mu(x) = \sum_{j=1}^{N} d\mu_j(x-j+1), \tag{9}$$

where each μ_j is a measure supported by $[\alpha, \alpha+1)$ for $j = 1, \dots, N-1$, and $\operatorname{supp}\mu_N = [\alpha, \alpha+1]$. Then (6) is equivalent to

$$\int_{[\alpha,\alpha+1]} e^{-2\pi int} \sum_{j=1}^{N} (t+j-1)^\gamma\, d\mu_j(t) = 0, \quad \text{for every } \gamma \in \Gamma, n \in \mathbb{Z}.$$

It follows from Lemma 3 that μ_j satisfy the system of N equations

$$\sum_{j=1}^{N}(t+j-1)^{\gamma}d\mu_j(t) = C_{\gamma}(\delta_{\alpha}-\delta_{\alpha+1}), \quad \text{for every } \gamma \in \Gamma. \tag{10}$$

The corresponding matrix on the left hand-side is $V((t,\dots,t+N-1),\Gamma)$. As we mentioned above, the subset $S \subset (\alpha,\alpha+1)$ of the zeros of its determinant is finite. Therefore, (10) implies that each measure $\mu_j, 1 \le j < N$, may only be concentrated at $\{\alpha\}$ and on S, while the support of μ_N belongs to $\{\alpha,\alpha+1\}\cup S$. We may therefore write:

$$d\mu_j = \sum_{s\in S} c_{s,j}\delta_s + a_j\delta_{\alpha}, \quad 1 \le j \le N-1;$$

$$d\mu_N = \sum_{s\in S} c_{s,N}\delta_s + a_N\delta_{\alpha} + a_{N+1}\delta_{\alpha+1}.$$

This and (9) proves part (i) of the lemma, where ν and μ_j are defined in (7).

Finally, part (ii) easily follows from (10). $\square$

4 Uniqueness sets for lacunary polynomials

In this section we will prove Theorem 2. Clearly, if $S(t_1,\dots,t_N)$ is a uniqueness set for $P(M)$, then so is $-S(t_1,\dots,t_N)$, since $P(-x)\in P(M)$ whenever $P(x)\in P(M)$. Therefore, it suffices to prove that $S(t_1,\dots,t_N)$ is a uniqueness set for every space $P(M), \#M = N$.

Assume a polynomial $P \in P(M)$ vanishes on $S(t_1,\dots,t_N)$. If P is even or odd, we have $P(t_k) = 0, 1 \le k \le N$, and by the Descartes' rule of signs we deduce that $P \equiv 0$. Thus, we can assume that $P \not\equiv 0$ is neither even nor odd and derive a contradiction from there.

Consider the polynomials

$$P_e(x) = \sum_{m_j\in M, 2|m_j} c_j x^{m_j} = \frac{1}{2}(P(x)+P(-x))$$

and

$$P_o(x) = \sum_{m_j\in M, 2\nmid m_j} c_j x^{m_j} = \frac{1}{2}(P(x)-P(-x)).$$

If one of them is identically zero, then P is even or odd and we are done. Let M have K even elements and $N-K$ odd elements. Then P_e has at most $K-1$ positive roots and P_o has at most $N-K-1$ positive roots by the Descartes' rule of signs. We are going to show that P_e and P_o together have at least $N-1$ positive roots thus getting the contradiction we need.

Let us consider the graphs of $P(x)$, $-P(x)$ and $P(-x)$, see Figure 1. Since we assumed that P is neither even nor odd, these are three different polynomials.

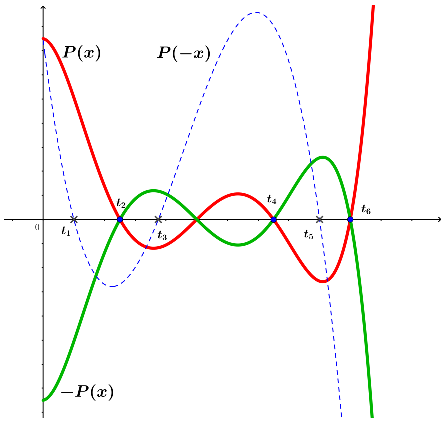

Figure 1: $P(-x)$ has many intersections with either $P(x)$ or $-P(x)$

For simplicity we first cover the case when $P(x)$ and $P(-x)$ do not have common positive zeroes. We indicate t_k with odd indices by crosses.

By assumption each cross except the first one and the last one is separated from the other crosses by the zeroes of $P(x)$. That is, it is contained in a connected component bounded by the pieces of the curves $y = P(x)$ and $y = -P(x)$. Thus, to get from the cross number m to the cross number $m+1$ we have to exit the component containing the first and enter the next one, giving us at least two intersections of the curve $y = P(-x)$ with curves $y = P(x)$ and $y = -P(x)$. Additionally, if N is even, then we also have to exit the last connected component as well, since there must be at least one more zero of $P(x)$ after the last cross. In total we will always have at least $N-1$ intersections, that is P_e and P_o together have at least $N-1$ positive roots as we wanted.

Now, we indicate the necessary changes in the case when $P(x)$ and $P(-x)$ have common positive roots. If we have two crosses which are not zeroes of $P(x)$ but between them there is a zero of $P(x)$, then the curve $y = P(-x)$ can go directly from the connected component of the first cross to the connected component of the second cross through this zero. But if $P(x_0) = P(-x_0) = 0$ then x_0 is a zero for both P_e and P_o, thus we anyway get two zeroes.

It remains to consider the case when we have a cross which is also a zero of $P(x)$. Assume that crosses from the number m to $m+l$ are zeroes of $P(x)$ and crosses number $m-1$ and $m+l+1$ are not (or there are no crosses with these indices). Then each of these $l+1$ zeroes are both zeroes for P_e and P_o, thus giving us two intersections. Finally, since the $m+l$'th cross is separated from

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$m+l+1$'st by at least one more zero of $P(x)$ we have to enter the connected component corresponding to this zero and the same between m'th and $m-1$'st zero, thus giving us the same number of intersections as in the case when $P(x)$ and $P(-x)$ did not have common zeroes.

5 Completeness of $E(\mathbb{Z},\Gamma)$ in $L^2(a,b)$

Part (i) of Theorem 1 follows from

Theorem 3. *Given any finite set $\Gamma \subset \mathbb{N}_0$, the system $E(\mathbb{Z},\Gamma)$ is complete in $L^2(a,b)$ if and only if $b-a \leq \#\Gamma$.*

Proof. (i) Assume $b-a \leq N := \#\Gamma$. It is then a simple consequence of Lemma 4 that $E(\mathbb{Z},\Gamma)$ is complete in $L^2(a,b)$. Indeed, if the system is not complete then there exists non-trivial $f \in L^2(a,b)$ which is orthogonal to our system. Therefore, the measure $f\,dx$ is also orthogonal to the system, but it can not be a sum of delta measures unless f is identically zero.

(ii) Assume that $b-a > N$. We have to prove that $E(\mathbb{Z},\Gamma)$ is not complete in $L^2(a,b)$, i.e. that there is a non-trivial function $F \in L^2(a,b)$ such that

$$\int_a^b t^\gamma e^{-2\pi i n t}\,F(t)\,dt = 0, \qquad \text{for every } \gamma \in \Gamma, n \in \mathbb{Z}. \tag{11}$$

The existence of such a function follows essentially from elementary linear algebra.

We have $b = a+N+\delta$, for some $\delta > 0$, and may assume that $\delta < 1$. Write F in the form

$$F(t) = \sum_{j=0}^{N} F_j(t-j), \quad t \in (a, a+N+\delta),$$

where $F_j(t) := F(t+j)\mathbf{1}_{(a,a+1)}(t)$ vanish outside $(a,a+1)$ for $j = 0, ..., N-1$, and f_N vanishes outside $(a,a+\delta)$. Here $\mathbf{1}_{(a,a+1)}$ is the characteristic function of $(a,a+1)$. Clearly, to prove (11) it suffices to find $N+1$ non-trivial functions F_j as above satisfying for a.e. $t \in (a,a+1)$ the system of N equations

$$\sum_{j=0}^{N} (t+j)^\gamma F_j(t) = 0, \quad \text{for all } \gamma \in \Gamma,\ t \in (a,a+1).$$

Rewrite this system in the matrix form

$$V(t)\cdot(F_0(t), ..., F_{N-1}(t))^T = -((t+N)^{\gamma_1}, ..., (t+N)^{\gamma_N})^T \cdot F_N(t), \quad \Gamma = \{\gamma_1, ..., \gamma_N\},$$

where $V(t) := V((t, t+1, \ldots, t+(N-1); \Gamma)$ is a generalized Vandermonde matrix defined above, whose determinant has only finite number of real zeroes. Therefore, there is an interval $I \subset (a, a+\delta)$ where $V(t)$ is invertible and satisfies

$$\sup_{t\in I} \sup_{\mathbf{x}\in\mathbb{R}^N,\, \|x\|=1} \|V^{-1}(t)\cdot\mathbf{x}\| < \infty.$$

Now, one can simply choose $F_N(t) := \mathbf{1}_I(t)$ and set

$$(F_0(t), ..., F_{N-1}(t))^T := -V^{-1}(t) \cdot ((t+N)^{\gamma_1}, ..., (t+N)^{\gamma_N})^T \cdot \mathbf{1}_I(t).$$

□

Remark 1. *One can check that the above result on completeness of $E(\mathbb{Z}, \Gamma)$ in $L^2(a, b)$ remain true for the space $L^p(a, b)$, $1 \le p < \infty$.*

6 Completeness of $E(\mathbb{Z}, \Gamma)$ in $C([-a, a])$

Theorem 4. *$E(\mathbb{Z}, \Gamma)$ is complete in $C([-a, a])$ if and only if $a < r(\Gamma)$.*

Clearly, this theorem implies $CR_C(\mathbb{Z}, \Gamma) = r(\Gamma)$.

Proof. **1**. Suppose $a \ge r(\Gamma)$. We have to check that the system is not complete in $C([-a, a])$. Clearly, it suffices to produce a bounded measure μ on $[-r(\Gamma), r(\Gamma)]$ which satisfies (6).

Set $\mathbb{O} := \#\Gamma_{odd}, \mathbb{E} := \#\Gamma_{even}$ and

$$f(x) = \begin{cases} \sin(\pi x) + \sum_{k=1}^{\mathbb{O}} \alpha_k \sin\left((2k+1)\,\pi x\right), \text{ if } \mathbb{O} < \mathbb{E}, \\ 1 + \sum_{k=1}^{\mathbb{E}} \alpha_k \cos\left(2\pi k x\right), \text{ if } \mathbb{O} \ge \mathbb{E}, \end{cases} \tag{12}$$

where $\{\alpha_k\} \subset \mathbb{R}$.

Lemma 5. *There exist numbers α_k in (12) such that f satisfies*

$$f^{(\gamma)}(n) = 0, \quad \gamma \in \Gamma,\, n \in \mathbb{N}. \tag{13}$$

It is easy to check that f in (12) is the Fourier-Stieltjes transform of a measure supported by $[-r(\Gamma), r(\Gamma)]$. One may therefore easily see that Lemma 5 proves the necessity in part (i) of Theorem 4.

Proof of Lemma 5. Consider the case $\mathbb{E} \le \mathbb{O}$.

We wish to find α_k so that the function

$$f(x) = 1 + \alpha_1 \cos(2\pi x) + \cdots + \alpha_{\mathbb{E}} \cos(2\pi \mathbb{E} x)$$

satisfies (13).

It is clear that every *odd* derivative of f vanishes on $\mathbb{Z}$. Therefore, it suffices to find the coefficients so that $f^{(\gamma)}$ vanishes on $\mathbb{Z}$ for every $\gamma \in \Gamma_{even}$ (in particular, for $\gamma = 0$). This is equivalent to saying that the coefficients must satisfy the following system of $\mathbb{E}$ linear equations:

$$\gamma = 0: \quad \alpha_1 + \cdots + \alpha_{\mathbb{E}} = -1$$

and

$$\gamma \in \Gamma_{even}, \gamma \ne 0: \quad (2\pi)^\gamma \alpha_1 + (4\pi)^\gamma \alpha_2 \cdots + (2\pi\mathbb{E})^\gamma \alpha_{\mathbb{E}} = 0.$$

This system has a *unique non-trivial solution* by Proposition 1.

The case $\mathbb{E} > \mathbb{O}$ is similar, and we leave the proof to the reader. □

We return now to the proof of Theorem 4.

2. Assume $a < r(\Gamma)$. We have to show that $E(\mathbb{Z},\Gamma)$ is complete in $C([-a,a])$, i.e. that the only measure μ on $[-a,a]$ which satisfies (6) is trivial.

We will consider the case $\mathbb{E} \le \mathbb{O}$, i.e. $r(\Gamma) = \mathbb{E}$. Clearly, we may assume that $\mathbb{E} = \mathbb{O}$ and so $\mathbb{E} = N/2$, where $N := \#\Gamma$ is an even number. Also, to avoid trivial remarks, we assume that $N \ge 4$.

Assume that μ is concentrated on $[-a,a]$ and satisfies (6). By (7) and Lemma 4, since $\mu(\{\pm N/2\}) = 0$, we have

$$d\mu = \sum_{s\in S} d\mu_s + d\nu = \sum_{s\in S}\sum_{j=1}^{N} c_{s,j}\delta_{s+j-1} + \sum_{j=2}^{N} a_j\delta_{-N/2+j-1},$$

where S is a finite subset of $(-N/2, -N/2+1)$ and the coefficients $c_{s,j}$ satisfy for every $s \in S$ the system of equations (8). By part (ii) of Lemma 2, this system has only trivial solutions $c_{s,j} = 0, j = 1, ..., N, s \in S \setminus (1/2)\mathbb{Z}$, and so

$$\mu = \nu_1 + \nu, \quad d\nu_1 := \sum_{j=1}^{N} c_j\delta_{-N/2+j-1/2},$$

where ν and ν_1 both are orthogonal to $E(\mathbb{Z},\Gamma)$.

Let us check that $\nu = 0$. It is more convenient to write ν in the form

$$\nu = \sum_{k=-N/2+1}^{N/2-1} b_k\delta_k, \quad b_k := a_{N/2+k+1}.$$

Then clearly, (6) is equivalent to the system of $N-1$ equations:

$$\sum_{k=-N/2+1}^{N/2-1} k^\gamma b_k = 0, \qquad \text{for every } \gamma \in \Gamma.$$

This is equivalent to the following systems:

$$\sum_{k=0}^{N/2-1} k^\gamma(b_{-k} + b_k) = 0,\ \gamma \in \Gamma_{even}, \quad \sum_{k=1}^{N/2-1} k^\gamma(b_{-k} - b_k) = 0,\ \gamma \in \Gamma_{odd}.$$

One may now use Proposition 1 to deduce that $b_{-k} + b_k = b_{-k} - b_k = 0$, for every k, thus $b_k = b_{-k} = 0$ for every k, that is $\nu = 0$. Similarly, one may check that $\nu_1 = 0$, and so $\mu = 0$.

The proof of the case $\mathbb{O} < \mathbb{E}$ is similar is left to the reader. □

Remark 2. *One can prove that for $a \in [r(\Gamma), \#\Gamma/2]$, the deficiency of $E(\mathbb{Z},\Gamma)$ in $C([-a,a])$ is always finite.*

7 Frame Property of $E(\mathbb{Z},\Gamma)$

The frame property of $E(\mathbb{Z},\Gamma)$ in $L^2(a,b)$ is closely connected with the completeness property of $E(\mathbb{Z},\Gamma)$ in $C([a,b])$:

Theorem 5. *Assume $a<b$ and $\epsilon>0$.*
(i) *If $E(\mathbb{Z},\Gamma)$ is complete in $C([a,b])$, then $E(\mathbb{Z},\Gamma)$ is a frame in $L^2(a,b)$.*
(ii) *If $E(\mathbb{Z},\Gamma)$ is not complete in $C([a,b])$, then $E(\mathbb{Z},\Gamma)$ is not a frame in $L^2(a-\epsilon,b+\epsilon)$.*

Observe that to finish the proof of Theorem 1, it remains to show that $FR(\mathbb{Z},\Gamma)=r(\Gamma)$. This follows easily from Theorem 4 and Theorem 5.

Proof of Theorem 5. (i) Assume that the system $E(\mathbb{Z},\Gamma)$ is complete in $C([a,b])$. We have to show that it is a frame in $L^2(a,b)$.

Recall that $E(\mathbb{Z},\Gamma)$ is a frame in $L^2(a,b)$ if there are positive constants A,B such that

$$A\|F\|_2^2 \le \sum_{n\in\mathbb{Z}}\sum_{\gamma\in\Gamma}|\langle F, t^\gamma e^{2\pi int}\rangle|^2 \le B\|F\|_2^2, \quad \text{for every } F\in L^2(a,b). \tag{14}$$

Using the Fourier transform, this is equivalent to the condition

$$A\|f\|_2^2 \le \sum_{n\in\mathbb{Z}}\sum_{\gamma\in\Gamma}|f^{(\gamma)}(n)|^2 \le B\|f\|_2^2, \tag{15}$$

where f is the inverse Fourier transform of F.

It is standard to check that the right hand-side inequality in (14) (and in (15)) holds for every interval (a,b), see e.g. [7], Lecture 2. So, we only prove the left hand-side inequality.

By Theorem 1, $E(\mathbb{Z},\Gamma)$ is not complete, and so is not a frame in $L^2(a,b)$ when $b-a>N:=\#\Gamma$. Therefore, in what follows we may assume that $a+k-1<b\le a+k$, for some $k\in\mathbb{N}, k\le N$.

Write

$$F(t)=\sum_{j=0}^{k-1}F_j(t-j), \quad F_j(t):=F(t+j)\cdot \mathbf{1}_{(a,a+1)}(t). \tag{16}$$

Then we have

$$\langle F, t^\gamma e^{2\pi int}\rangle = \int_a^{a+1} e^{2\pi int}\left(\sum_{j=0}^{k-1}(t+j)^\gamma F_j(t)\right)dt.$$

Hence,

$$\sum_{n\in\mathbb{Z}}|\langle F, t^\gamma e^{2\pi int}\rangle|^2 = \|\sum_{j=0}^{k-1}(t+j)^\gamma F_j(t)\|_2^2.$$

We see that the left hand-side inequality in (14) is equivalent to

$$\|V_k(t) \cdot (F_0(t), \dots, F_{k-1}(t))^T\|_2^2 \geq A\|F\|_2^2, \tag{17}$$

where

$$V_k(t) := V(t, \dots, t+k-1; \Gamma)^T$$

denotes the $k \times N$ matrix which consists of the first k columns of $V(t, \dots, t + N - 1; \Gamma)$, and we set

$$\|(G_1, \dots, G_k)^T\|_2^2 := \|G_1\|_2^2 + \dots + \|G_k\|_2^2.$$

Let us first consider the case $b = a+k$. Since $E(\mathbb{Z}, \Gamma)$ is complete in $C([a, b])$, there is no measure μ on $[a, b]$ orthogonal to this system. Then, since any measure of the form

$$d\mu = \sum_{j=0}^{k-1} x_j \delta_{t+j}, \quad (x_1, \dots, x_k) \in \mathbb{R}^k \setminus \{\mathbf{0}\}, \ t \in [a, a+1],$$

is not orthogonal to $E(\mathbb{Z}, \Gamma)$, we see that $V_k(t) \cdot \mathbf{x}^T \neq \mathbf{0}$, for every $\mathbf{x} \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ and $t \in [a, b]$. Therefore, there is a constant A such that

$$\|V_k(t) \cdot \mathbf{x}^T\|^2 \geq A\|\mathbf{x}\|^2, \quad t \in [a, a+1],$$

which implies (17).

Now, let us assume that $b = a + k - 1 + \delta$, where $0 < \delta < 1$. Then the function F_{k-1} in (16) satisfies $F_{k-1}(t) = 0, \delta < t < 1$. Similarly to above, for every vectors $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^{k-1}$ we have

$$\|V_k(t) \cdot \mathbf{x}\| \geq A_1\|\mathbf{x}\|, \ t \in [a, a+\delta], \ \|V_{k-1}(t) \cdot \mathbf{y}\| \geq A_2\|\mathbf{y}\|, \ t \in [a+\delta, a+1],$$

from which (17) follows.

(ii) Assume that the system $E(\mathbb{Z}, \Gamma)$ is not complete in $C([a, b])$. We have to show that it is not a frame in $L^2(a - \epsilon, b + \epsilon)$, for every $\epsilon > 0$. We may assume that $0 < \epsilon < 1/2$.

Let g be the inverse Fourier transform of a measure μ on $[a, b]$ that is orthogonal to the system. Then $g^{(\gamma)}$ vanishes on $\mathbb{Z}$, for every $\gamma \in \Gamma$.

Choose any r, $0 < r < \epsilon$, and consider the function

$$f(x) := g(x)\varphi(x), \quad \varphi(x) := \frac{\sin(\pi r x)}{\pi r x}.$$

Then, clearly, f is the (inverse) Fourier transform of an absolutely continuous measure on $(a - r, b + r) \subset (a - \epsilon, b + \epsilon)$, and

$$\|f\|_2 > C > 0, \quad \text{where } C \text{ does not depend on } \epsilon. \tag{18}$$

We will need

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Lemma 6. *There is a constant C such that*

$$\sum_{n\in\mathbb{Z}} |\varphi^{(j)}(n)|_2^2 \le C^j r^j, \quad j \in \mathbb{N}. \tag{19}$$

The proof of the lemma follows from two observations:

(i) φ is the Fourier transform of $\mathbf{1}_{(-r/2,r/2)}(t)/r$, and so $\varphi^{(j)}$ is the Fourier transform of

$$(-2\pi it)^j \mathbf{1}_{(-r/2,r/2)}(t)/r.$$

It easily follows that $\|\varphi^{(j)}\|_2^2 \le Cr^j, j \in \mathbb{N}$.

(ii) The sum in (19) is equal to the norm $\|\varphi^{(j)}\|_2^2$.

Using (19), since $g^{(\gamma)}, \gamma \in \Gamma$, vanishes on $\mathbb{Z}$ and the functions $g^{(j)}, j \in \mathbb{N}$, are bounded on $\mathbb{R}$, one can easily check that

$$\sum_{n\in\mathbb{Z}}\sum_{\gamma\in\Gamma} |f^{(\gamma)}(n)|^2 = \sum_{n\in\mathbb{Z}}\sum_{\gamma\in\Gamma} |(g\varphi)^{(\gamma)}(n)|^2 \le Cr,$$

for some C. This and (18) imply that the left hand-side inequality in (15) is not true for all small enough values of r. $\square$

Remark 3. *Observe that by Theorem* 3*, $E(\mathbb{Z},\Gamma)$ is not complete in $C([a,b])$ whenever $b-a > N := \#\Gamma$. Let us state two results on the completeness of $E(\mathbb{Z},\Gamma)$ in $C([a,b])$ when $a \ge 0$:*

(i) *Using part* (i) *of Lemma* 2 *and Lemma* 4*, one may check that $E(\mathbb{Z},\Gamma)$ is complete in $C([a,b])$ whenever $b-a < N$ and if $a > 0$ then we don't need the assumption $0 \in \Gamma$.*

(ii) *One may also prove that $E(\mathbb{Z},\Gamma)$ is complete in $C([a,a+N])$ if and only if $a \notin \mathbb{N}_0$.*

Remark 4. *Let us come back to the exponential systems $E(\mathbb{Z},\Gamma(n))$ defined in the beginning of Section* 2*. Here we present a simple example which illustrates that such systems may have strikingly different completeness properties in L^2-spaces and C-spaces.*

Let $f(x) = \sin(\pi x/2)$. Then $f^{(2k)}(2n) = f^{(2k+1)}(2n+1) = 0$, for every $k \in \mathbb{N}_0, n \in \mathbb{Z}$. Then, since f is the inverse Fourier transform of $(\delta_{1/4} - \delta_{-1/4})/2i$, the system

$$\{t^{2k}e^{4\pi int} : k \in \mathbb{N}_0, n \in \mathbb{Z}\} \bigcup \{t^{2k+1}e^{2\pi i(2k+1)t} : k \in \mathbb{N}_0, n \in \mathbb{Z}\}$$

is not complete in $C([-1/4,1/4])$. On the other hand, one may check that it is complete in $L^2(I)$ on every finite interval $I \subset \mathbb{R}$.

8 Acknowledgements

The authors want to thank Fedor Petrov and Pavel Zatitskiy for valuable discussions about this paper.

Aleksei Kulikov was supported by Grant 275113 of the Research Council of Norway, by BSF Grant 2020019, ISF Grant 1288/21, and by The Raymond and Beverly Sackler Post-Doctoral Scholarship.

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