# Bridgeland Stability Conditions and the Hilbert Scheme of Skew Lines in Projective Space

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# Introduction

Bridgeland stability conditions are powerful tools for studying derived categories, with several applications to algebraic geometry. They were introduced by Bridgeland in 2002 [Bri07], who was motivated by Douglas' work on Π-stability of $D$-branes [Dou02] in the context of string theory. Bridgeland showed that the set $\mathrm{Stab}(\mathcal{D})$ of stability conditions on a triangulated category $\mathcal{D}$ is a complex manifold, a result of extreme importance and central to all mathematical applications of this field of study. But in order to use this concept of stability conditions in string theory (as intended by Bridgeland), one needs to prove the existence of stability conditions on the bounded derived category $D^b(X)$ of a compact Calabi-Yau threefold $X$. This task is far from easy, as it took more than a decade before the first example was produced for the smooth quintic threefold by Li in [Li18]. This achievement came into fruition thanks to the extensive amount of work in the domain over this period of time, where the existence of stability conditions was progressively established for arbitrary smooth projective varieties of dimension one [Bri07, Oka06, Mac07], dimension two [Bri08, AB13], and then some dimension three cases (see Section 1.3).

One of the main applications of stability conditions on $D^b(X)$ (for an arbitrary variety $X$) is to study the geometry of moduli spaces of coherent sheaves over $X$ with some Chern character $v$ via the strategy known as "wall crossing". In loose terms, a "wall" is a codimension one submanifold of $\mathrm{Stab}(D^b(X))$ such that by changing stability conditions along a continuous path in $\mathrm{Stab}(D^b(X))$ that goes through the wall causes the moduli space of sheaves over $X$ with Chern character $v$ to transform. When $X$ is of dimension two, we have a solid control over wall crossing thanks to Bayer–Macrì [BM11], who provided a full understanding of how moduli spaces of sheaves change as we cross walls, as well as knowing the exact geometrical relationship these walls have with the underlying surface. In addition the precise structure of the walls is known and there are effective techniques to detect them. This thorough picture of wall crossing in dimension two is demonstrated through various complete studies of moduli spaces of sheaves over surfaces [AB13, ABCH13, Mea12].

Comparatively, the situation in dimension three is nowhere as good, as even the existence of stability conditions on smooth projective threefolds is still an open question, with a positive answer only for a few threefolds. Moreover, most known methods for finding walls that work in dimension two are extremely hard to replicate in dimension three, due to severe technical obstacles. Schmidt came up with a useful way of circumventing these technical issues in his work [Sch15], by connecting $\mathrm{Stab}(D^b(X))$ with tilt-stability[1] given the right conditions, so that one can search for walls in tilt-stability and "lift" them to $\mathrm{Stab}(D^b(X))$. This is one of the few reliable wall crossing techniques that exist for dimension three up to date. Schmidt illustrates this with an example of a wall crossing of the Hilbert scheme of twisted cubics in $\mathbb{P}^3$; he finds three walls, including a "simple-resolution" wall that comes from the two-term resolution of ideal sheaves of twisted cubics. In general, this wall exists for Chern characters of the form

$$i\,\mathrm{ch}(\mathcal{O}_{\mathbb{P}^3}(m)) - j\,\mathrm{ch}(\mathcal{O}_{\mathbb{P}^3}(n)) \tag{0.0.1}$$

with $i, j \in \mathbb{Z}_{>0}$ and $m, n \in \mathbb{Z}$ satisfying $n < m$ (which includes the Chern

[1] an intermediate notion used to construct stability conditions on threefolds, where wall crossing is as nicely-behaved as in dimension two.

character of twisted cubics and its deformations). Arguing via quiver representations, Schmidt shows that on one side of this "simple-resolution" wall the moduli space of stable objects in $D^b(\mathbb{P}^3)$ with Chern character of type (0.0.1) is a smooth irreducible projective variety, while on the other side it is empty.

Following Schmidt's work, we study in this thesis the wall crossing of the Hilbert scheme $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ parametrizing subschemes of $\mathbb{P}^3$ with Hilbert polynomial $2m+2$. It has two irreducible smooth components, one parametrizing plane conics union a point, and the other pairs of skew lines. Observe that although a pair of skew lines degenerates in a similar fashion as a twisted cubic, the Chern character of its ideal sheaves is not of the form (0.0.1) and thus is not covered by Schmidt's work. In particular, there is no such "simple-resolution" wall for skew lines.

We apply Schmidt's technique to find two walls that divide a certain open connected subset of $\mathrm{Stab}(D^b(\mathbb{P}^3))$ into three "chambers". We then proceed to describe the moduli space of stable objects in each chamber set-theoretically first, then geometrically. We shall in fact prove that the moduli spaces in the second and third chambers (denoted $\mathcal{M}^{\mathrm{II}}$ and $\mathcal{M}^{\mathrm{III}}$ respectively) are at least obtained as smooth algebraic spaces via a result by Artin, and then show that they are smooth projective varieties using Mori theory, based on the work by Chen–Coskun–Nollet in [CCN11]. Along the way, we construct flat families of sheaves over the moduli spaces $\mathcal{M}^{\mathrm{II}}$ and $\mathcal{M}^{\mathrm{III}}$. We witness an interesting phenomenon that these families are flat ideal sheaves of (very) non flat varieties.

The following is an overview of the content of this text:

Chapter 1 contains an introduction to Bridgeland stability conditions as well as a gentle reminder of the definitions of cycles on a variety, Mori cone, nef cone and Mori's contraction theorem. It has no original results of ours. This chapter should be seen as a complement to the preliminary section of our joint paper with Gulbrandsen, exhibited in Chapter 2, bringing more detail to some notions. Note that there are a few overlapping definitions that could not be avoided.

Chapter 2 is the main part of this thesis, and consists of our joint article with Gulbrandsen planned to be published in a journal, which we integrate to this text (note that the number of each section from the article gets a "2." attached to it in this text). It is organized as follows: in Section 2.2 we give a bit more details on the Hilbert scheme $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ and its components, then we briefly list the necessary definitions and results from the Bridgeland stability machinery. In Section 2.3 we find our two walls using Schmidt's strategy, then describe the set of stable objects in each chamber. In Section 2.4 we realize the moduli spaces in the second and third chambers as smooth algebraic spaces, plus we construct a certain flat family over these spaces. And finally, in Section 2.5 we use Mori theory to show that the moduli space in the second chamber is a smooth projective variety.

We conclude with an appendix (Chapter 3), where we prove that the Ext space of certain families of sheaves is one dimensional. We think that the proof is interesting as it uses some advanced arguments, including Leray spectral sequences. The appendix is divided into two parts: in Section 3.1 we introduce some background on spectral sequences, then in Section 3.2 we state the result and prove it.

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Sammy Alaoui Soulimani

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# Chapter 1

# Generalities

## 1.1 Bridgeland stability conditions

This section is divided into three parts: first, we quickly recall the notion of a t-structure on a triangulated category and its heart, a crucial element in the definition of stability conditions. Then, we give a brief summary of the ideas behind the definition of stability conditions, based on its source of origin, i.e. [Bri07]. Finally, we touch upon the space of stability conditions and its manifold structure. We omit all proofs and refer the reader to [Bri07] and [Bri08] for all the details. We remind the reader that this chapter has no original results of ours.

Throughout, $\mathcal{D}$ is a triangulated category with its shift functor $[n]\colon \mathcal{D} \to \mathcal{D}$, $E \mapsto E[n]$. Moreover we assume that the class of objects in $\mathcal{D}$ is a set (i.e. $\mathcal{D}$ is *essentially small*).

### t-structure and hearts

Since their introduction by Beilinson–Bernstein–Deligne, t-structures proved to be a very useful tool with implications for algebra and geometry. For instance, they are used to detect the different abelian categories inside an arbitrary triangulated category. Another application of t-structures is to cut up objects of a triangulated category $\mathcal{D}$ into cohomology objects (indexed by the integers) that live in an abelian subcategory of $\mathcal{D}$ called the heart of the t-structure. We give the definition of a t-structure following [GM96, Definition IV.4.2]:

**Definition 1.1.1.** A *t-structure* on $\mathcal{D}$ is a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of full subcategories in $\mathcal{D}$ satisfying the following (we write $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$):

1. $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$.
2. $\mathrm{Hom}_{\mathcal{D}}(A, B) = 0$ for all $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.
3. For all objects $E \in \mathcal{D}$ there exists a triangle
$$A \longrightarrow E \longrightarrow B \longrightarrow A[1]$$
with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

A consequence of the above definition is the existence of the functors

$$\tau^{\leq n} : \mathcal{D} \to \mathcal{D}^{\leq n}$$

and

$$\tau^{\geq n} : \mathcal{D} \to \mathcal{D}^{\geq n}$$

called the *truncation functors*, where $\tau^{\leq n}$ is right adjoint to the inclusion functor $\mathcal{D}^{\leq n} \to \mathcal{D}$ and $\tau^{\geq n}$ is left adjoint to the inclusion functor $\mathcal{D}^{\geq n} \to \mathcal{D}$ (see [GM96, Lemma IV.4.5(a)]).

**Definition 1.1.2.** The *heart* of a t-structure on $\mathcal{D}$ is the full subcategory of $\mathcal{D}$ given by

$$\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}.$$

It is known that the heart is an abelian category [GM96, Theorem IV.4.4].

A t-structure on $\mathcal{D}$ is *bounded* if for every object $E \in \mathcal{D}$ there exist integers $m \leq n$ such that $E \in \mathcal{D}^{\geq m} \cap \mathcal{D}^{\leq n}$. A bounded t-structure is uniquely determined by its heart $\mathcal{A} \subset \mathcal{D}$. Also, the heart of a t-structure is characterized by the following properties:

**Proposition 1.1.3** ([Bri07], Lemma 3.2)**.** *Let $\mathcal{A} \subset \mathcal{D}$ be a full additive subcategory. Then $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$ if and only if the following two conditions hold:*

1. *if $k_1 > k_2$ are integers then $\mathrm{Hom}_{\mathcal{D}}(A[k_1], B[k_2]) = 0$ for all $A$, $B$ of $\mathcal{A}$,*
2. *for every nonzero object $E \in \mathcal{D}$ there are a finite sequence of integers*

$$k_1 > k_2 > \ldots > k_n$$

*and a collection of triangles*

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[r] \arrow[dl] & \ldots \arrow[r] & E_{n-1} \arrow[rr] \arrow[dl] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed, "{[1]}"] & & \cdots \arrow[ul, dashed, "{[1]}"] & & A_n \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd} \tag{1.1.1}$$

*with $A_j \in \mathcal{A}[k_j]$ for all $j$.*

In particular, part (2) of the above proposition means that every bounded t-structure comes with a filtration of an object $E \in \mathcal{D}$ by "cohomology pieces" indexed by the integers.

**Example 1.1.4.** Let $\mathcal{A}$ be an abelian category and $D(\mathcal{A})$ its derived category. A t-structure can be defined on $D(\mathcal{A})$ by taking $\mathcal{D}^{\leq 0}$ as the full subcategory of $D(\mathcal{A})$ consisting of complexes with vanishing cohomology in all positive degrees, and $\mathcal{D}^{\geq 0}$ as the full subcategory of complexes with vanishing cohomology in all negative degrees. This is known as the standard t-structure on $D(\mathcal{A})$. Its heart consists of complexes with cohomology vanishing everywhere except at degree zero, and is in fact equal to $\mathcal{A}$.

Furthermore, for any complex $E \in D(\mathcal{A})$ the truncations $\tau^{\leq j}(E)$ sit in a collection of triangles

$$\begin{tikzcd}
\dots \arrow[r] & \tau^{\leq j-1}(E) \arrow[rr] & & \tau^{\leq j}(E) \arrow[rr] \arrow[dl] & & \tau^{\leq j+1}(E) \arrow[r] \arrow[dl] & \dots \\
& & A_j \arrow[ul, dashed, "{[1]}"] & & A_{j+1} \arrow[ul, dashed, "{[1]}"] & &
\end{tikzcd} \tag{1.1.2}$$

breaking down $E$ into its shifted cohomology objects $A_j = H^j(E)[-j]$.

The *cohomology functors* of a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ with heart $\mathcal{A}$ are defined as follows:

$$\mathcal{H}^0 = \tau^{\leq 0} \circ \tau^{\geq 0} \colon \mathcal{D} \to \mathcal{A}, \quad \mathcal{H}^i(X) = \mathcal{H}^0(X[i]).$$

## Motivation and definition of stability conditions

Roughly speaking, a stability condition on $\mathcal{D}$ is the concatenation of a heart of a bounded t-structure on $\mathcal{D}$ and a group homomorphism $Z$ from the Grothendieck group of $\mathcal{D}$ to the complex numbers such that any nonzero object $E \in \mathcal{D}$ has a unique "generalized" Harder-Narasimhan filtration. The motivation behind this definition starts with slope stability of coherent sheaves over a smooth projective curve $X$: a sheaf $E \in \mathrm{Coh}(X)$ is (semi)stable if $\mu(F) < (\leq)\mu(E)$ for every subsheaf $F \subset E$, where $\mu(E)$ is the slope function given by

$$\mu(E) = \frac{\deg(E)}{\operatorname{rk}(E)} \in \mathbb{R} \cup \{\infty\}.$$

Also, it is known that every coherent sheaf $E$ admits a unique Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_{n-1} \subset E_n = E$$

where each factor $G_i = E_i/E_{i-1}$ is semistable with descending slope (i.e. $\mu(G_i) \leq \mu(G_{i-1})$ for all $i$).

As stated in [Bri07], one can extend this notion of stability to complexes of sheaves in the derived category $D^b(X)$: define the group homomorphism

$$\begin{aligned} Z\colon K(X) &\longrightarrow \mathbb{C} \\ E &\mapsto -\deg(E) + i \operatorname{rk}(E) \end{aligned} \tag{1.1.3}$$

and let the phase of $E$ be the real number $\phi(E) = \arg(Z(E))/\pi$ in $\big(0, 1\big]$. Now, construct a generalized "Harder-Narasimhan" filtration for a nonzero object $E \in D^b(X)$ by first breaking it down into its (shifted) cohomology sheaves (via the same process shown in Example 1.1.4), then break each cohomology sheaf further down to the semistable factors from its respective Harder-Narasimhan filtration. Arranging these factors gives the sought after filtration of $E$ by shifts of semistable sheaves; setting $\phi(E[k]) = \phi(E) + k$ for each integer $k$ gives the factors in this "generalized" filtration decreasing phases.

An abstraction of these "generalized" filtrations to arbitrary triangulated categories is given by the notion of slicing:

**Definition 1.1.5** ([Bri07], Definition 3.3)**.** A *slicing* $\mathcal{P}$ of a triangulated category $\mathcal{D}$ consists of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$ satisfying the following axioms:

1. for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ ,
2. if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,
3. for each nonzero object $E \in \mathcal{D}$ there are a finite sequence of real numbers

   $$\phi_1 > \phi_2 > \ldots > \phi_n$$

   and a collection of triangles

   $$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[r] \arrow[dl] & \ldots \arrow[r] & E_{n-1} \arrow[rr] \arrow[dl] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed, "{[1]}"] & & \ldots \arrow[ul, dashed, "{[1]}"] & & A_n \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

   where $A_j$ are nonzero objects in $\mathcal{P}(\phi_j)$.

We see in part (3) of the above definition that slicings propose a finer filtration for nonzero $E \in \mathcal{D}$ by objects indexed by real numbers $\phi_j$. Define the real numbers $\phi^+(E) = \phi_1$ and $\phi^-(E) = \phi_n$.

Let $I \subset \mathbb{R}$ be an interval. Given a slicing $\mathcal{P}$ of $\mathcal{D}$, denote by $\mathcal{P}(I)$ the subcategory of $\mathcal{D}$ closed under extensions and generated by the subcategories $\mathcal{P}(\phi)$ for all $\phi \in I$. As stated in [Bri07], if $I = \big(a, b\big)$ with $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$, then the objects of $\mathcal{P}(I)$ are the zero objects of $\mathcal{D}$ together with the nonzero objects $E \in \mathcal{D}$ satisfying $a < \phi^-(E) \leq \phi^+(E) < b$. The pair $(\mathcal{P}(> \phi), \mathcal{P}(\leq \phi + 1))$ defines a bounded t-structure in $\mathcal{D}$ whose heart is $\mathcal{P}(\big(\phi, \phi + 1\big])$ for all $\phi \in \mathbb{R}$, where $\mathcal{P}(> \phi) = \mathcal{P}(\big(\phi, +\infty\big))$ and $\mathcal{P}(\leq \phi + 1) = \mathcal{P}(\big(-\infty, \phi + 1\big])$.

We now define stability conditions on $\mathcal{D}$:

**Definition 1.1.6** ([Bri07], Definition 5.1)**.** A *stability condition* on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{P})$ comprised of a slicing $\mathcal{P}$ and a group homomorphism

$$Z \colon K(\mathcal{D}) \to \mathbb{C}$$

called the central charge, such that for any nonzero $E \in \mathcal{P}(\phi)$, $Z(E) = m(E)e^{\phi\pi i}$ for some real number $m(E) > 0$. The nonzero objects of $\mathcal{P}(\phi)$ are said to be *semistable* (or *$\sigma$-semistable*) of phase $\phi$, and the simple[1] objects of $\mathcal{P}(\phi)$ are called *stable* (or *$\sigma$-stable*).

One can equivalently define stability conditions using the language of "hearts" instead of "slicings". For this, we need one last ingredient (following [Bri07, Definitions 2.1, 2.2, 2.3]):

**Definition 1.1.7.** Let $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z \geq 0\} \setminus \mathbb{R}_{\geq 0}$ and $\mathcal{A}$ an abelian category. A *stability function* is a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ such that for all nonzero $E \in \mathcal{A}$ the complex number $Z(E)$ lies in $\mathbb{H}$. Moreover, for all nonzero $E \in \mathcal{A}$

1. the *phase* of $E$ is the real number $\phi_Z(E) = \dfrac{1}{\pi}\arg(Z(E))$ in $\big(0,1\big]$.
2. $E$ is *Z-(semi)stable* if every subobject $F \subsetneq E$ satisfies $\phi_Z(F)(\leq) < \phi_Z(E)$.
3. $Z$ has the *Harder-Narasimhan property* if every nonzero $E \in \mathcal{A}$ has a Harder-Narasimhan filtration, i.e. a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset \ldots \subset E_{n-1} \subset E_n = E$$

whose factors $G_i = E_i/E_{i-1}$ are semistable objects of $\mathcal{A}$ with

$$\phi_Z(G_1) > \phi_Z(G_2) > \ldots > \phi_Z(G_n).$$

**Proposition 1.1.8** ([Bri07], Proposition 5.3)**.** *Giving a stability condition $(Z, \mathcal{P})$ on $\mathcal{D}$ is equivalent to giving the heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{D}$ and a stability function $Z\colon K(\mathcal{A}) \to \mathbb{C}$ with the Harder-Narasimhan property.*

[1]i.e. nonzero objects $E \in \mathcal{P}(\phi)$ with exactly two subobjects, namely $E$ and the zero object of $\mathcal{P}(\phi)$ (the latter is an abelian category by [Bri07, Lemma 5.2])).

We omit the full details of the proof and only mention some of its key steps: given a stability condition $(Z, \mathcal{P})$, take $\mathcal{A}$ to be the heart $\mathcal{P}(\big(0,1\big])$ of the bounded t-structure $(\mathcal{P}(>0), \mathcal{P}(\leq 1))$. The central charge $Z$ is a stability function on $\mathcal{A}$ when identifying $K(\mathcal{A}) = K(\mathcal{D})$, with the $Z$-semistable objects in $\mathcal{A}$ being precisely the objects of $\mathcal{P}(\phi)$ for $\phi \in \big(0,1\big]$. Lastly, the decomposition of objects of $\mathcal{A}$ given by Definition 1.1.5 (3) gives $Z$ the Harder-Narasimhan property.

For the other direction, define a stability condition $(Z, \mathcal{P})$ as follows: for every $\phi \in \big(0,1\big]$, let $\mathcal{P}(\phi)$ be the full subcategory of $\mathcal{A}$ comprised of the $Z$-semistable objects $E$ in $\mathcal{A}$ with phase $\phi$ as well as the zero objects of $\mathcal{D}$. Definition 1.1.5 (1) is easily satisfied, thus determining $\mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$, and Definition 1.1.5 (2) follows from Proposition 1.1.3 (1). Next, for any nonzero object $E \in \mathcal{D}$ there is a filtration as in Definition 1.1.5 (3) obtained by considering the decomposition of $E$ given in Proposition 1.1.3 (2) and then taking the Harder-Narasimhan filtration of each object $A_j \in \mathcal{A}$ that figures in that decomposition.

**Example 1.1.9.** Let $X$ be a smooth projective curve on an algebraically closed field $k$ of characteristic zero. Consider the group homomorphism $Z$ defined in (1.1.3). Then by Proposition 1.1.8 the pair $(Z, \mathrm{Coh}(X))$ is a stability condition on $D^b(X)$.

### Stability manifold

Bridgeland defines a natural topology on the set of stability conditions [Bri07, Section 6]. Denote by $\mathrm{Stab}(\mathcal{D})$ the subspace of stability conditions satisfying a certain property called local finiteness [Bri07, Definition 5.7].

Bridgeland's main result [Bri07, Theorem 1.2] says that for each connected component $\Sigma \subset \mathrm{Stab}(\mathcal{D})$ there is a local homeomorphism $\mathcal{Z}$ from $\Sigma$ to a linear subspace $V(\sigma) \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$ sending a stability condition $(Z, \mathcal{P})$ to its central charge $Z$. In other words, $\mathrm{Stab}(\mathcal{D})$ is a complex manifold, possibly of infinite dimension. The local finiteness assumption ensures that deformations of the central charge $Z$ lift to deformations of stability conditions, and without it, the map $\mathcal{Z}$ is only locally injective [Bri07, Lemma 6.4].

To avoid dealing with situations where $\mathrm{Stab}(\mathcal{D})$ is infinite-dimensional, we focus on the subspace $\mathrm{Stab}_{\Lambda}(\mathcal{D})$ of stability conditions $(Z, \mathcal{P})$ whose central charge $Z$ factors through a group homomorphism $\boldsymbol{cl}\colon K(X) \to \Lambda$, for some free $\mathbb{Z}$-module $\Lambda$ of finite rank. If the datum $(\boldsymbol{cl}, \Lambda)$ exists for $\mathcal{D}$, then $\mathrm{Stab}_{\Lambda}(\mathcal{D})$ is a complex manifold of finite dimension [Bri07, Corollary 1.3].

In most practical cases, the datum $(\boldsymbol{cl}, \Lambda)$ does in fact exist for $D^b(X)$, as the expression of the commonly used central charge $Z\colon K(X) \to \mathbb{C}$ is given in terms of intersection numbers of Chern classes, and thus factors through some finite rank lattice[2] $\Lambda$. This factorization is incorporated into the definition of stability conditions on $D^b(X)$.

## 1.2 Walls, chambers and moduli spaces

We give the proper formal definition of a wall in our main article (see Definition 2.2.1 in Chapter 2). To avoid redundancy, we only give an informal description of walls, chambers and the main strategy used in wall crossing in this section. Moreover, we say a few words on the moduli space of “Bridgeland” stable objects and list some of its properties that we use later on.

Let $X$ be a smooth projective variety over an algebraically closed field $k$. Consider the manifold of stability conditions $\mathrm{Stab}_{\Lambda}(X) = \mathrm{Stab}_{\Lambda}(D^b(X))$ on $X$ together with the data $(\Lambda, \boldsymbol{cl})$, and suppose that it is nonempty. For a fixed class $v \in \Lambda$, denote by $\mathcal{M}_{\sigma}(v)$ the coarse moduli space parametrizing the

[2] $\Lambda$ is often a quotient of the Grothendieck group.

$\sigma$-stable objects $E$ with $\boldsymbol{cl}(E) = v$, for some $\sigma \in \mathrm{Stab}_\Lambda(X)$.

Vaguely speaking, a *wall* is a closed codimension one submanifold of $\mathrm{Stab}_\Lambda(X)$ with a boundary, and a *chamber* is a connected component in the complement of every (finite) union of walls. A (possibly empty) collection of walls exists in $\mathrm{Stab}_\Lambda(X)$ for every class $v \in \Lambda$. By deforming a stability condition $\sigma \in \mathrm{Stab}_\Lambda(X)$ along a path that traverses a wall, an object that is $\sigma$-stable on one side of the wall may become unstable on the other side. Thus, crossing a wall may cause $\mathcal{M}_\sigma(v)$ to change. The key fact that leads to applications is that $\mathcal{M}_\sigma(v)$ can be identified with the moduli space of slope stable sheaves $E \in \mathrm{Coh}(X)$ with $\boldsymbol{cl}(E) = v$ for some appropriately chosen stability condition $\sigma$.

By Bridgeland's result (see [Bri08, Section 9]) the wall and chamber structure in $\mathrm{Stab}_\Lambda(X)$ has the following important properties: walls are locally finite, i.e. every compact subset in $\mathrm{Stab}_\Lambda(X)$ intersects a finite collection of walls. A stable object may become unstable only by crossing a wall. The set of stable objects remains the same in every chamber.

**Remark 1.2.1.** Let $X$ be a smooth projective threefold such that $\mathrm{Stab}_\Lambda(X)$ is nonempty. For a stability condition $\sigma \in \mathrm{Stab}_\Lambda(X)$ and a class $v \in \Lambda$, the coarse moduli space $\mathcal{M}_\sigma(v)$ associated to the moduli stack of $\sigma$-stable objects $E$ with $\boldsymbol{cl}(E) = v$ is a proper algebraic space [PT19, Corollary 4.23]. We recall the properties of $\mathcal{M}_\sigma(v)$ (as a coarse moduli space) that are relevant to us:

(I). The points of $\mathcal{M}_\sigma(v)$ are the $\sigma$-stable objects $E \in D^b(X)$.

(II). The tangent space at a point $E$ is isomorphic to $\mathrm{Ext}^1_X(E, E)$.

## 1.3 Tilting and the construction of stability conditions

This section has two main parts: first, a brief introduction to the concept of tilting and torsion pairs, which is the main tool for producing hearts of bounded t-structures for stability conditions on surfaces and threefolds. Then, we summarize the strategy behind the "double tilt" construction due to Bayer–Macrì–Toda [BMT14] for making stability conditions on threefolds. We do not go through the details of the construction here, as that is covered in Section 2.2.3 in Chapter 2.

### Torsion pairs and tilting

We start with the definition of a torsion pair (following [Bri08, Definition 3.2]):

**Definition 1.3.1.** A torsion pair in an abelian category $\mathcal{A}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{T}^{\perp})$ such that

1. $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{T}^{\perp}$, and
2. for all $X \in \mathcal{A}$ there is a short exact sequence
$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$
with $T \in \mathcal{T}$ and $F \in \mathcal{T}^{\perp}$.

The extension closed subcategory $\mathcal{A}^{\#} = \langle \mathcal{T}^{\perp}[1], \mathcal{T} \rangle \subset D^b(\mathcal{A})$ is called the *tilt* of $\mathcal{A}$ with respect to $(\mathcal{T}, \mathcal{T}^{\perp})$, and is an abelian category [HRS96].

The following is an explicit characterization of the abelian category $\mathcal{A}^{\#}$ in terms of cohomology objects:

**Proposition 1.3.2** ([HRS96], Proposition 2.1)**.** *Suppose $\mathcal{A}$ is the heart of a bounded t-structure on a triangulated category $\mathcal{D}$, and denote by $\mathcal{H}^i$ the cohomology functors associated to this t-structure. Let $(\mathcal{T}, \mathcal{T}^{\perp})$ be a torsion pair on $\mathcal{A}$. Then the full subcategory*

$$\mathcal{A}^{\#} = \{E \in \mathcal{D} : \mathcal{H}^{-1}(E) \in \mathcal{T}^{\perp}, \mathcal{H}^0(E) \in \mathcal{T}, \mathcal{H}^i(E) = 0 \text{ for all } i \neq -1, 0\},$$

*is the heart of a bounded t-structure on $\mathcal{D}$.*

As a consequence of the above proposition, every object $E \in \mathcal{A}^{\#}$ is isomorphic to a two-term complex

$$\mathcal{E}^{-1} \xrightarrow{d} \mathcal{E}^0$$

with $\mathrm{Coker}(d) \in \mathcal{T}$ and $\mathrm{Ker}(d) \in \mathcal{T}^{\perp}$. So we have the following short exact sequence in $\mathcal{A}^{\#}$

$$0 \to \mathcal{H}^{-1}(E)[1] \to E \to \mathcal{H}^0(E) \to 0.$$

### Overview on the construction of stability conditions

For $\dim X = 1$, $\mathrm{Stab}_{\Lambda}(X)$ was completely described: first for elliptic curves by Bridgeland [Bri07] where it was shown to be isomorphic to $\mathbb{C}^2$ (thus it is connected), then for all smooth projective curves [Oka06, Mac07].

In dimension two, Bridgeland constructed stability conditions on K3 surfaces [Bri08]. Roughly, his strategy goes as follows: for an ample class $\omega \in \mathrm{NS}(X)_{\mathbb{R}}$ and a class $B \in \mathrm{NS}(X)_{\mathbb{R}}$, consider the central charge

$$Z_{\omega,B} = -\int_X e^{-i\omega} \, \mathrm{ch}^B$$

Create a heart $\mathrm{Coh}^B(X)$ for this central charge by tilting $\mathrm{Coh}(X)$ (with respect to an appropriately chosen torsion pair). It then follows from a certain numerical constraint on slope semistable sheaves (called the Bogomolov-Gieseker inequality [BMT14, Theorem 3.1.4]) that the pair $(Z_{\omega,B}, \mathrm{Coh}^B(X))$ is a Bridgeland stability condition on the surface $X$. This strategy is not exclusive to K3 surfaces and can be generalized to any smooth projective surface [AB13].

In dimension three, a candidate for a stability condition is proposed in Bayer–Macrì–Toda [BMT14] that consists on the central charge $Z_{\omega,B}$ and a "double tilted" heart $\mathcal{A}^{\omega,B}$ obtained by tilting $\mathrm{Coh}^B(X)$ with respect to a certain torsion pair. The key to proving that the pair $(Z_{\omega,B}, \mathcal{A}^{\omega,B})$ is a Bridgeland stability condition is a conjectural Bogomolov-Gieseker type inequality:

**Conjecture 1.3.3** ([BMT14], Conjecture 1.3.1)**.** *For any $\nu_{\omega,B}$-semistable object $E \in \mathrm{Coh}^B(X)$ satisfying $\nu_{\omega,B}(E) = 0$, we have the following generalized Bogomolov-Gieseker inequality*

$$\mathrm{ch}_3^B(E) \leq \frac{\omega^2}{18}\,\mathrm{ch}_1^B(E),$$

*where*

$$\nu_{\omega,B}(E) = \begin{cases} \frac{\omega\,\mathrm{ch}_2^B(E) - \frac{\omega^3}{6}\,\mathrm{ch}_0^B(E)}{\omega^2\,\mathrm{ch}_1^B(E)} & \text{if } \omega^2\,\mathrm{ch}_1^{\mathrm{B}}(\mathrm{E}) \neq 0, \\ +\infty \;\; \text{else} & \end{cases}$$

*and a nonzero $E \in \mathrm{Coh}^B(X)$ is $\nu_{\omega,B}$-semistable if $\nu_{\omega,B}(F) \leq \nu_{\omega,B}(E)$ for every subobject $F \subsetneq E$.*

One should note that this inequality (as stated in [BMT14] and [BMS16]) does not hold for all threefolds, as shown through the counterexamples using blow-ups [Sch17, MS19]. However, some recent works came up with "modified" versions of this inequality which lead to the first successful examples of stability conditions on Calabi-Yau threefolds: the quintic threefold [Li18], weighted hypersurfaces in the weighted projective spaces $\mathbb{P}(1,1,1,1,2)$ and $\mathbb{P}(1,1,1,1,4)$ respectively [Kos22], and a smooth complete intersection of quadratic and quartic hypersurfaces in $\mathbb{P}^5$ [Liu21a]. We also list the non-Calabi-Yau threefolds for which stability conditions exist (as of November 2022): Fano threefolds [Mac14, Sch14, Li19, Piy17, BMSZ17], principally polarized abelian threefolds [MP15, MP16, BMS16], Kummer type threefolds [BMS16], threefolds with nef tangent bundles [Kos20], product varieties of a curve with a surface [Liu21b], and rank two projective bundles over a smooth curve [Sun22b, Sun22a].

In general, as stated in [BMT14] one could use this "tilting" method to produce stability conditions $(Z_{\omega,B}, \mathcal{A}^{\omega,B}_{n-1})$ on $X$ for any dimension $n$ by tilting $\mathrm{Coh}(X)$ $(n-1)$-times to get the heart $\mathcal{A}^{\omega,B}_{n-1}$ [BMT14, Conjecture 2.1.2]. However, at each step one has to prove a condition similar to the inequality in Conjecture 1.3.3.

**Remark 1.3.4.** When $\mathrm{NS}(X) \cong \mathbb{Z}[H]$, we have $\omega = \alpha H$ and $B = \beta H$ for $\alpha \in \mathbb{R}_{>0}$, $\beta \in R$. In this case, the tilted category $\mathrm{Coh}^B(X)$ is denoted $\mathrm{Coh}^\beta(X)$ instead, as we shall see in Chapter 2.

## 1.4 Mori cone and the contraction theorem

In this section we give a brief introduction to the basic tools from Mori theory which we use in Section 2.5 in Chapter 2. Rather informally, we recall the notions of intersection of cycles on a scheme $X$, the Mori cone $X$ and its dual relation with the nef cone. Finally, we state Mori's contraction theorem. For a detailed account of Mori theory, refer to Debarre's lecture notes [Deb10].

Throughout, $X$ is a smooth projective variety of dimension $n$ over an algebraically closed field of characteristic zero. Recall that an *$r$-cycle* on $X$ is an element of the free abelian group (denoted $Z_r X$) generated by all the (irreducible reduced) $r$-dimensional subvarieties of $X$. An $r$-cycle $\sum m_V V$ is *effective* if $m_V \in \mathbb{Z}_{\geq 0}$ for all $V \subset X$. The $(n-1)$-cycles are precisely the Cartier divisors on $X$.

Let $C_1$ and $C_2$ be two curves in $\mathbb{P}^2$ of respective degrees $d_1$ and $d_2$. Then Bézout's theorem says that these two curves intersect in $d_1 d_2$ points (counted with multiplicity), and we write $C_1 \cdot C_2 = d_1 d_2$. This is generalized to define the intersection number of a Cartier divisor and a 1-cycle on $X$ (see [Deb10, Section 3.4]).

**Definition 1.4.1** (Numerical equivalence)**.** An $r$-cycle $R$ on $X$ is numerically equivalent to zero if

$$R \cdot T = 0$$

for every $(n-r)$-cycle $T$ on $X$.

Denote by $Z_r\, X^0 \subset Z_r\, X$ the subgroup of $r$-cycles that are numerically equivalent to zero, and define the following free $\mathbb{Z}$-modules

$$N^1(X)_{\mathbb{Z}} = Z_{(n-1)}\, X / Z_{(n-1)}\, X^0$$
$$N_1(X)_{\mathbb{Z}} = Z_1\, X / Z_1\, X^0$$

of Cartier divisors (respectively 1-cycles) on $X$ modulo numerical equivalence, and consider the real vector spaces

$$N^1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{Z}} \otimes \mathbb{R} \text{ and } N_1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{Z}} \otimes \mathbb{R}$$

These spaces are dual with respect to the intersection pairing, with finite dimension.

**Definition 1.4.2.** The Mori cone of curves $\overline{\mathrm{NE}}(X) \subset N_1(X)_{\mathbb{R}}$ is the closure of the convex cone spanned by all the classes of effective 1-cycles on $X$.

**Definition 1.4.3.**

1. A divisor class $D \in N^1(X)_{\mathbb{R}}$ is nef (numerically eventually free) if $D \cdot C \geq 0$ for every curve $C \subset X$. The nef cone $\mathrm{Nef}(X)$ is the closed cone spanned by all nef divisor classes in $N^1(X)_{\mathbb{R}}$.

2. A divisor class $D$ is ample if, for every sheaf $\mathcal{F}$ on $X$, the sheaf $\mathcal{F} \otimes$

$\mathcal{O}_X(mD)$ is generated by its global sections for all $m \gg 0$. Denote by $\mathrm{Amp}(X)$ the cone generated by ample divisor classes in $N^1(X)_{\mathbb{R}}$.

Kleiman's criterion [Deb10, Theorem 4.10] gives a numerical characterization to ample divisors, namely: a divisor class $D \in N^1(X)_{\mathbb{R}}$ is ample if and only if $D \cdot z > 0$ for all nonzero $z \in \overline{\mathrm{NE}}(X)$. In particular, it says that $\mathrm{Nef}(X)$ is the closure of $\mathrm{Amp}(X)$, and that $\mathrm{Amp}(X)$ is equal to the interior of $\mathrm{Nef}(X)$. Most importantly to us, it implies the useful fact that $\mathrm{Nef}(X)$ is the dual space of $\overline{\mathrm{NE}}(X)$.

**Definition 1.4.4.** An *extremal ray* $R$ in $\overline{\mathrm{NE}}(X)$ is a one dimensional closed convex subcone of $\overline{\mathrm{NE}}(X)$ such that for every $\alpha_1, \alpha_2 \in \overline{\mathrm{NE}}(X)$, if $\alpha_1 + \alpha_2 \in R$ then $\alpha_1, \alpha_2 \in R$.

The canonical divisor class of $X$ is denoted $K_X$. The following is the contraction theorem of extremal rays [Deb10, Corollary 8.4], [Mor82, Theorem 3.1]:

**Theorem 1.4.5** (Contraction theorem)**.** *Let $\alpha \in N_1(X)_{\mathbb{R}}$ be a curve class which spans an extremal ray in $\overline{\mathrm{NE}}(X)$. If $K_X \cdot \alpha < 0$ (i.e. $\alpha$ is $K_X$-negative), then there exists a contraction $f\colon X \to Y$ to a unique normal projective variety $Y$ such that:*

1. *$f_*\mathcal{O}_X = \mathcal{O}_Y$,*
2. *for any irreducible curve $C \subset X$, $[C] \in \alpha$ if and only if $\dim f(C) = 0$.*

# Chapter 2

# Main article

# Bridgeland stability conditions and skew lines on $\mathbb{P}^3$

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**Abstract**

Inspired by Schmidt's work on twisted cubics [Sch15], we study wall crossings in Bridgeland stability that start with the Hilbert scheme $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ parametrizing pairs of skew lines and plane conics union a point. We find two walls. Each wall crossing corresponds to a contraction of a divisor in the moduli space and the contracted space remains smooth. Building on work by Chen–Coskun–Nollet [CCN11] we moreover prove that the contractions are $K$-negative extremal in the sense of Mori theory and so the moduli spaces are projective.

## 2.1 Introduction

Bridgeland introduced the notion of stability conditions on triangulated categories in [Bri07] and showed that the set of stability conditions form a complex manifold.

One of the main applications of Bridgeland stability conditions is the study of the birational geometry of moduli spaces using wall crossing. In this work we consider the Hilbert scheme $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ of subschemes $Y \subset \mathbb{P}^3$ with Hilbert polynomial $2m+2$. It has two smooth components $\mathcal{C}$ and $\mathcal{S}$: a general point in $\mathcal{C}$ is a conic-union-a-point $Y = C \cup \{P\}$ and a general point in $\mathcal{S}$ is a pair of skew lines $Y = L_1 \cup L_2$. Note that when a line pair is deformed until the two lines meet, the result is a pair of intersecting lines with an embedded point at the intersection, and this can also be viewed as a degenerate case of a conic union a point.

For an appropriately chosen Bridgeland stability condition on the bounded derived category of coherent sheaves $D^b(\mathbb{P}^3)$, the ideal sheaves $\mathcal{I}_Y$ can be viewed as the stable objects with fixed Chern character, say $v = \mathrm{ch}(\mathcal{I}_Y)$. When deforming the stability condition, we identify two walls, separating three chambers. Getting slightly ahead of ourselves, the situation is illustrated in Figure 2.2 in Section 2.3.2: $\alpha$ and $\beta$ are parameters for the stability conditions considered and we restrict ourselves to the region to the immediate left in the picture of the hyperbola $\beta^2 - \alpha^2 = 4$ (the role of this boundary curve is explained in Section 2.2.4). In this region we have the two walls $W_1$ and $W_2$ separating three chambers, labeled by Roman numerals as in the figure. Let $\mathcal{M}^{\mathrm{I}}$, $\mathcal{M}^{\mathrm{II}}$ and $\mathcal{M}^{\mathrm{III}}$ be the moduli spaces of Bridgeland stable objects with Chern character $v$ in each chamber, considered as algebraic spaces [PT19, Corollary 4.23].

Our first main result contains the set-theoretical description of these moduli spaces:

**Theorem 2.1.1.**

*(I)* $\mathcal{M}^{\mathrm{I}}$ *is* $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ *with its two components* $\mathcal{S}$ *and* $\mathcal{C}$ *described above.*

*(II)* $\mathcal{M}^{\mathrm{II}}$ *consists of:*

- *(i) Ideal sheaves* $\mathcal{I}_Y$ *for* $Y \in \mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ *not contained in a plane.*
- *(ii) Non-split extensions* $\mathcal{F}_{P,V}$ *in*

$$0 \longrightarrow \mathcal{I}_{P/V}(-2) \longrightarrow \mathcal{F}_{P,V} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow 0 \tag{2.1.1}$$

*for* $V \subset \mathbb{P}^3$ *a plane and* $P \in V$*. Moreover,* $\mathcal{F}_{P,V}$ *is uniquely determined up to isomorphism by the pair* $(P, V)$*.*

*(III)* $\mathcal{M}^{\mathrm{III}}$ *consists of:*

- *(i) Ideal sheaves* $\mathcal{I}_Y$ *for* $Y \in \mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ *a pair of disjoint lines or a pure double line.*
- *(ii) Non-split extensions* $\mathcal{G}_{P,V}$ *in*

$$0 \longrightarrow \mathcal{O}_V(-2) \longrightarrow \mathcal{G}_{P,V} \longrightarrow \mathcal{I}_P(-1) \longrightarrow 0 \tag{2.1.2}$$

*with* $P$ *and* $V$ *as above. Moreover,* $\mathcal{G}_{P,V}$ *is uniquely determined by the pair* $(P, V)$*.*

The method we employ to locate walls and classify stable objects is due to Schmidt [Sch15], and involves "lifting" walls from an intermediate notion of tilt stability. Schmidt considers as an application the Hilbert scheme $\mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$: it parametrizes twisted cubics and plane cubics union a point. This was our starting point and we can apply many of Schmidt's results directly, although modified or new arguments are needed as well. The end result is closely analogous in the two cases, with two wall crossings of the same nature. In the twisted cubic situation, however, Schmidt also finds an additional "final wall crossing" where all objects are destabilized. This has no analogy in our case.

Next we describe the moduli spaces geometrically, guided by the set theoretic classification of objects above; this leads to contractions of the two smooth components $\mathcal{C}$ and $\mathcal{S}$ of $\mathcal{M}^{\mathrm{I}} = \mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$. First introduce notation for the loci that are destabilized by the two wall crossings according to the above classification:

**Notation 2.1.2.**

(a) Let $E \subset \mathcal{C}$ be the divisor[1] consisting of all planar $Y \in \mathcal{C}$.

(b) Let $F \subset \mathcal{S}$ be the divisor[2] consisting of all $Y \in \mathcal{S}$ having an embedded point.

Thus the locus (II)(i) is $(\mathcal{C} \setminus E) \cup \mathcal{S}$ and the locus (III)(i) is $\mathcal{S} \setminus F$. On the other hand both loci (II)(ii) and (III)(ii) are parametrized by the incidence variety

$$I := \{(P, V) \in \mathbb{P}^3 \times \check{\mathbb{P}}^3 \mid P \in V\} \tag{2.1.3}$$

where $\check{\mathbb{P}}^3$ is the dual projective space. The process of replacing $E$ and $F$ by $I$ can be realized as contractions of algebraic spaces: $E$ and $F$ may be viewed as projective bundles over $I$, and in Section 2.4 we apply Artin's contractibility criterion to obtain smooth algebraic spaces $\mathcal{C}'$ and $\mathcal{S}'$ each containing the incidence variety $I$ as a closed subspace, and birational morphisms

$$\phi \colon \mathcal{C} \to \mathcal{C}'$$
$$\psi \colon \mathcal{S} \to \mathcal{S}'$$

[1] This is explained in Section 2.4.1.
[2] This is explained in Section 2.4.3.

which are isomorphisms outside of $E$, respectively $F$, and restrict to the natural maps $E \to I$, respectively $F \to I$. Moreover $E \subset \mathcal{C}$ is disjoint from $\mathcal{S}$, so the union $\mathcal{C}' \cup \mathcal{S}$ makes sense as the gluing together of $(\mathcal{C} \setminus E) \cup \mathcal{S}$ and $\mathcal{C}'$. We can then state our second main result:

**Theorem 2.1.3.**

*(a)* $\mathcal{M}^{\mathrm{II}}$ *is isomorphic to* $\mathcal{C}' \cup \mathcal{S}$*.*

*(b)* $\mathcal{M}^{\mathrm{III}}$ *is isomorphic to* $\mathcal{S}'$*.*

To prove the theorem it suffices to treat the contracted spaces as algebraic spaces. However, they turn out to be projective varieties: the contractions are in fact $K$-negative extremal contractions in the sense of Mori theory. The case of $\mathcal{S} \to \mathcal{S}'$ can be found in previous work by Chen–Coskun–Nollet [CCN11] and in fact it turns out that $\mathcal{S}'$ is a Grassmannian; see Section 2.4.2. Inspired by this work, we exhibit in Section 2.5 the map $\mathcal{C} \to \mathcal{C}'$ as a $K$-negative extremal contraction. This may be contrasted with Schmidt's approach in the twisted cubic situation [Sch15], where projectivity of the moduli spaces is proved by viewing them as moduli of quiver representations.

In Section 2.2, we list the background results that we need, in particular, we briefly recall the construction of stability conditions on threefolds, along with the notion of tilt-stability. In Section 2.3 we apply Schmidt's machinery to prove Theorem 2.1.1. In Section 2.4 we study universal families and prove Theorem 2.1.3. Finally, in Section 2.5 we work out the Mori cone of $\mathcal{C}$.

We work over $\mathbb{C}$. Throughout and in particular in Section 2.4, intersections and unions of subschemes are defined by the sum and intersection of ideals, respectively, and inclusions and equalities between subschemes are meant in the scheme theoretic sense.

## 2.2 Preliminaries

After detailing the two components of $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$, we collect notions and results from the literature surrounding Bridgeland stability and wall crossings for smooth projective threefolds. There are no original results in this section.

### 2.2.1 The Hilbert scheme and its two components

It is known that $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ has two smooth components $\mathcal{C}$ and $\mathcal{S}$, whose general points are conics union a point and pairs of skew lines, respectively. A quick parameter count yields $\dim\mathcal{C} = 11$ and $\dim\mathcal{S} = 8$. We refer to Lee [Lee00] for an overview, to Chen–Nollet [CN12] for the smoothness of $\mathcal{C}$ and to Chen–Coskun–Nollet [CCN11] for the smoothness of $\mathcal{S}$. In fact, the referenced works show that $\mathcal{C}$ is the blowup

$$\mathcal{C} \to \mathbb{P}^3 \times \mathrm{Hilb}^{2m+1}(\mathbb{P}^3)$$

along the universal conic $\mathcal{Z} \subset \mathbb{P}^3 \times \mathrm{Hilb}^{2m+1}(\mathbb{P}^3)$ and $\mathcal{S}$ is the blowup

$$\mathcal{S} \to \mathrm{Sym}^2(G(2,4))$$

along the diagonal in the symmetric square of the Grassmannian $G(2,4)$ of lines in $\mathbb{P}^3$. In other words, it is the Hilbert scheme $\mathrm{Hilb}^2(G(2,4))$ of finite subschemes in $G$ of length two.

Following [Lee00], we next list all elements $Y \in \mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$, including degenerate cases.

By a curve $C \subset \mathbb{P}^3$ *with an embedded point* at $P \in C$ we mean a subscheme $Y \subset \mathbb{P}^3$ such that $C \subset Y$ and the relative ideal $\mathcal{I}_{C/Y}$ is isomorphic to $k(P)$. This makes sense even when we allow $C$ to be singular or nonreduced.

The component $\mathcal{C}$ parametrizes subschemes $Y$ of the following form: let $C$ be a conic in a plane $V \subset \mathbb{P}^3$, possibly a union of two lines or a planar double line. Then $Y$ is either the disjoint union of $C$ and a point $P \in \mathbb{P}^3$, or $C$ with an embedded point at $P \in C$. If $C$ is nonsingular at $P$, embedded points correspond to normal directions, parametrized by a $\mathbb{P}^1$. Since even degenerate conics are complete intersections, also embedded point structures at a singular or nonreduced point $P$ form a $\mathbb{P}^1$. Among these, precisely one is planar ($Y$ is contained in a plane) and precisely one is spatial ($Y$ contains the first order infinitesimal neighborhood of $P$ in $\mathbb{P}^3$).

The component $\mathcal{S}$ parametrizes pairs $Y = L_1 \cup L_2$ of skew lines, together with its degenerations. These are of the following three types: (1) a pair of incident lines $L_1 \cup L_2$ with a spatial embedded point at the intersection point, (2) a planar double line with a spatial embedded point, or (3) a pure double line in a nonsingular quadric. Clearly, then, $\mathcal{C} \cap \mathcal{S}$ consists of the incident lines or planar double lines with a spatial embedded point.

### 2.2.2 Stability conditions and walls

Let $X$ be a smooth projective threefold over $\mathbb{C}$ and fix a finite rank lattice $\Lambda$ equipped with a homomorphism $K(X) \to \Lambda$ from the Grothendieck group of coherent sheaves modulo short exact sequences. On $\mathbb{P}^3$ we will take $\Lambda = \mathbb{Z} \oplus \mathbb{Z} \oplus \frac{1}{2}\mathbb{Z} \oplus \frac{1}{6}\mathbb{Z}$ equipped with the Chern character map $\mathrm{ch}\colon K(\mathbb{P}^3) \to \Lambda$.

Recall [Bri07, BMT14, BMS16] that a Bridgeland stability condition $\sigma = (\mathcal{A}, Z)$ on $X$ (with respect to $\Lambda$) consists of

(i) an abelian subcategory $\mathcal{A} \subset D^b(X)$, which is the heart of a bounded $t$-structure, and

(ii) a stability function $Z$, which is a group homomorphism

$$Z\colon \Lambda \to \mathbb{C}$$

whose value on any nonzero object $\mathcal{E} \in \mathcal{A}$ is in the upper half plane

$$\mathbb{H} = \left\{ z \in \mathbb{C} \,\middle|\, \Im z \geq 0 \right\} \setminus \mathbb{R}_{\geq 0}.$$

This is subject to a list of axioms which we will not give (see [BMS16, Section 8]). We may then partially order the nonzero objects in $\mathcal{A}$ by their *slope*

$$\lambda_\sigma = -\Re(Z)/\Im(Z) \in \mathbb{R} \cup \{+\infty\}.$$

This yields a notion of $\sigma$-stability and $\sigma$-semistability for objects in $\mathcal{A}$ in the usual way by comparing the slope of an object with that of its sub- or quotient objects. These notions extend to $D^b(X)$ by shifting in the sense of the $t$-structure.

Bridgeland's result [Bri07, Theorem 1.2] gives the set $\mathrm{Stab}_\Lambda(X)$ of stability conditions the structure of a complex manifold, and for a given $u \in \Lambda$, it admits a wall and chamber structure:

**Definition 2.2.1.** Fix a primitive $u \in \Lambda$.

i. *Numerical walls*: Let $v \in \Lambda$ be an element not parallel to $u$. A numerical wall $W_v^u$ for $u$ with respect to $v$ is the solution set

$$W_v^u = \{\sigma \in \mathrm{Stab}_\Lambda(X) \mid \lambda_\sigma(u) = \lambda_\sigma(v)\}.$$

ii. *Actual walls*: A subset $V \subset W_v^u$ of a numerical wall is an actual wall if for each point $\sigma \in V$, there is a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

in $\mathcal{A}$ of $\sigma$-semistable objects such that $\mathrm{ch}(\mathcal{E}) = u$, and $\lambda_\sigma(\mathcal{F}) = \lambda_\sigma(\mathcal{G}) = \lambda_\sigma(\mathcal{E})$ with $\mathrm{ch}(\mathcal{F}) = \mathrm{ch}(\mathcal{G}) = v$.

When the context is clear, we drop the term "actual" and just say "wall".

We say that a short exact sequence as in (ii) above *defines* the wall. Relaxing this, an unordered pair $(\mathcal{F}, \mathcal{G})$ *defines* the wall if there is a short exact sequence in either direction (i.e. we allow the roles of sub and quotient objects to be swapped) as in (ii).

Given a union of walls, we refer to each connected component of its complement in $\mathrm{Stab}_\Lambda(X)$ as a *chamber*. By the arguments in [Bri08, Section 9] there is a locally finite collection of (actual) walls in $\mathrm{Stab}_\Lambda(X)$, each being a closed codimension one manifold with boundary, such that the set of stable objects in $\mathcal{A}$ with Chern character $u$ remains constant within each chamber, and there are no strictly semistable objects in a chamber.

**Remark 2.2.2.** A *very weak* stability condition $(\mathcal{A}, Z)$ is a weakening of the above concept (see Piyaratne–Toda [PT19]) where $Z$ is allowed to map nonzero objects in $\mathcal{A}$ to zero. One may define an associated slope function $\lambda$ as before, with the convention that $\lambda(\mathcal{E}) = +\infty$ also when $Z(\mathcal{E}) = 0$. An object $\mathcal{E} \in \mathcal{A}$ is declared to be stable if every nontrivial subobject $\mathcal{F} \subsetneq \mathcal{E}$ satisfies $\lambda(\mathcal{F}) < \lambda(\mathcal{E}/\mathcal{F})$, and semistable when nonstrict inequality is allowed. With this definition one avoids the need to treat cases where $Z(\mathcal{F}) = 0$ or $Z(\mathcal{E}/\mathcal{F}) = 0$ separately. We will not need to go into further detail.

### 2.2.3 Construction of stability conditions on threefolds

We next recall the "double tilt" construction of stability conditions by Bayer–Macrì–Toda [BMT14]. For this it is necessary to assume that the threefold $X$ satisfies a certain "Bogomolov inequality" type condition [BMS16, Conjecture 4.1]), which is known in several cases including $\mathbb{P}^3$ [Mac14]. Fix a polarization

$H$ on $X$; on $\mathbb{P}^3$ this will be a (hyper-) plane.

### Slope stability

Let $\beta \in \mathbb{R}$. The twisted Chern character of a sheaf or a complex $\mathcal{E}$ on $X$ is defined by $\mathrm{ch}^\beta(\mathcal{E}) = e^{-\beta H}\,\mathrm{ch}(\mathcal{E})$. Its homogeneous components are

$$\begin{aligned}
\mathrm{ch}_0^\beta(\mathcal{E}) &= \mathrm{ch}_0(\mathcal{E}) = \mathrm{rk}(\mathcal{E})\\
\mathrm{ch}_1^\beta(\mathcal{E}) &= \mathrm{ch}_1(\mathcal{E}) - \beta H\,\mathrm{ch}_0(\mathcal{E})\\
\mathrm{ch}_2^\beta(\mathcal{E}) &= \mathrm{ch}_2(\mathcal{E}) - \beta H\,\mathrm{ch}_1(\mathcal{E}) + \frac{\beta^2H^2}{2}\,\mathrm{ch}_0(\mathcal{E})\\
\mathrm{ch}_3^\beta(\mathcal{E}) &= \mathrm{ch}_3(\mathcal{E}) - \beta H\,\mathrm{ch}_2(\mathcal{E}) + \frac{\beta^2H^2}{2}\,\mathrm{ch}_1(\mathcal{E}) - \frac{\beta^3H^3}{6}\,\mathrm{ch}_0(\mathcal{E}).
\end{aligned}$$

The twisted slope stability function on the abelian category $\mathrm{Coh}(X)$ of coherent sheaves is given by

$$\mu_\beta(\mathcal{E}) = \begin{cases} \frac{H^2ch_1^\beta(\mathcal{E})}{H^3ch_0^\beta(\mathcal{E})} & \text{if } \mathrm{rk}(\mathcal{E}) \neq 0,\\ +\infty & \text{else.}\end{cases} \tag{2.2.1}$$

This is the slope of a very weak stability condition. Notice that $\mu_\beta(\mathcal{E}) = \mu(\mathcal{E}) - \beta$, where $\mu(\mathcal{E})$ is the classical slope stability function. A sheaf $\mathcal{E} \in \mathrm{Coh}(X)$ which is (semi)stable with respect to this very weak stability condition is called $\mu_\beta$-(semi)stable (or slope (semi)stable).

**Tilt stability**

Next, define the following full subcategories of $\mathrm{Coh}(X)$

$$\mathcal{T}_\beta = \{\mathcal{E} \in \mathrm{Coh}(X) \mid \text{Any quotient } \mathcal{E} \twoheadrightarrow \mathcal{G} \text{ satisfies } \mu_\beta(\mathcal{G}) > 0\},$$
$$\mathcal{T}_\beta^\perp = \{\mathcal{E} \in \mathrm{Coh}(X) \mid \text{Any subsheaf } \mathcal{F} \subset \mathcal{E} \text{ satisfies } \mu_\beta(\mathcal{F}) \leq 0\}.$$

The pair $(\mathcal{T}_\beta, \mathcal{T}_\beta^\perp)$ is a torsion pair (Definition 1.3.1) in $\mathrm{Coh}(X)$. Tilt the category $\mathrm{Coh}(X)$ with respect to this torsion pair and denote the obtained heart by $\mathrm{Coh}^\beta(X) = \left\langle \mathcal{T}_\beta^\perp[1], \mathcal{T}_\beta \right\rangle$. Thus every object $\mathcal{E} \in \mathrm{Coh}^\beta(X)$ fits in a short exact sequence

$$0 \longrightarrow \mathcal{H}^{-1}(\mathcal{E})[1] \longrightarrow \mathcal{E} \longrightarrow \mathcal{H}^0(\mathcal{E}) \longrightarrow 0 \tag{2.2.2}$$

with $\mathcal{H}^{-1}(\mathcal{E}) \in \mathcal{T}_\beta^\perp$ and $\mathcal{H}^0(\mathcal{E}) \in \mathcal{T}_\beta$.

Let $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$, and let

$$Z^{\mathrm{tilt}}_{\alpha,\beta}(\mathcal{E}) = -Hch_2^\beta(\mathcal{E}) + \frac{\alpha^2}{2} H^3 ch_0^\beta(\mathcal{E}) + iH^2 ch_1^\beta(\mathcal{E}). \tag{2.2.3}$$

The associated slope function is

$$\nu_{\alpha,\beta}(\mathcal{E}) = \begin{cases} \frac{Hch_2^\beta(\mathcal{E}) - \frac{\alpha^2}{2} H^3 ch_0^\beta(\mathcal{E})}{H^2 ch_1^\beta(\mathcal{E})} & \text{if } \mathrm{H}^2\mathrm{ch}_1^\beta(\mathcal{E}) \neq 0, \\ +\infty & \text{else.} \end{cases}$$

By [BMS16, Proposition B.2 (case $B = \beta H$)], the pair $(\mathrm{Coh}^\beta(X), Z^{\mathrm{tilt}}_{\alpha,\beta})$ is a very weak stability condition, and the set of such very weak stability conditions is continuously parametrized by $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$. An object $\mathcal{E} \in \mathrm{Coh}^\beta(X)$ which is (semi)stable with respect to this very weak stability condition is called $\nu_{\alpha,\beta}$-(semi)stable (or tilt (semi)stable). Moreover the parameter space $\mathbb{R}_{>0} \times \mathbb{R}$ admits a wall and chamber structure, in which walls are nested semicircles centered on the $\beta$-axis, or vertical lines (we view $\alpha$ as the vertical axis) [Sch15, Theorem 3.3]. We refer to them as "tilt-stability walls" or "$\nu$-walls" interchangeably.

The following is the Bogomolov inequality for tilt-stability:

**Proposition 2.2.3.** *[BMT14, Corollary 7.3.2] Any $\nu_{\alpha,\beta}$-semistable object $\mathcal{E} \in \mathrm{Coh}^{\beta}(X)$ satisfies*

$$\overline{\Delta}_H(\mathcal{E}) := \left(H^2 \operatorname{ch}_1^{\beta}(\mathcal{E})\right)^2 - 2H^3 \operatorname{ch}_0^{\beta}(\mathcal{E}) H \operatorname{ch}_2^{\beta}(\mathcal{E}) \geq 0.$$

**Bridgeland stability**

Define the following full subcategories of $\mathrm{Coh}^{\beta}(X)$:

$$\mathcal{T}'_{\alpha,\beta} = \{\mathcal{E} \in \mathrm{Coh}^{\beta}(X) \mid \text{Any quotient } \mathcal{E} \twoheadrightarrow \mathcal{G} \text{ satisfies } \nu_{\alpha,\beta}(\mathcal{G}) > 0\},$$
$$\mathcal{T}'^{\perp}_{\alpha,\beta} = \{\mathcal{E} \in \mathrm{Coh}^{\beta}(X) \mid \text{Any subsheaf } \mathcal{F} \subset \mathcal{E} \text{ satisfies } \nu_{\alpha,\beta}(\mathcal{F}) \leq 0\}.$$

They form a torsion pair. Tilting $\mathrm{Coh}^{\beta}(X)$ with respect to this pair yields stability conditions $(\mathcal{A}_{\alpha,\beta}(X), Z_{\alpha,\beta,s})$ ([BMS16, Theorem 8.6, Lemma 8.8]) on $X$, where $\mathcal{A}_{\alpha,\beta}(X) = \left\langle \mathcal{T}'^{\perp}_{\alpha,\beta}[1], \mathcal{T}'_{\alpha,\beta} \right\rangle$ and

$$Z_{\alpha,\beta,s} = -\operatorname{ch}_3^{\beta} + \alpha^2\left(\frac{1}{6} + s\right)H^2 \operatorname{ch}_1^{\beta} + i\left(H \operatorname{ch}_2^{\beta} - \frac{\alpha^2}{2} H^3 \operatorname{ch}_0^{\beta}\right). \tag{2.2.4}$$

The slope function of $Z_{\alpha,\beta,s}$ is given by

$$\lambda_{\alpha,\beta,s}(\mathcal{E}) = \frac{\operatorname{ch}_3^{\beta}(\mathcal{E}) - \alpha^2(\frac{1}{6} + s)H^2 \operatorname{ch}_1^{\beta}(\mathcal{E})}{H \operatorname{ch}_2^{\beta}(\mathcal{E}) - \frac{\alpha^2}{2} H^3 \operatorname{ch}_0^{\beta}(\mathcal{E})}$$

with $\lambda_{\alpha,\beta,s}(\mathcal{E}) = +\infty$ when $H \operatorname{ch}_2^{\beta}(\mathcal{E}) = \frac{\alpha^2}{2} H^3 \operatorname{ch}_0^{\beta}(\mathcal{E})$.

An object $\mathcal{E} \in \mathcal{A}_{\alpha,\beta}(X)$ which is (semi)stable with respect to this stability condition is called $\lambda_{\alpha,\beta,s}$-(semi)stable.

By [BMS16, Proposition 8.10], there is a family of stability conditions $(\mathcal{A}_{\alpha,\beta}(X), Z_{\alpha,\beta,s})$ in $\mathrm{Stab}_\Lambda(X)$ continuously parametrized by $(\alpha, \beta, s) \in \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}_{>0}$. We refer to walls in $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}_{>0}$ as "$\lambda$-walls".

The following lemma allows us to identify moduli spaces of slope-stable sheaves with moduli spaces of tilt-stable sheaves, given the right conditions:

**Lemma 2.2.4.** *[GHS16, Lemma 1.4] On $\mathbb{P}^3$, let $v = (v_0, v_1, v_2, v_3) \in \Lambda$ satisfy $\mu_\beta(v) > 0$ and assume $(v_0, v_1)$ is primitive. Then an object $\mathcal{E} \in \mathrm{Coh}^\beta(X)$ with $\mathrm{ch}(\mathcal{E}) = v$ is $\nu_{\alpha,\beta}$-stable for all $\alpha \gg 0$ if and only if $\mathcal{E}$ is a slope stable sheaf.*

### 2.2.4 Comparison between $\nu$-stability and $\lambda$-stability — after Schmidt

Let $\mathcal{E}$ be an object in $D^b(X)$. Throughout this section, let $(\alpha_0, \beta_0) \in \mathbb{R}_{>0} \times \mathbb{R}$ satisfy $\nu_{\alpha_0,\beta_0}(\mathcal{E}) = 0$, and fix $s > 0$. We shall summarize a series of results by Schmidt [Sch15] enabling us to compare walls and chambers with respect to $\nu$-stability with those of $\lambda$-stability. (Looking ahead to our application illustrated in Figures 2.1 and 2.2, the dashed hyperbola is the solution set to $\nu_{\alpha,\beta}(\mathcal{E}) = 0$.)

Consider the following conditions on $\mathcal{E}$:

1. $\mathcal{E}$ is a $\nu_{\alpha_0,\beta_0}$-stable object in $\mathrm{Coh}^{\beta_0}(X)$.
2. $\mathcal{E}$ is a $\lambda_{\alpha,\beta,s}$-stable object in $\mathcal{A}^{\alpha,\beta}(X)$, for all $(\alpha, \beta)$ in an open neighborhood of $(\alpha_0, \beta_0)$ with $\nu_{\alpha,\beta}(\mathcal{E}) > 0$.
3. $\mathcal{E}$ is a $\lambda_{\alpha,\beta,s}$-semistable object in $\mathcal{A}^{\alpha,\beta}(X)$, for all $(\alpha, \beta)$ in an open neighborhood of $(\alpha_0, \beta_0)$ with $\nu_{\alpha,\beta}(\mathcal{E}) > 0$.
4. $\mathcal{E}$ is a $\nu_{\alpha_0,\beta_0}$-semistable object in $\mathrm{Coh}^{\beta_0}(X)$.

Obviously there are implications (1) $\implies$ (4) and (2) $\implies$ (3). The following says that, under a mild condition on $\operatorname{ch}(\mathcal{E})$, there are in fact implications

$$(1) \implies (2) \implies (3) \implies (4)$$

so that $\lambda$-stability in a certain sense refines $\nu$-stability:

**Theorem 2.2.5** (Schmidt)**.** *The implication* (1) $\implies$ (2) *above always holds. If* $H^2 \operatorname{ch}_1^{\beta_0}(\mathcal{E}) > 0$ *and* $\overline{\Delta}_H(\mathcal{E}) > 0$ *then also the implication* (3) $\implies$ (4) *holds.*

For the proof we refer to Schmidt [Sch15]: the first implication follows from Lemma 6.2 in *loc. cit.* and the second follows from Lemmas 6.3 and 6.4.

Schmidt furthermore compares walls for $\nu$-stability and $\lambda$-stability, for objects $\mathcal{E}$ in some fixed class $v \in \Lambda$. Let

$$\mathcal{F} \to \mathcal{E} \to \mathcal{G} \to \mathcal{F}[1] \tag{2.2.5}$$

be a triangle in $D^b(X)$ with $\mathcal{E}$ in class $v$.

- Say that (2.2.5) *defines a $\nu$-wall* through $(\alpha_0, \beta_0)$ if $\mathcal{F}$,$\mathcal{E}$,$\mathcal{G}$ are $\nu_{\alpha_0,\beta_0}$-semistable objects in $\operatorname{Coh}^{\beta_0}(X)$ and $\nu_{\alpha_0,\beta_0}(\mathcal{F}) = \nu_{\alpha_0,\beta_0}(\mathcal{G})$ (which is thus zero).

- Say that (2.2.5) *defines a $\lambda$-wall at the $\nu$-positive side of* $(\alpha_0, \beta_0)$ if there is an open neighborhood $U$ of $(\alpha_0, \beta_0)$ such that, writing

$$W = \{(\alpha, \beta) \in U \mid \nu_{\alpha,\beta}(v) > 0 \text{ and } \lambda_{\alpha,\beta,s}(\mathcal{F}) = \lambda_{\alpha,\beta,s}(\mathcal{G})\}$$

  the following holds: $(\alpha_0, \beta_0)$ is in the closure of $W$ and $\mathcal{F}, \mathcal{E}, \mathcal{G}$ are $\lambda_{\alpha,\beta,s}$-semistable objects in $\mathcal{A}^{\alpha,\beta}(X)$ for all $(\alpha, \beta) \in W$.

Note that the assumption that $\mathcal{F}, \mathcal{E}, \mathcal{G}$ are all in $\operatorname{Coh}^{\beta_0}(X)$ or in $\mathcal{A}^{\alpha,\beta}(X)$

implies that the triangle (2.2.5) is a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

in that abelian category.

**Theorem 2.2.6** (Schmidt)**.** *Let $\mathcal{E}$ be an object in $D^b(X)$ and let $(\alpha_0, \beta_0) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that $\nu_{\alpha_0,\beta_0}(\mathcal{E}) = 0$ and $\mathrm{ch}_1^{\beta_0}(\mathcal{E}) > 0$.*

1. *If a triangle (2.2.5) defines a $\lambda$-wall on the $\nu$-positive side of $(\alpha_0, \beta_0)$, then it also defines a $\nu$-wall through $(\alpha_0, \beta_0)$.*
2. *Suppose a triangle (2.2.5) defines a $\nu$-wall through $(\alpha_0, \beta_0)$ and $\mathcal{F}$, $\mathcal{G}$ are $\nu_{\alpha_0,\beta_0}$-stable. Moreover let*

$$W = \{(\alpha, \beta) \mid \nu_{\alpha,\beta}(v) > 0 \text{ and } \lambda_{\alpha,\beta,s}(\mathcal{F}) = \lambda_{\alpha,\beta,s}(\mathcal{G})\}$$

*and suppose there are points $(\alpha, \beta) \in W$ arbitrarily close to $(\alpha_0, \beta_0)$ such that $\nu_{\alpha,\beta}(\mathcal{F}) > 0$ and $\nu_{\alpha,\beta}(\mathcal{G}) > 0$. Then (2.2.5) defines a $\lambda$-wall on the $\nu$-positive side of $(\alpha_0, \beta_0)$, namely $W$.*

For the proof we refer to Schmidt [Sch15]: part (1) is Schmidt's Theorem 6.1(1) and part (2) is the special case $n = 1$ of Schmidt's Theorem 6.1(4). To align the notation, in part (1) Schmidt's $\mathcal{F}, \mathcal{E}, \mathcal{G}$ are our $\mathcal{F}[1], \mathcal{E}[1], \mathcal{G}[1]$. To apply Theorem 6.1(1) these are required to be $\lambda_{\alpha_0,\beta_0,s}$-semistable objects in $\mathcal{A}^{\alpha_0,\beta_0}(X)$; this is ensured by Schmidt's Lemma 6.3.

To control how the set of stable objects changes as a $\lambda$-wall is crossed, we take advantage of the fact that the $\lambda$-walls we obtain are defined by short exact sequences with stable sub- and quotient objects (in other words, only two Jordan–Hölder factors on the wall) and apply:

**Proposition 2.2.7.** *Suppose $\mathcal{F}$ and $\mathcal{G}$ are $\lambda_{\alpha,\beta,s}$-stable objects in $\mathcal{A}^{\alpha,\beta}(X)$. Then there is a neighborhood $U$ of $(\alpha, \beta)$ such that for all $(\alpha', \beta') \in U$ and all* *nonsplit extensions*

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

*the object* $\mathcal{E}$ *is* $\lambda_{\alpha',\beta',s}$*-stable if and only if* $\lambda_{\alpha',\beta',s}(\mathcal{F}) < \lambda_{\alpha',\beta',s}(\mathcal{G})$.

This result is stated and proved (for arbitrary Bridgeland stability conditions) in Schmidt [Sch15, Lemma 3.11], and credited there also to Bayer–Macrì [BM11, Lemma 5.9].

## 2.3 Wall and chamber structure

The starting point for the entire discussion that follows is a simple minded observation. Namely, let $V \subset \mathbb{P}^3$ be a plane and let $Y$ be the union of a conic in $V$ and a point $P$ also in $V$. Then there is a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{I}_Y \to \mathcal{I}_{P/V}(-2) \to 0 \tag{2.3.1}$$

(read $\mathcal{O}_{\mathbb{P}^3}(-1)$ as the ideal of $V$ and $\mathcal{I}_{P/V}(-2)$ as the relative ideal of $Y \subset V$). If we instead let $Y$ be the union of a conic in $V$ and a point $P$ outside of $V$ then there is a short exact sequence

$$0 \to \mathcal{I}_P(-1) \to \mathcal{I}_Y \to \mathcal{O}_V(-2) \to 0 \tag{2.3.2}$$

(read $\mathcal{I}_P(-1)$ as the ideal of $\{P\} \cup V$ and $\mathcal{O}_V(-2)$ as the relative ideal of a conic in $V$). The claim is that in a certain region of the stability manifold of $\mathbb{P}^3$, there are exactly two walls with respect to the Chern character $\mathrm{ch}(\mathcal{I}_Y) = (1, 0, -2, 2)$, and they are defined precisely by the two pairs of sub and quotient objects appearing in the short exact sequences (2.3.1) and (2.3.2).

Mimicking Schmidt's work for twisted cubics (and their deformations), we argue via tilt stability. Since $\nu_{\alpha,\beta}$-stability only involves Chern classes of codimension at least one, and the above two short exact sequences are indistinguishable in codimension one, they give rise to one and the same wall in the tilt stability parameter space. Making this precise is the content of Section 2.3.1. Moving on to $\lambda_{\alpha,\beta,s}$-stability, we apply Schmidt's method to see that the single $\nu_{\alpha,\beta}$-wall "sprouts" two distinct $\lambda_{\alpha,\beta,s}$-walls corresponding to (2.3.1) and (2.3.2). This is carried out in Section 2.3.2.

### 2.3.1 $\nu$-stability wall

Throughout we specialize to $X = \mathbb{P}^3$ with $H$ a (hyper)plane. Let $v = (1, 0, -2, 2)$ be the Chern character of ideal sheaves $\mathcal{I}_Y$ of subschemes $Y \in \mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$.

For $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$ we have the tilted abelian category $\mathrm{Coh}^{\beta}(\mathbb{P}^3)$ and the slope function $\nu_{\alpha,\beta}$. We concentrate on the region $\beta < 0$, in which any ideal $\mathcal{I}_Y$ of a subscheme $Y \subset \mathbb{P}^3$ of dimension $\leq 1$ satisfies

$$\mu_{\beta}(\mathcal{I}_Y) = \frac{c_1(\mathcal{I}_Y)}{\mathrm{rk}(\mathcal{I}_Y)} - \beta = -\beta > 0.$$

As $\mathcal{I}_Y$ is $\mu$-stable also $\mu_{\beta}(\mathcal{G}) > 0$ for every quotient $\mathcal{I}_Y \twoheadrightarrow \mathcal{G}$ and so $\mathcal{I}_Y \in \mathcal{T}_{\beta}$. In particular $\mathcal{I}_Y \in \mathrm{Coh}^{\beta}(\mathbb{P}^3)$.

We begin by establishing that there is exactly one tilt-stability wall in the region $\beta < 0$. The result as well as the argument is analogous to the analysis for twisted cubics by Schmidt [Sch15, Theorem 5.3], except that twisted cubics come with a second wall that destabilizes all objects — for our skew lines there is no such final wall.

**Proposition 2.3.1.** *There is exactly one tilt-stability wall for objects with Chern character $v = (1, 0, -2, 2)$ in the region $\beta < 0$: it is the semicircle*

$$W\colon \quad \alpha^2 + (\beta + \tfrac{5}{2})^2 = (\tfrac{3}{2})^2.$$

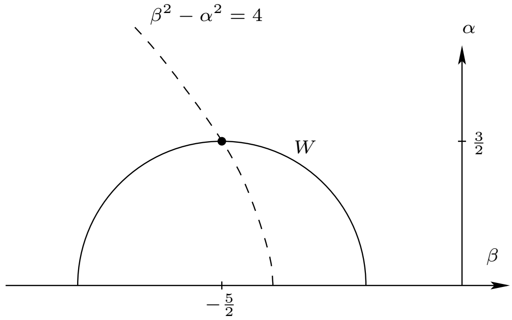


Figure 2.1: The semicircular $\nu$-wall

*The wall is defined by exactly the unordered pairs of the following two types:*

*(1)* $\{\mathcal{I}_P(-1), \mathcal{O}_V(-2)\}$*, where* $V \subset \mathbb{P}^3$ *is a plane and* $P \in V$*, and*

*(2)* $\{\mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{I}_{P/V}(-2)\}$*, where* $V \subset \mathbb{P}^3$ *is a plane and* $P \notin V$*.*

*Moreover, the four sheaves figuring in the above unordered pairs are* $\nu_{\alpha,\beta}$*-stable objects in* $\mathrm{Coh}^\beta(\mathbb{P}^3)$ *for all* $(\alpha, \beta)$ *on* $W$*.*

The wall $W$ and the hyperbola $\nu_{\alpha,\beta}(v) = 0$, intersecting at $(\alpha, \beta) = (3/2, -5/2)$, are shown in Figure 2.1. Note that we visualize the $\alpha$-axis as the vertical one.

We first prove the final claim in the proposition. Here is a slightly more general statement:

**Lemma 2.3.2.**

1. *Let* $Z \subset \mathbb{P}^3$ *be a finite, possibly empty subscheme. Then the ideal sheaf* $\mathcal{I}_Z(-1)$ *is a* $\nu_{\alpha,\beta}$*-stable object in* $\mathrm{Coh}^\beta(\mathbb{P}^3)$ *for all* $\alpha > 0$ *and* $\beta < -1$*.*
2. *Let* $V \subset \mathbb{P}^3$ *be a plane and* $Z \subset V$ *be a finite, possibly empty subscheme. Then the relative ideal sheaf* $\mathcal{I}_{Z/V}(-2)$ *is a* $\nu_{\alpha,\beta}$*-stable object in* $\mathrm{Coh}^\beta(\mathbb{P}^3)$ *for all* $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$ *such that*

$$\alpha^2 + (\beta + \tfrac{5}{2})^2 > (\tfrac{1}{2})^2.$$

**Remark 2.3.3.** The condition on $(\alpha, \beta)$ in part (2) is necessary because of a wall for $\mathcal{I}_{Z/V}(-2)$. For simplicity let $Z$ be empty. There is a short exact sequence of coherent sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_V(-2) \to 0$$

which yields a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_V(-2) \to \mathcal{O}_{\mathbb{P}^3}(-3)[1] \to 0$$

in $\mathrm{Coh}^\beta(\mathbb{P}^3)$ when $-3 < \beta < -2$. The condition $\nu_{\alpha,\beta}(\mathcal{O}_{\mathbb{P}^3}(-2)) < \nu_{\alpha,\beta}(\mathcal{O}_V(-2))$ is exactly the inequality in (2).

*Proof of Lemma 2.3.2.* The sheaf $\mathcal{I}_Z(-1)$ is $\mu$-stable and satisfies $\mu_\beta(\mathcal{I}_Z(-1)) = -1 - \beta$. For all $\beta < -1$ it is thus an object in $\mathcal{T}_\beta$ and so also in $\mathrm{Coh}^\beta(\mathbb{P}^3)$. Since $\mathcal{I}_{Z/V}(-2)$ is a torsion sheaf it too belongs to $\mathcal{T}_\beta$ and so to $\mathrm{Coh}^\beta(\mathbb{P}^3)$, for all $\beta$.

We reduce to the situation $Z = \emptyset$. First consider $\mathcal{I}_Z(-1)$ and assume $\beta < -1$. Note that $\mathcal{I}_Z(-1)$ is a subobject of $\mathcal{O}_{\mathbb{P}^3}(-1)$ also in $\mathrm{Coh}^\beta(\mathbb{P}^3)$ since the torsion sheaf $\mathcal{O}_Z$ belongs to that category and hence

$$0 \to \mathcal{I}_Z(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_Z \to 0$$

is a short exact sequence in $\mathrm{Coh}^\beta(\mathbb{P}^3)$. Suppose $\mathcal{O}_{\mathbb{P}^3}(-1)$ is $\nu_{\alpha,\beta}$-stable. Let $\mathcal{F} \subset \mathcal{I}_Z(-1)$ be a proper nonzero subobject in $\mathrm{Coh}^\beta(\mathbb{P}^3)$ with quotient $\mathcal{G}$. View $\mathcal{F}$ also as a subobject of $\mathcal{O}_{\mathbb{P}^3}(-1)$, with quotient $\mathcal{G}'$. Then $\nu_{\alpha,\beta}$ cannot distinguish between $\mathcal{G}$ and $\mathcal{G}'$. Thus if $\mathcal{O}_{\mathbb{P}^3}(-1)$ is $\nu_{\alpha,\beta}$-stable then

$$\nu_{\alpha,\beta}(\mathcal{F}) < \nu_{\alpha,\beta}(\mathcal{G}') = \nu_{\alpha,\beta}(\mathcal{G})$$

and so $\mathcal{I}_Z(-1)$ is $\nu_{\alpha,\beta}$-stable as well. The reduction from $\mathcal{I}_{Z/V}(-2)$ to $\mathcal{O}_V(-2)$

is completely analogous.

$\nu_{\alpha,\beta}$-stability of the line bundle $\mathcal{O}_{\mathbb{P}^3}(-1)$ is a consequence of $\overline{\Delta}_H(\mathcal{O}_{\mathbb{P}^3}(-1)) = 0$, by [BMT14, Proposition 7.4.1].

The main task is to establish $\nu_{\alpha,\beta}$-stability of $\mathcal{O}_V(-2)$ in the region defined in part (2). By point (3) of [Sch15, Theorem 3.3], the ray $\beta = -\frac{5}{2}$ intersects all potential semicircular $\nu$-walls for $\mathrm{ch}(\mathcal{O}_V(-2))$ at their top point, meaning they must be centered at $(0, -\frac{5}{2})$. All such semicircles of radius bigger than $\frac{1}{2}$ will intersect the ray $\beta = -2$ (as well as $\beta = -3$). Thus it suffices to prove that $\mathcal{O}_V(-2)$ is $\nu_{\alpha,\beta}$-stable for all $\alpha > 0$ and all *integers* $\beta$.

For such $(\alpha, \beta)$, suppose

$$0 \to \mathcal{F} \to \mathcal{O}_V(-2) \to \mathcal{G} \to 0$$

is a short exact sequence in $\mathrm{Coh}^\beta(\mathbb{P}^3)$ with $\mathcal{F} \neq 0$. We claim that $\mathrm{ch}_1^\beta(\mathcal{G}) = 0$. This yields the result, since then $\nu_{\alpha,\beta}(\mathcal{G}) = \infty$ and so $\mathcal{O}_V(-2)$ is $\nu_{\alpha,\beta}$-stable.

Let $r_\mathcal{F} = H^3 \mathrm{ch}_0(\mathcal{F})$ and $c_\mathcal{F} = H^2 \mathrm{ch}_1(\mathcal{F})$, i.e. the rank and first Chern class considered as integers. Also let $r_\mathcal{G} = H^3 \mathrm{ch}_0(\mathcal{G})$ and $c_\mathcal{G} = H^2 \mathrm{ch}_1(\mathcal{G})$. By the short exact sequence we have

$$r_\mathcal{F} + r_\mathcal{G} = 0 \quad \text{and} \quad c_\mathcal{F} + c_\mathcal{G} = 1.$$

The induced long exact cohomology sequence of sheaves shows that $\mathcal{H}^{-1}(\mathcal{F}) = 0$, so from the short exact sequence

$$0 \longrightarrow \mathcal{H}^{-1}(\mathcal{F})[1] \longrightarrow \mathcal{F} \longrightarrow \mathcal{H}^0(\mathcal{F}) \longrightarrow 0$$

we see that $\mathcal{F} \cong \mathcal{H}^0(\mathcal{F})$ is a coherent sheaf in $\mathcal{T}_\beta$. The remaining long exact sequence is

$$0 \to \mathcal{H}^{-1}(\mathcal{G}) \to \mathcal{F} \to \mathcal{O}_V(-2) \to \mathcal{H}^0(\mathcal{G}) \to 0$$

The leftmost (nonzero) map cannot be an isomorphism, since $\mathcal{H}^{-1}(\mathcal{G})$ is in $\mathcal{T}_\beta^\perp$ and $\mathcal{F}$ is in $\mathcal{T}_\beta$ and is nonzero by assumption. Therefore the map in the middle is nonzero and so the rightmost sheaf $\mathcal{H}^0(\mathcal{G})$ is a proper quotient of $\mathcal{O}_V(-2)$ and so is a torsion sheaf supported in dimension $\leq 1$. Thus only $\mathcal{H}^{-1}(\mathcal{G})$ contributes to $r_\mathcal{G}$ and $c_\mathcal{G}$.

Suppose $r_\mathcal{F} \neq 0$. As $\mathcal{F} \in \mathcal{T}_\beta$ and $\mathcal{H}^{-1}(\mathcal{G}) \in \mathcal{T}_\beta^\perp$ we have

$$\begin{cases} \mu_\beta(\mathcal{F}) > 0 \\ \mu_\beta(\mathcal{H}^{-1}(\mathcal{G})) \leq 0 \end{cases} \Rightarrow \begin{cases} \dfrac{c_\mathcal{F}}{r_\mathcal{F}} - \beta > 0 \\ \dfrac{c_\mathcal{G}}{r_\mathcal{G}} - \beta \leq 0 \end{cases} \Rightarrow \begin{cases} \dfrac{c_\mathcal{G} - 1}{r_\mathcal{G}} - \beta > 0 \\ \dfrac{c_\mathcal{G}}{r_\mathcal{G}} - \beta \leq 0 \end{cases}$$

and since $r_\mathcal{G} = -r_\mathcal{F}$ is negative we get $0 \leq c_\mathcal{G} - \beta r_\mathcal{G} < 1$ and since these are integers we must have $c_\mathcal{G} - \beta r_\mathcal{G} = 0$. Thus

$$\mathrm{ch}_1^\beta(\mathcal{G}) = (c_\mathcal{G} - \beta r_\mathcal{G})H = 0$$

as claimed.

If on the other hand $r_\mathcal{F} = 0$ then also $\mathcal{H}^{-1}(\mathcal{G})$ has rank zero and hence must be zero as there are no torsion sheaves in $\mathcal{T}_\beta^\perp$. Thus also $\mathcal{G} = \mathcal{H}^0(\mathcal{G})$ is a sheaf, with vanishing rank and first Chern class. Again $\mathrm{ch}_1^\beta(\mathcal{G}) = 0$ as claimed. This completes the proof. □

By explicit computation (see [Sch15, Theorem 3.3]), all numerical tilt walls with respect to $v = (1, 0, -2, 2)$ in the region $\beta < 0$ are nested semicircles. More precisely, each is centered on the axis $\alpha = 0$ and has top point on the curve $\nu_{\alpha,\beta}(v) = 0$, that is the hyperbola

$$\beta^2 - \alpha^2 = 4. \tag{2.3.3}$$

In particular every tilt wall must intersect the ray $\beta = -2$.

We establish in the following lemma that there is at most one tilt stability wall intersecting the ray $\beta = -2$ for Chern character $v$ and $\beta < 0$. We also give the possible Chern characters of sub- and quotient objects that define it. This lemma is tightly analogous to Schmidt [Sch15, Lemma 5.5]. We use an asterisk $*$ to denote an unspecified numerical value.

**Lemma 2.3.4.** *Let $\beta_0 = -2$ and let $\alpha > 0$ be arbitrary. Suppose there is a short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

*of $\nu_{\alpha,\beta_0}$-semistable objects in $\mathrm{Coh}^{\beta_0}(\mathbb{P}^3)$ with $\mathrm{ch}(\mathcal{E}) = (1, 0, -2, *)$ and $\nu_{\alpha,\beta_0}(\mathcal{F}) = \nu_{\alpha,\beta_0}(\mathcal{G})$. Then*

$$\mathrm{ch}^{\beta_0}(\mathcal{F}) = (1, 1, \tfrac{1}{2}, *) \quad \textit{and} \quad \mathrm{ch}^{\beta_0}(\mathcal{G}) = (0, 1, -\tfrac{1}{2}, *)$$

*or the other way around.*

*Proof.* Keep $\beta_0 = -2$ throughout. We compute $\mathrm{ch}^{\beta_0}(\mathcal{E}) = (1, 2, 0, *)$. Let $\mathrm{ch}^{\beta_0}(\mathcal{F}) = (r, c, d, *)$ with $r, c \in \mathbb{Z}$ and $d \in \frac{1}{2}\mathbb{Z}$. Then $\mathrm{ch}^{\beta_0}(\mathcal{G}) = (1 - r, 2 - c, -d, *)$.

Since the (very weak) stability function $Z^{\mathrm{tilt}}$ sends effective classes to the upper half plane $\mathbb{H} \cup \{0\}$ and $Z^{\mathrm{tilt}}(\mathcal{E}) = Z^{\mathrm{tilt}}(\mathcal{F}) + Z^{\mathrm{tilt}}(\mathcal{G})$ we have

$$0 \leq \Im Z^{\mathrm{tilt}}(\mathcal{F}) \leq \Im Z^{\mathrm{tilt}}(\mathcal{E}).$$

Since $\Im Z^{\mathrm{tilt}} = H\, \mathrm{ch}_1^{\beta_0}$ this gives $0 \leq c \leq 2$.

If $c = 0$ then $\nu_{\alpha,\beta_0}(\mathcal{F}) = \infty$ and $\nu_{\alpha,\beta_0}(\mathcal{G}) < \infty$, which is a contradiction. Similarly if $c = 2$ then $\nu_{\alpha,\beta_0}(\mathcal{F}) < \infty$ and $\nu_{\alpha,\beta_0}(\mathcal{G}) = \infty$, again a contradiction. Therefore $c = 1$.

With $c = 1$ we compute

$$\nu_{\alpha,\beta_0}(\mathcal{F}) = d - \tfrac{1}{2}\alpha^2 r \quad \text{and} \quad \nu_{\alpha,\beta_0}(\mathcal{G}) = -d - \tfrac{1}{2}\alpha^2(1-r)$$

and so the condition $\nu_{\alpha,\beta_0}(\mathcal{F}) = \nu_{\alpha,\beta_0}(\mathcal{G})$ says

$$\alpha^2 = \frac{4d}{2r-1} \tag{2.3.4}$$

so this expression must be strictly positive.

Suppose $r \geq 1$ and apply the Bogomolov inequality (Proposition 2.2.3) to $\mathcal{F}$:

$$0 \leq \overline{\Delta}_H(\mathcal{F}) = 1 - 2rd \quad \Longrightarrow \quad d \leq \frac{1}{2r}$$

When $r \geq 1$ this gives $d \leq \frac{1}{2}$. On the other hand the positivity of (2.3.4) gives $d > 0$ and as $d$ is a half integer this leaves only the possibility $d = \frac{1}{2}$ and $r = 1$.

Similarly suppose $r \leq 0$ and apply the Bogomolov inequality to $\mathcal{G}$:

$$0 \leq \overline{\Delta}_H(\mathcal{G}) = 1 + 2(1-r)d \quad \Longrightarrow \quad d \geq -\frac{1}{2(1-r)}$$

When $r \leq 0$ this gives $d \geq -\frac{1}{2}$. On the other hand the positivity of (2.3.4) gives $d < 0$ and as $d$ is a half integer this leaves only the possibility $d = -\frac{1}{2}$ and $r = 0$. □

*Proof of Proposition 2.3.1.* Assume there is a tilt stability wall for $v = (1, 0, -2, 2)$, i.e. there is a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

of $\nu_{\alpha,\beta}$-semistable objects in $\mathrm{Coh}^{\beta}(\mathbb{P}^3)$ with $\mathrm{ch}(\mathcal{E}) = (1, 0, -2, 2)$ and $\nu_{\alpha,\beta}(\mathcal{F}) = \nu_{\alpha,\beta}(\mathcal{G})$. As already pointed out, the same conditions then hold for some $(\alpha, \beta_0)$

with $\beta_0 = -2$. Then by Lemma 2.3.4, up to swapping $\mathcal{F}$ and $\mathcal{G}$, we have

$$\operatorname{ch}^{\beta_0}(\mathcal{F}) = (1, 1, \tfrac{1}{2}, *) \tag{2.3.5}$$

$$\operatorname{ch}^{\beta_0}(\mathcal{G}) = (0, 1, -\tfrac{1}{2}, *). \tag{2.3.6}$$

Given any pair $\mathcal{F}, \mathcal{G}$ of such objects, write out the condition $\nu_{\alpha,\beta}(\mathcal{F}) = \nu_{\alpha,\beta}(\mathcal{G})$ on $(\alpha, \beta)$ to obtain the equation for the wall in question; this yields the semicircle as claimed. Thus we have proved that there is at most one tilt-wall and found its equation.

A further result of Schmidt [Sch15, Lemma 5.4] (which requires $\beta$ to be integral, and so applies for $\beta_0 = -2$) says that the only $\nu_{\alpha,\beta_0}$-semistable objects in $\operatorname{Coh}^{\beta_0}(\mathbb{P}^3)$ with the invariants (2.3.5) and (2.3.6) are

$$\mathcal{F} \cong \mathcal{I}_Z(-1)$$

$$\mathcal{G} \cong \mathcal{I}_{Z'/V}(-2)$$

for a finite subscheme $Z \subset \mathbb{P}^3$, a plane $V \subset \mathbb{P}^3$ and a finite subscheme $Z' \subset V$ (where $Z$ and $Z'$ are allowed to be empty). Let $n$ and $n'$ denote the lengths of $Z$ and $Z'$, respectively. Again for $\beta_0 = -2$ we compute

$$\operatorname{ch}_3^{\beta_0}(\mathcal{F}) = \operatorname{ch}_3^{\beta_0}(\mathcal{I}_Z(-1)) = \tfrac{1}{6} - n$$

$$\operatorname{ch}_3^{\beta_0}(\mathcal{G}) = \operatorname{ch}_3^{\beta_0}(\mathcal{I}_{Z'/V}(-2)) = \tfrac{1}{6} - n'$$

and moreover $\operatorname{ch}_3^{\beta_0}(\mathcal{E}) = -\frac{2}{3}$. Thus from $\operatorname{ch}_3^{\beta_0}(\mathcal{E}) = \operatorname{ch}_3^{\beta_0}(\mathcal{F}) + \operatorname{ch}_3^{\beta_0}(\mathcal{G})$ we find

$$n + n' = 1$$

and so either $Z$ is empty and $Z'$ is a point, or $Z$ is a point and $Z'$ is empty. This proves that only the two listed pairs of semistable objects $\mathcal{F}, \mathcal{G}$ may occur in a short exact sequence defining the wall.

To finish the proof it only remains to show that both pairs of objects listed do in fact realize the wall. By Lemma 2.3.2, the sheaves $\mathcal{I}_Z(-1)$ and $\mathcal{I}_{Z'/V}(-2)$ are in $\mathrm{Coh}^\beta(\mathbb{P}^3)$ and are $\nu_{\alpha,\beta}$-semistable (in fact $\nu_{\alpha,\beta}$-stable) for all $(\alpha, \beta)$ on the semicircle. Also, the ideal $\mathcal{E} = \mathcal{I}_Y$ of any $Y \in \mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ is an object in $\mathrm{Coh}^\beta(\mathbb{P}^3)$ (when $\beta < 0$) and since any ideal is $\mu$-stable it is $\nu_{\alpha,\beta}$-stable for $\alpha \gg 0$ (by Proposition 2.2.4). Hence it is $\nu_{\alpha,\beta}$-stable outside the semicircle and at least $\nu_{\alpha,\beta}$-semistable on the semicircle. Thus, short exact sequences of the types (2.3.1) and (2.3.2) define the wall and we are done. □

### 2.3.2 $\lambda$-stability walls

Next we apply Schmidt's Theorem 2.2.6 to the single $\nu_{\alpha,\beta}$-wall found in Proposition 2.3.1; this yields two $\lambda_{\alpha,\beta,s}$-walls.

We set up notation first: for $(\alpha, \beta, s) \in \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}_{>0}$ we have the doubly tilted category $\mathcal{A}^{\alpha,\beta}(\mathbb{P}^3)$ and the slope function $\lambda_{\alpha,\beta,s}$. Once and for all we fix an arbitrary value $s > 0$ and view the $(\alpha, \beta)$-plane $\mathbb{R}_{>0} \times \mathbb{R}$ as parametrizing both $\nu_{\alpha,\beta}$-stability and $\lambda_{\alpha,\beta,s}$-stability; as before we restrict to $\beta < 0$. Walls and chambers are taken with respect to the Chern character $v = (1, 0, -2, 2)$.

Write $P_v \subset \mathbb{R}_{>0} \times \mathbb{R}$ for the open subset defined by $\nu_{\alpha,\beta}(v) > 0$ and $\beta < 0$; this is the region to the left of the hyperbola (2.3.3) in Figure 2.2. Theorem 2.2.6 addresses walls in $P_v$ close to the boundary hyperbola.

**Proposition 2.3.5.** *There are exactly two $\lambda_{\alpha,\beta,s}$-walls with respect to $v = (1, 0, -2, 2)$ in $P_v$ whose closure intersect the hyperbola (2.3.3). They are defined exactly by the two pairs of objects listed in Proposition 2.3.1.*

This means that the two walls are

$$W_1 = \{(\alpha, \beta) \in P_v \mid \lambda_{\alpha,\beta,s}(\mathcal{O}_{\mathbb{P}^3}(-1)) = \lambda_{\alpha,\beta,s}(\mathcal{I}_{Q/V}(-2))\} \tag{2.3.7}$$

and

$$W_2 = \{(\alpha, \beta) \in P_v \mid \lambda_{\alpha,\beta,s}(\mathcal{I}_P(-1)) = \lambda_{\alpha,\beta,s}(\mathcal{O}_V(-2))\} \tag{2.3.8}$$

and the pair of objects defining each wall (close to $(\alpha_0, \beta_0)$) is unique.

We refrain from writing out the (quartic) equations defining them. They do depend on $s$, but independently of $s$ they both intersect the hyperbola (2.3.3) in $(\alpha_0, \beta_0) = (\frac{3}{2}, -\frac{5}{2})$ and as we will show in the following proof, $W_1$ has negative slope at $(\alpha_0, \beta_0)$ whereas $W_2$ has positive slope there. Thus $W_1$ lies above $W_2$ ($\alpha$ bigger) in the intersection between $P_v$ and a small open neighborhood of $(\alpha_0, \beta_0)$.

*Proof.* We apply Schmidt's Theorem 2.2.6. Firstly, when $\mathrm{ch}(\mathcal{E}) = v$ we have $\mathrm{ch}_1^\beta(\mathcal{E}) = v_1 - \beta v_0 = -\beta > 0$ and $\overline{\Delta}_H(v) = v_1^2 - 2v_0v_2 = 4 > 0$ so the theorem applies. The first part of the Theorem says that any $\lambda$-wall in $P_v$, having a point $(\alpha_0, \beta_0)$ with $\nu_{\alpha_0,\beta_0}(v) = 0$ in its closure, must be defined by one of the two pairs $(\mathcal{F}, \mathcal{G})$ listed in Proposition 2.3.1. This leaves $W_1$ and $W_2$ as the only candidates. Moreover the sub- and quotient objects $\mathcal{F}$ and $\mathcal{G}$ appearing are $\nu_{\alpha_0,\beta_0}$-stable by Lemma 2.3.2. Thus the second part of the theorem says that conversely, $W_1$ and $W_2$ are indeed $\lambda$-walls, provided they contain points $(\alpha, \beta)$ arbitrarily close to $(\alpha_0, \beta_0) = (\frac{3}{2}, -\frac{5}{2})$ such that $\nu_{\alpha,\beta}(\mathcal{F}) > 0$ and $\nu_{\alpha,\beta}(\mathcal{G}) > 0$. It remains to check this last condition. We verify this by comparing slopes at $(\alpha_0, \beta_0)$.

So let $(\mathcal{F}, \mathcal{G})$ be one of the pairs $(\mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{I}_{P/V}(-2))$ or $(\mathcal{I}_P(-1), \mathcal{O}_V(-2))$. The region $P_v$ is bounded by the hyperbola $\nu_{\alpha_0,\beta_0}(v) = 0$ and implicit differentiation readily shows that this has slope $\frac{d\alpha}{d\beta} = -5/3$ at $(\alpha_0, \beta_0)$. Similarly $\nu_{\alpha_0,\beta_0}(\mathcal{F}) = 0$ has slope $-1$ and $\nu_{\alpha_0,\beta_0}(\mathcal{G}) = 0$ is just the line $\beta = -5/2$, and in each case $\nu_{\alpha,\beta} > 0$ is the region to the left of these boundary curves. Thus it suffices to show that our walls have slope $> -1$ at $(\alpha_0, \beta_0)$. Now each wall

$W_i$ is defined by the condition $\lambda_{\alpha,\beta,s}(\mathcal{F}) = \lambda_{\alpha,\beta,s}(\mathcal{G})$, which is equivalent to

$$(\Re Z(\mathcal{F}))(\Im Z(\mathcal{G})) = (\Re Z(\mathcal{G}))(\Im Z(\mathcal{F})) \tag{2.3.9}$$

where $Z = Z_{\alpha,\beta,s}$ is the stability function defined in (2.2.4) from Section 2.2.3. Implicit differentation of this equation at $(\alpha_0, \beta_0)$ gives, after some work, that $W_1$ has slope

$$-\left(\frac{27s}{16}+1\right)^{-1} \in (-1,0) \tag{2.3.10}$$

and $W_2$ has slope

$$\left(\frac{27s}{4}+1\right)^{-1} \in (0,1) \tag{2.3.11}$$

both of which are $> -1$, and we are done. $\square$

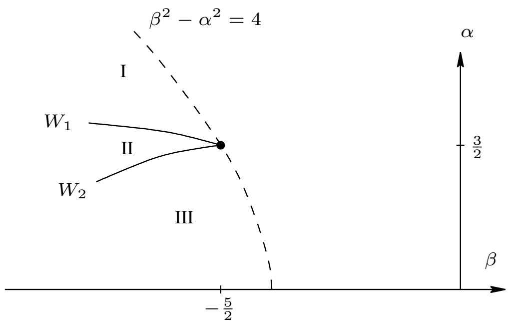


Figure 2.2: Walls and chambers for $\lambda_{\alpha,\beta,s}$ for fixed $s$

We are now in position to prove Theorem 2.1.1. By Proposition 2.3.5 there exists an open (connected) neighborhood $N \subset \mathbb{R}_{>0} \times \mathbb{R}$ around the $\beta < 0$ branch of the hyperbola (2.3.3), such that the only $\lambda$-walls in $N \cap P_v$ are $W_1$ and $W_2$, defined in (2.3.7) and (2.3.8). Moreover it follows from the slopes (2.3.10) and (2.3.11) that, after shrinking $N$ further if necessary, $W_1$ lies above ($\alpha$ bigger) $W_2$ throughout $N \cap P_v$. Thus the two walls separate $N$ into three chambers, which we label I, II and III in order of decreasing $\alpha$.

*Proof of Theorem 2.1.1 (I).* By Proposition 2.3.1 there is a single semicircular wall (in the region $\beta < 0$) for $\nu$-stability. It follows from Theorem 2.2.5 that the class of $\lambda_{\alpha,\beta,s}$-stable objects in $\mathcal{A}^{\alpha,\beta}(\mathbb{P}^3)$ for $(\alpha, \beta)$ in chamber I coincides with the class of $\nu_{\alpha,\beta}$-stable objects in $\mathrm{Coh}^{\beta}(\mathbb{P}^3)$ for $(\alpha, \beta)$ outside the single $\nu$-wall (up to shrinking $N$ even further if necessary).

Moreover, for $\alpha$ sufficiently big, the $\nu$-stable objects in $\mathrm{Coh}^{\beta}(\mathbb{P}^3)$ are exactly the $\mu$-stable coherent sheaves (Proposition 2.2.4). For Chern character $v = (1, 0, -2, 2)$ these are the ideals $\mathcal{I}_Y$ with $Y \in \mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$. □

*Proof of Theorem 2.1.1(II).* Let $\mathcal{E}$ be $\lambda_{\alpha,\beta,s}$-stable for $(\alpha, \beta)$ in chamber II. Since semistability is a closed property, $\mathcal{E}$ is semistable on the wall $W_1$. If $\mathcal{E}$ is stable on the wall, then it is also stable in chamber I hence it is an ideal sheaf in $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ by part (I). Such an ideal remains stable on the wall if and only if it is not an extension of the type (2.3.1), that is if and only if it is the ideal of a nonplanar subscheme. This is case (II)(i) in the Theorem.

If on the other hand $\mathcal{E}$ is stable in chamber II, but strictly semistable on $W_1$, then by Proposition 2.3.5 it is a nonsplit extension of the pair

$$\big(\mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{I}_{P/V}(-2)\big) \tag{2.3.12}$$

and we determine the direction of the extension (which object is the subobject and which is the quotient) as follows: we claim that

$$\lambda_{\alpha,\beta,s}(\mathcal{I}_{P/V}(-2)) < \lambda_{\alpha,\beta,s}(\mathcal{O}_{\mathbb{P}^3}(-1)) \tag{2.3.13}$$

for all $(\alpha, \beta)$ in chamber II sufficiently close to $(\alpha_0, \beta_0)$. Granted this, it follows that for $\mathcal{E}$ to be stable in chamber II it must be a nonsplit extension as in case (II)(ii) in the Theorem. Conversely it follows from Proposition 2.2.7 that every such nonsplit extension is indeed stable in chamber II. To verify (2.3.13) we let

$$\Phi(\alpha, \beta) = \Re Z(\mathcal{F})\Im Z(\mathcal{G}) - \Re Z(\mathcal{G})\Im Z(\mathcal{F}) \tag{2.3.14}$$

with $Z = Z_{\alpha,\beta,s}$, $\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(-1)$ and $\mathcal{G} = \mathcal{I}_{P/V}(-2)$. Thus $W_1$ is defined by $\Phi(\alpha,\beta) = 0$ and (2.3.13) is equivalent to $\Phi(\alpha,\beta) < 0$. It thus suffices to check that the partial derivative of $\Phi$ with respect to $\alpha$ is positive at $(\alpha_0,\beta_0)$. An explicit computation yields in fact

$$\frac{\partial \Phi}{\partial \alpha}(\alpha_0,\beta_0) = 2 + \frac{27s}{8} > 0.$$

It remains only to show uniqueness of the nonsplit extensions $\mathcal{F}_{P,V}$, that is

$$\dim \operatorname{Ext}^1_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{I}_{P/V}(-2)) = 1.$$

But this space is $H^1(\mathcal{I}_{P/V}(-1))$, which is isomorphic to $H^0(k(P)) = k$ via the short exact sequence

$$0 \to \mathcal{I}_{P/V}(-1) \to \mathcal{O}_V(-1) \to k(P) \to 0.$$

$\square$

*Proof of Theorem 2.1.1(*III*).* Let $\mathcal{E}$ be $\lambda_{\alpha,\beta,s}$-stable for $(\alpha,\beta)$ in chamber III. Since semistability is a closed property, $\mathcal{E}$ is semistable on the wall $W_2$. If $\mathcal{E}$ is stable on $W_2$, then it is stable in chamber II. This means two things: first, by part (II) of the Theorem $\mathcal{E}$ is either an ideal sheaf of a nonplanar subscheme or a nonsplit extension $\mathcal{F}_{P,V}$ as in case (II)(ii). Second, to remain stable on $W_2$, the object $\mathcal{E}$ cannot be in a short exact sequence of the type (2.3.2) ruling out ideal sheaves of plane conics union a point. Also the sheaves $\mathcal{F}_{P,V}$ sit in short exact sequences of this type, as we show in Lemma 2.3.6 below (the vertical short exact sequence in the middle), and so are ruled out as well. Hence $\mathcal{E}$ is an ideal sheaf of a disjoint pair of lines as claimed in (III)(i).

If on the other hand $\mathcal{E}$ is strictly semistable on $W_2$, then by Proposition 2.3.5

$\mathcal{E}$ is a nonsplit extension (in either direction) of the pair

$$\big(\mathcal{I}_P(-1), \mathcal{O}_V(-2)\big).$$

Now we claim that

$$\lambda_{\alpha,\beta,s}(\mathcal{O}_V(-2) < \lambda_{\alpha,\beta,s}(\mathcal{I}_P(-1))$$

for all $(\alpha, \beta)$ in chamber III sufficiently close to $(\alpha_0, \beta_0)$. We prove this as in part II above, by partial differentiation of $\Phi$ defined in (2.3.14), this time with $\mathcal{F} = \mathcal{I}_P(-1)$ and $\mathcal{G} = \mathcal{O}_V(-2)$. We find

$$\frac{\partial \Phi}{\partial \alpha}(\alpha_0, \beta_0) = \frac{1}{2} + \frac{27s}{8} > 0.$$

As before we conclude that $\mathcal{E}$ is a nonsplit extension as in (III)(ii) and by Proposition 2.2.7 all such extensions are stable.

It remains to verify uniqueness of the extensions $\mathcal{G}_{P,V}$, i.e.

$$\dim \operatorname{Ext}^1_{\mathbb{P}^3}(\mathcal{I}_P(-1), \mathcal{O}_V(-2)) = 1$$

when $P \in V$. For this first apply $\operatorname{Hom}(-, \mathcal{O}_V(-1))$ to the short exact sequence

$$0 \to \mathcal{I}_P \to \mathcal{O}_{\mathbb{P}^3} \to k(P) \to 0$$

to obtain a long exact sequence which together with the vanishing of $H^1(\mathcal{O}_V(-1))$ and $H^2(\mathcal{O}_V(-1))$ gives an isomorphism

$$\operatorname{Ext}^1_{\mathbb{P}^3}(\mathcal{I}_P, \mathcal{O}_V(-1)) \cong \operatorname{Ext}^2(k(P), \mathcal{O}_V(-1))$$

and ignoring twists, as these are not seen by $k(P)$, the right hand side is Serre dual to $\operatorname{Ext}^1(\mathcal{O}_V, k(P))$. This is one dimensional as is seen by applying

$\mathrm{Hom}(-, k(P))$ to the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_V \to 0.$$

$\square$

### 2.3.3 The special sheaves

Let $\mathcal{F} = \mathcal{F}_{P,V}$ and $\mathcal{G} = \mathcal{G}_{P,V}$ denote sheaves given by nonsplit extensions of the form (2.1.1) and (2.1.2), respectively. The definition through (unique) nonsplit extensions is indirect and it is useful to have alternative constructions available. We give such constructions here and compute the spaces of first order infinitesimal deformations.

**Lemma 2.3.6.** *There is a commutative diagram with exact rows and columns as follows:*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{I}_P(-1) \arrow[r, equal] \arrow[d] & \mathcal{I}_P(-1) \arrow[d] & \\
0 \arrow[r] & \mathcal{I}_{P/V}(-2) \arrow[r] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_{P/V}(-2) \arrow[r] & \mathcal{O}_V(-2) \arrow[r] \arrow[d] & k(P) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

*Proof.* Up to identifying the skyscraper sheaf $k(P)$ with any of its twists, there are canonical short exact sequences as in the bottom row and the rightmost column. The diagram can then be completed by letting $\mathcal{F}$ be the fiber product as laid out by the square in the bottom right corner. It remains only to verify that the middle row is nonsplit. But if it were split the middle column twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$ would be a short exact sequence of the form

$$0 \to \mathcal{I}_P \to \mathcal{I}_{P/V}(-1) \oplus \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_V(-1) \to 0.$$

Taking global sections this yields a contradictory left exact sequence in which all terms vanish except $H^0(\mathcal{O}_{\mathbb{P}^3}) = k$. □

**Proposition 2.3.7.** *We have* $\dim \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) = 11$.

*Proof.* We will actually only prove that the dimension is at most 11. The opposite inequality may be shown by similar techniques, although it follows from viewing $\operatorname{Ext}^1(\mathcal{F}, \mathcal{F})$ as a Zariski tangent space to the 11-dimensional moduli space $\mathcal{M}^{\mathrm{II}}$ studied in the next section.

Apply $\operatorname{Hom}(-, \mathcal{F})$ to the middle row in the diagram in Lemma 2.3.6. This yields a long exact sequence

$$\cdots \to H^1(\mathcal{F}(1)) \to \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) \to \operatorname{Ext}^1(\mathcal{I}_{P/V}(-2), \mathcal{F}) \to H^2(\mathcal{F}(1)) \to \cdots$$

and from the middle column of the diagram we compute $H^1(\mathcal{F}(1)) = H^2(\mathcal{F}(1)) = 0$. Thus we proceed to show that $\dim \operatorname{Ext}^1(\mathcal{I}_{P/V}(-2), \mathcal{F}) \leq 11$.

Apply $\operatorname{Hom}(\mathcal{I}_{P/V}(-2), -)$ to the middle row in the diagram. This yields a long exact sequence:

$$\cdots \to \operatorname{Ext}^1(\mathcal{I}_{P/V}, \mathcal{I}_{P/V}) \to \operatorname{Ext}^1(\mathcal{I}_{P/V}(-2), \mathcal{F}) \to \operatorname{Ext}^1(\mathcal{I}_{P/V}(-1), \mathcal{O}_{\mathbb{P}^3}) \to \cdots$$

The space on the right is Serre dual to $H^2(\mathcal{I}_{P/V}(-5)) \cong H^2(\mathcal{O}_V(-5))$, which again on $V$ is Serre dual to $H^0(\mathcal{O}_V(2))$. This has dimension 6. At least heuristically, the space on the left should have dimension 5, as it may be viewed as a tangent space to the incidence variety $I \subset \mathbb{P}^3 \times \check{\mathbb{P}}^3$ (defined in (2.1.3)) seen as a moduli space for the sheaves $\mathcal{I}_{P/V}$. More directly we may apply $\operatorname{Hom}(\mathcal{I}_{P/V}, -)$ to the Koszul complex on $V$

$$0 \to \mathcal{O}_V(-2) \to \mathcal{O}_V(-1)^{\oplus 2} \to \mathcal{I}_{P/V} \to 0$$

to obtain a long exact sequence

$$\cdots \to \underbrace{\operatorname{Ext}^1(\mathcal{I}_{P/V}, \mathcal{O}_V(-1))^{\oplus 2}}_{2\cdot 2} \to \operatorname{Ext}^1(\mathcal{I}_{P/V}, \mathcal{I}_{P/V}) \to \underbrace{\operatorname{Ext}^2(\mathcal{I}_{P/V}, \mathcal{O}_V(-2))}_{1} \to \cdots$$

where the indicated dimensions may be computed by applying $\operatorname{Hom}(\mathcal{I}_{P/V}(d), -)$ (for $d = 1, 2$) to the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_V \to 0.$$

We skip further details. It follows then that $\dim \operatorname{Ext}^1(\mathcal{I}_{P/V}, \mathcal{I}_{P/V}) \le 5$ and so $\dim \operatorname{Ext}^1(\mathcal{I}_{P/V}(-2), \mathcal{F})$ is at most $5 + 6 = 11$. $\square$

**Lemma 2.3.8.** *There is a commutative diagram with exact rows and columns as follows:*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}_{\mathbb{P}^3}(-2) \arrow[r, equal] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-2) \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_V(-2) \arrow[r] \arrow[d, equal] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{I}_P(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_V(-2) \arrow[r] & \Omega^1_V \arrow[r] \arrow[d] & \mathcal{I}_{P/V}(-1) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

*Proof.* From the Euler sequence

$$0 \to \Omega^1_V \to \mathcal{O}_V(-1)^{\oplus 3} \to \mathcal{O}_V \to 0 \tag{2.3.15}$$

on $V \cong \mathbb{P}^2$ it follows that $\Omega^1_V(2)$ has a unique section (up to scale) vanishing at $p \in V$. This leads to the short exact sequence in the bottom row. Moreover there is a canonical short exact sequence as in the rightmost column. The rest of the diagram can then be formed by taking $\mathcal{G}$ to be the fiber product as laid out by the bottom right square. It just remains to verify that the middle row is indeed nonsplit. But if it were split the middle column would be a short exact sequence of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_V(-2) \oplus \mathcal{I}_P(-1) \to \Omega^1_V \to 0.$$

This sequence implies that $H^1(\Omega^1_V(-1))$ is isomorphic to $H^1(\mathcal{I}_P(-2))$, which is one dimensional. But the Euler sequence shows that in fact $H^1(\Omega^1_V(-1)) = 0$. □

**Proposition 2.3.9.** *We have* $\dim \operatorname{Ext}^1(\mathcal{G}, \mathcal{G}) = 8$.

*Proof.* We will be using the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{G} \to \Omega^1_V \to 0 \tag{2.3.16}$$

which sits as the middle column in Lemma 2.3.8. As preparation we observe that all (dimensions of) $H^i(\Omega^1_V(d))$ may be computed from the Euler sequence, and this enables us to compute several $H^i(\mathcal{G}(d))$ from (2.3.16). We use these results freely below without writing out further details.

Apply $\operatorname{Hom}(-, \mathcal{G})$ to (2.3.16) to produce a long exact sequence

$$\begin{aligned} 0 \to \underbrace{\operatorname{Hom}(\Omega^1_V, \mathcal{G})}_{0} \to \underbrace{\operatorname{Hom}(\mathcal{G}, \mathcal{G})}_{1} \to \underbrace{H^0(\mathcal{G}(2))}_{4} \\ \to \operatorname{Ext}^1(\Omega^1_V, \mathcal{G}) \to \operatorname{Ext}^1(\mathcal{G}, \mathcal{G}) \to \underbrace{H^1(\mathcal{G}(2))}_{0} \to \cdots \end{aligned}$$

in which we have indicated some of the dimensions: $H^i(\mathcal{G}(2))$ are computed from (2.3.16) as sketched above, and since $\mathcal{G}$ is simple we have $\operatorname{Hom}(\mathcal{G}, \mathcal{G}) = k$. For the same reason (2.3.16) is nonsplit, which implies $\operatorname{Hom}(\Omega^1_V, \mathcal{G}) = 0$. It thus remains to see that the dimension of $\operatorname{Ext}^1(\Omega^1_V, \mathcal{G})$ is 11.

Next apply $\operatorname{Hom}(-, \mathcal{G})$ to the Euler sequence. This gives a long exact sequence

$$\begin{aligned} \cdots \to \underbrace{\operatorname{Hom}(\Omega^1_V, \mathcal{G})}_{0} \to \underbrace{\operatorname{Ext}^1(\mathcal{O}_V, \mathcal{G})}_{1} \to \underbrace{\operatorname{Ext}^1(\mathcal{O}_V, \mathcal{G}(1))^{\oplus 3}}_{4\cdot 3} \\ \to \operatorname{Ext}^1(\Omega^1_V, \mathcal{G}) \to \underbrace{\operatorname{Ext}^2(\mathcal{O}_V, \mathcal{G})}_{0} \to \cdots \end{aligned}$$

where again dimensions have been indicated: the vanishing of $\operatorname{Hom}(\Omega^1_V, \mathcal{G})$ has already been noted, and there remain several spaces of the form $\operatorname{Ext}^i(\mathcal{O}_V, \mathcal{G}(d))$. These may be computed from $H^i(\mathcal{G}(d))$ and the long exact sequence resulting from applying $\operatorname{Hom}(-, \mathcal{G}(d))$ to

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_V \to 0.$$

It follows that $\dim \operatorname{Ext}^1(\Omega^1_V, \mathcal{G}) = 11$ and we are done. $\square$

## 2.4 Moduli spaces and universal families

By the classification of stable objects in chamber II, the moduli space $\mathcal{M}^{\mathrm{II}}$ is at least obtained as a set from $\operatorname{Hilb}^{2m+2}(\mathbb{P}^3) = \mathcal{C} \cup \mathcal{S}$ by just replacing the divisor $E \subset \mathcal{C}$, parametrizing conics union a point inside a plane, with the incidence variety $I$. Similarly, the moduli space $\mathcal{M}^{\mathrm{III}}$ of stable objects in chamber III is obtained from $\mathcal{M}^{\mathrm{II}}$ set-theoretically by removing $\mathcal{M}^{\mathrm{II}} \setminus \mathcal{S}$ and replacing the divisor $F \subset \mathcal{S}$, parametrizing pairs of incident lines with a spatial embedded point at the intersection, with $I$. We shall carry out these replacements as a contraction, i.e. a blow-down, and prove that this indeed yields $\mathcal{M}^{\mathrm{II}}$ and $\mathcal{M}^{\mathrm{III}}$, essentially by writing down a universal family for each case.

### 2.4.1 The contraction $\mathcal{C} \to \mathcal{C}'$

Recall that $\mathcal{C}$ is isomorphic to the blow-up of $\mathbb{P}^3 \times \mathrm{Hilb}^{2m+1}(\mathbb{P}^3)$ along the universal curve $\mathcal{Z}$, where $\mathrm{Hilb}^{2m+1}(\mathbb{P}^3)$ is the Hilbert scheme of plane conics in $\mathbb{P}^3$ ([Lee00]). The exceptional divisor $E'$ is comprised of plane conics with an embedded point.

It is helpful to keep an eye at the following diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "b"] & \mathbb{P}^3 \times \mathrm{Hilb}^{2m+1}(\mathbb{P}^3) \arrow[r, "pr_2"] \arrow[d, "pr_1"] & \mathrm{Hilb}^{2m+1}(\mathbb{P}^3) \arrow[d, "\pi"] \\ & \mathbb{P}^3 & \check{\mathbb{P}}^3 \end{tikzcd} \tag{2.4.1}$$

where $\pi$ sends a conic $C \in \mathrm{Hilb}^{2m+1}(\mathbb{P}^3)$ to the plane $V \in \check{\mathbb{P}}^3$ it spans and $b$ is the blowup along the universal family $\mathcal{Z}$ of conics.

**Remark 2.4.1.** It will sometimes be useful to resort to explicit computation in local coordinates. For this let $U \subset \check{\mathbb{P}}^3$ be the affine open subset of planes $V \subset \mathbb{P}^3$ with equation of the form

$$x_3 = c_0x_0 + c_1x_1 + c_2x_2. \tag{2.4.2}$$

Furthermore the $\mathbb{P}^5$ of symmetric $3 \times 3$ matrices $(s_{ij})$ parametrizes plane conics

$$\sum_{0 \leq i,j \leq 2} s_{ij}x_ix_j = 0 \tag{2.4.3}$$

so that $\mathrm{Hilb}^{2m+1}(\mathbb{P}^3)\big|_U \cong \mathbb{P}^5 \times U$ with universal family defined by the two equations (2.4.2) and (2.4.3). This is also the center for the blowup $b$, and we note that it is nonsingular.

**Lemma 2.4.2.** *$\pi$ is a Zariski locally trivial $\mathbb{P}^5$-bundle. More precisely, let*

*$I \subset \mathbb{P}^3 \times \check{\mathbb{P}}^3$ be the incidence variety and let*

$$\mathcal{E} = pr_{2*}(pr_1^*\mathcal{O}_{\mathbb{P}^3}(2)|_I).$$

*Then $\mathcal{E}$ is locally free of rank 6 and $\mathrm{Hilb}^{2m+1}(\mathbb{P}^3) \cong \mathbb{P}(\mathcal{E}^\vee)$ over $\check{\mathbb{P}}^3$.*

Here $\mathbb{P}(\mathcal{E}^\vee)$ denotes the projective bundle parametrizing lines in the fibers of $\mathcal{E}$. Starting with the observation that the fiber of $\mathcal{E}$ over $V \in \check{\mathbb{P}}^3$ is $H^0(V, \mathcal{O}_V(2))$ (note that $H^1(V, \mathcal{O}_V(2)) = 0$, so base change in cohomology applies) the Lemma is straight forward and we refrain from writing out details.

Now let $E \subset \mathcal{C}$ be the locus of planar elements of $\mathcal{C}$. The condition on a disjoint union $Y = C \cup \{P\}$ to be in $E$ is just that $P$ is in the plane $V$ spanned by $C$. For a conic with an embedded point the condition $Y \subset V$ also singles out the scheme structure at the embedded point (refer to Figure 2.3 for simple illustrations of the types of elements in $E$ and $E'$). View $E$ as a variety over the incidence variety $I \subset \mathbb{P}^3 \times \check{\mathbb{P}}^3$ via the morphism $(\mathrm{id}_{\mathbb{P}^3} \times \pi) \circ b$. In the following proposition we show that $E$ is a $\mathbb{P}^5$-bundle over $I$, thus $E$ is a *divisor* in $\mathcal{C}$: this is true because $\dim I = 5$ and a fiber over $(P, V) \in I$ is of the form $\mathbb{P}(\mathrm{Ext}^1(\mathcal{I}_{P/V}(-2), \mathcal{O}_{\mathbb{P}^3}(-1)))$ with the dimension of the latter Ext space (computed in the proof of Proposition 2.3.7) being six.

**Proposition 2.4.3.** *$E$ is a Zariski locally trivial $\mathbb{P}^5$-bundle over the incidence variety $I \subset \mathbb{P}^3 \times \check{\mathbb{P}}^3$. The restriction $\mathcal{O}_{\mathcal{C}}(E)|_{\mathbb{P}^5}$ to a fiber is isomorphic to $\mathcal{O}_{\mathbb{P}^5}(-1)$.*

Before giving the proof, we harvest our application:

**Corollary 2.4.4.** *There exist a smooth algebraic space $\mathcal{C}'$, a morphism $\phi\colon \mathcal{C} \to \mathcal{C}'$ and a closed embedding $I \subset \mathcal{C}'$, such that $\phi$ restricts to an isomorphism from $\mathcal{C} \setminus E$ to $\mathcal{C}' \setminus I$ and to the given projective bundle structure $E \to I$. Moreover $\phi$ is the blowup of $\mathcal{C}'$ along $I$.*

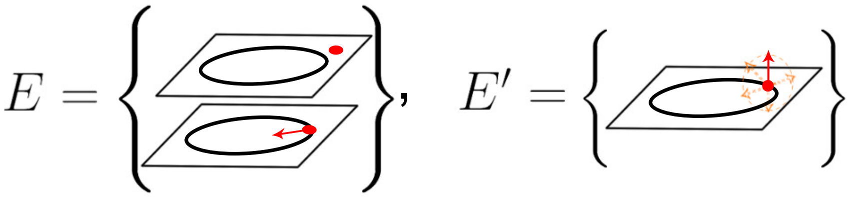


Figure 2.3: A Circle represents a conic contained in a plane that is shown as a parallelogram, and a red dot is a point, possibly embedded in the conic. The arrow is the direction vector at an embedded point. Note that in the left illustration, the arrow is strictly contained in the plane.

It is well known that the condition verified in Proposition 2.4.3 implies the contractibility of $E/I$ in the above sense. In the category of analytic spaces this is the Moishezon [Moi67] or Fujiki–Nakano [Nak70, FN71] criterion. In the category of algebraic spaces the contractibility is due to Artin [Art70, Corollary 6.11], although the statement there lacks the identification with a blowup. Lascu [Las69, Théorème 1] however shows that once the contracted space $\mathcal{C}'$ as well as the image $I \subset \mathcal{C}'$ of the contracted locus are both smooth, it does follow that the contracting morphism is a blowup. Strictly speaking Lascu works in the category of varieties, but our $\mathcal{C}'$ turns out to be a variety anyway:

**Remark 2.4.5.** The algebraic space $\mathcal{C}'$ is in fact a projective variety. We prove this in Section 2.5 using Mori theory. As the arguments there and in the present section are largely independent we separate the statements.

We also point out that the smooth contracted space $\mathcal{C}'$ is unique once it exists: in general, suppose $\phi\colon X \to U$ and $\psi\colon X \to V$ are proper birational morphisms between separated algebraic spaces (say, of finite type over $k$) with $U$ and $V$ normal. Moreover assume that $\phi(x_1) = \phi(x_2)$ if and only if $\psi(x_1) = \psi(x_2)$. Let $\Gamma$ denote the image of $(\phi, \psi)\colon X \to U \times V$. Then each of the projections from $\Gamma$ to $U$ and $V$ is birational and bijective and hence an isomorphism by Zariski's Main Theorem (for this in the language of algebraic spaces we refer to the Stacks Project [dJ22, Tag 05W7].

*Proof of Proposition 2.4.3.* Consider the divisor

$$\overline{E} = \{(P, C) \in \mathbb{P}^3 \times \mathrm{Hilb}^{2m+1}(\mathbb{P}^3) \mid P \text{ in the plane spanned by C}\}.$$

It follows from Lemma 2.4.2 that

$$\mathbb{P}^3 \times \mathrm{Hilb}^{2m+1}(\mathbb{P}^3) \xrightarrow{id_{\mathbb{P}^3} \times \pi} \mathbb{P}^3 \times \check{\mathbb{P}}^3$$

is a $\mathbb{P}^5$-bundle, hence its restriction to $\left(id_{\mathbb{P}^3} \times \pi\right)^{-1}(I) = \overline{E}$ is a $\mathbb{P}^5$-bundle over $I \subset \mathbb{P}^3 \times \check{\mathbb{P}}^3$.

Now, $E$ is the strict transform of $\overline{E}$, i.e. its blow-up along $\mathcal{Z} \subset \overline{E}$. But this is a Cartier divisor, since $\overline{E}$ is smooth, and so $E \cong \overline{E}$. This proves the first claim.

Again using that $\overline{E}$ is smooth, its strict transform $E$ satisfies the linear equivalence

$$E = b^*(\overline{E}) - E'. \tag{2.4.4}$$

The term $b^*(\overline{E}) = b^*(\mathrm{id}_{\mathbb{P}^3} \times \pi)^*(I)$ is a pullback from the base of the $\mathbb{P}^5$-bundle, so its restriction to any fiber is trivial. Thus it suffices to see that $E'$ restricted to a fiber $\mathbb{P}^5$ is a hyperplane. Now the isomorphism $b\colon E \cong \overline{E}$ identifies $E \cap E' \subset E$ with $\mathcal{Z} \subset E$. In the local coordinates from Remark 2.4.1 the divisor $E$ is given by equation (2.4.2) and $\mathcal{Z}$ is given by the additional equation (2.4.3). Here $(s_{ij})$ are the coordinates on the fiber $\mathbb{P}^5$ and clearly (2.4.3) defines a hyperplane in each fiber — it is the linear condition on the space of plane conics given by passage through a given point. □

**Remark 2.4.6.** The locus $\mathcal{C} \cap \mathcal{S}$ consists of pairs of intersecting lines with a *spatial* embedded point at the intersection (and, as degenerate cases, planar double lines with a spatial embedded point). On the other hand $E$ consists only of planar objects, so $E$ is disjoint from $\mathcal{S}$. Thus we may extend the contraction $\phi$ to a morphism between algebraic spaces

$$(\phi \cup \mathrm{id}) \colon \mathcal{C} \cup \mathcal{S} \to \mathcal{C}' \cup \mathcal{S}$$

which is an isomorphism away from $E$ and restricts to the $\mathbb{P}^5$-bundle $E \to I$ as before.

### 2.4.2 Moduli in chamber II

In this section we shall modify the universal family on the Hilbert scheme $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3) = \mathcal{C} \cup \mathcal{S}$ in such a way that we replace its fibers over $E \subset \mathcal{C}$ with the objects $\mathcal{F}_{P,V}$ in Theorem 2.1.1. This family induces a morphism

$$\mathcal{C} \cup \mathcal{S} \to \mathcal{M}^{\mathrm{II}}$$

and we conclude via uniqueness of normal (in this case smooth) contractions that $\mathcal{M}^{\mathrm{II}}$ coincides with $\mathcal{C}' \cup \mathcal{S}$.

Here is the construction: let

$$\mathcal{Y} \subset \mathbb{P}^3 \times \mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$$

be the universal family and let

$$\mathcal{V} \subset \mathbb{P}^3 \times E$$

be the $E$-flat family whose fiber $\mathcal{V}_\xi \subset \mathbb{P}^3$ over a point $\xi$ mapping to $(P, V) \in I$ is the plane $V$. Clearly $\mathcal{V}$ can be written down as a pullback of the universal plane over $\check{\mathbb{P}}^3$. We view $\mathcal{V}$ as a closed subscheme of $\mathbb{P}^3 \times \mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$. Then our modified universal family is the ideal sheaf $\mathcal{I}_{\mathcal{Y} \cup \mathcal{V}}$.

**Remark 2.4.7.** We emphasize the (to us, at least) unusual situation that $\mathcal{I}_{\mathcal{Y} \cup \mathcal{V}}$ is the ideal of a very nonflat subscheme, yet as we show below it is flat as a coherent sheaf. Its fibers over points in $E$ are not ideals at all, but rather the objects $\mathcal{F}_{P,V}$.

**Theorem 2.4.8.** *As above let $\mathcal{Y}$ be the universal family over* $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ *and $\mathcal{V}$ the family of planes in $\mathbb{P}^3$ parametrized by $E$.*

*(i) $\mathcal{I}_{\mathcal{Y}\cup\mathcal{V}}$ is flat as a coherent sheaf over* $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$*. Its fibers $\mathcal{I}_{\mathcal{Y}\cup\mathcal{V}} \otimes k(\xi)$ over $\xi \in$* $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ *are stable objects for stability conditions in chamber II.*

*(ii) The morphism* $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3) \to \mathcal{M}^{\mathrm{II}}$ *determined by $\mathcal{I}_{\mathcal{Y}\cup\mathcal{V}}$ induces an isomorphism $\mathcal{C}' \cup \mathcal{S} \cong \mathcal{M}^{\mathrm{II}}$.*

We begin by showing that the fibers $\mathcal{I}_{\mathcal{Y}\cup\mathcal{V}} \otimes k(\xi)$ over $\xi \in E$ sit in a short exact sequence of the type (2.1.1). The mechanism producing such a short exact sequence is quite general. Note that when $Y = \mathcal{Y}_\xi$ is a conic with a (possibly embedded) point $P$ in a plane $V = \mathcal{V}_\xi$, we have $\mathcal{I}_{Y/V} \cong \mathcal{I}_{P/V}(-2)$ and $\mathcal{I}_V \cong \mathcal{O}_{\mathbb{P}^3}(-1)$.

**Lemma 2.4.9.** *Let $X$ be a projective scheme, $\mathcal{Y} \subset X \times S$ an $S$-flat subscheme, $E \subset S$ a Cartier divisor and $\mathcal{V} \subset E \times S$ an $E$-flat subscheme such that $\mathcal{Y} \cap (E \times S) \subset \mathcal{V}$. Let $\xi \in E$. Then there is a short exact sequence*

$$0 \to \mathcal{I}_{\mathcal{Y}_\xi/\mathcal{V}_\xi} \to \mathcal{I}_{\mathcal{Y}\cup\mathcal{V}} \otimes k(\xi) \to \mathcal{I}_{\mathcal{V}_\xi} \to 0.$$

*In particular if $S$ is integral in a neighborhood of $E$ then $\mathcal{I}_{\mathcal{Y}\cup\mathcal{V}}$ is flat over $S$.*

*Proof.* Observe that the last claim is implied by the first: outside of $E$ the ideal $\mathcal{I}_{\mathcal{Y}\cup\mathcal{V}}$ agrees with $\mathcal{I}_{\mathcal{Y}}$, which is flat. For $\xi \in E$ we have $\mathcal{Y}_\xi \subset \mathcal{V}_\xi$ and so a short exact sequence

$$0 \to \mathcal{I}_{\mathcal{V}_\xi} \to \mathcal{I}_{\mathcal{Y}_\xi} \to \mathcal{I}_{\mathcal{Y}_\xi/\mathcal{V}_\xi} \to 0.$$

The short exact sequence in the statement has the same sub and quotient objects in opposite roles, so the Hilbert polynomial of $\mathcal{I}_{\mathcal{Y}\cup\mathcal{V}} \otimes k(\xi)$ agrees with that of $\mathcal{I}_{\mathcal{Y}_\xi}$. Thus the Hilbert polynomial of the fibers of $\mathcal{I}_{\mathcal{Y}\cup\mathcal{V}}$ is constant; over an integral base this implies flatness.

Begin with the short exact sequence

$$0 \to \mathcal{I}_{\mathcal{Y}\cup\mathcal{V}} \to \mathcal{O}_{X\times S} \to \mathcal{O}_{\mathcal{Y}\cup\mathcal{V}} \to 0$$

and tensor with $\mathcal{O}_{X\times E}$ to obtain the exact sequence

$$0 \to \mathcal{T}or_1^{X\times S}(\mathcal{O}_{\mathcal{Y}\cup\mathcal{V}}, \mathcal{O}_{X\times E}) \to \mathcal{I}_{\mathcal{Y}\cup\mathcal{V}}|_E \to \mathcal{O}_{X\times E} \to \mathcal{O}_{\mathcal{V}} \to 0. \qquad (2.4.5)$$

The kernel of the rightmost map is clearly the ideal $\mathcal{I}_{\mathcal{V}} \subset \mathcal{O}_{X\times E}$. To compute the Tor-sheaf on the left use the short exact sequence

$$0 \to \mathcal{O}_S(-E) \to \mathcal{O}_S \to \mathcal{O}_E \to 0.$$

Pull this back to $X\times S$ and tensor with $\mathcal{O}_{\mathcal{Y}\cup\mathcal{V}}$ to see that $\mathcal{T}or_1^{X\times S}(\mathcal{O}_{\mathcal{Y}\cup\mathcal{V}}, \mathcal{O}_{X\times E})$ is isomorphic to the kernel of the homomorphism

$$\mathcal{O}_{\mathcal{Y}\cup\mathcal{V}}(-pr_2^*E) \to \mathcal{O}_{\mathcal{Y}\cup\mathcal{V}}$$

which locally is multiplication by an equation for $E$. Thus

$$\mathcal{T}or_1^{X\times S}(\mathcal{O}_{\mathcal{Y}\cup\mathcal{V}}, \mathcal{O}_{X\times E}) \cong \mathcal{J}(-pr_2^*E)$$

where $\mathcal{J} \subset \mathcal{O}_{\mathcal{Y}\cup\mathcal{V}}$ is the ideal locally consisting of elements annihilated by an equation for $E$.

We compute $\mathcal{J}$ in an open affine subset $\mathrm{Spec}A \subset X \times S$ in which $\mathcal{Y}$ and $\mathcal{V}$ are given by ideals $I_{\mathcal{Y}}$ and $I_{\mathcal{V}}$ respectively and $f \in A$ is (the pullback of) a local equation for $E$. Thus $\mathcal{J}$ corresponds to $(I_{\mathcal{Y}} \cap I_{\mathcal{V}} : f)/(I_{\mathcal{Y}} \cap I_{\mathcal{V}})$. Now

$f \in I_{\mathcal{V}}$ since $\mathcal{V} \subset X \times E$. This implies that for $g \in A$ the condition $fg \in I_{\mathcal{Y}}$ is equivalent to $fg \in I_{\mathcal{Y}} \cap I_{\mathcal{V}}$ and so $(I_{\mathcal{Y}} \cap I_{\mathcal{V}} : f) = (I_{\mathcal{Y}} : f)$. Moreover the latter equals $I_{\mathcal{Y}}$, since $\mathcal{Y}$ is flat over $S$, so that multiplication by the non-zero-divisor $f$ remains injective after tensor product with $\mathcal{O}_{\mathcal{Y}}$, that is $A/I_{\mathcal{Y}}$. Thus $\mathcal{J}$ is locally

$$\begin{aligned}(I_{\mathcal{Y}} \cap I_{\mathcal{V}} : f)/(I_{\mathcal{Y}} \cap I_{\mathcal{V}}) &= I_{\mathcal{Y}}/(I_{\mathcal{Y}} \cap I_{\mathcal{V}}) \\ &\cong (I_{\mathcal{Y}} + I_{\mathcal{V}})/I_{\mathcal{V}} = I_{\mathcal{Y}_E}/I_{\mathcal{V}}\end{aligned}$$

where we write $\mathcal{Y}_E$ for the restriction $\mathcal{Y} \cap (X \times E) = \mathcal{Y} \cap \mathcal{V}$. This shows

$$\mathcal{T}or_1^{X \times S}(\mathcal{O}_{\mathcal{Y} \cup \mathcal{V}}, \mathcal{O}_{X \times E}) \cong \mathcal{I}_{\mathcal{Y}_E/\mathcal{V}}(-pr_2^* E)$$

and (2.4.5) gives the short exact sequence

$$0 \to \mathcal{I}_{\mathcal{Y}_E/\mathcal{V}}(-pr_2^* E) \to \mathcal{I}_{\mathcal{Y} \cup \mathcal{V}}|_E \to \mathcal{I}_{\mathcal{V}} \to 0$$

on $X \times E$. Finally restrict to the fiber over a point $\xi \in E$: since $\mathcal{Y}_E$ and $\mathcal{V}$ are both $E$-flat this yields the short exact sequence in the statement. $\square$

Lemma 2.4.9 does not guarantee that the short exact sequence obtained is nonsplit. Showing this in the case at hand requires some work. Our strategy is to exhibit a certain quotient sheaf $\mathcal{I}_{\mathcal{Y} \cup \mathcal{V}} \otimes k(\xi) \twoheadrightarrow \mathcal{Q}$ and check that the split extension $\mathcal{I}_{\mathcal{V}_\xi} \oplus \mathcal{I}_{\mathcal{Y}_\xi/\mathcal{V}_\xi}$ admits no surjection onto $\mathcal{Q}$. In fact $\mathcal{Q} = \mathcal{O}_V(-2)$ will work:

**Lemma 2.4.10.** *Let $V \subset \mathbb{P}^3$ be a plane and $P \in V$ a point. Then there is no surjection from $\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{I}_{P/V}(-2)$ onto $\mathcal{O}_V(-2)$.*

*Proof.* Just note that $\mathrm{Hom}(\mathcal{I}_{P/V}(-2), \mathcal{O}_V(-2)) = k$ is generated by the (non-surjective) inclusion, whereas $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{O}_V(-2)) = 0$. $\square$

We will produce the required quotient sheaf by the following construction, which depends on the choice of a tangent direction at $\xi$ in $S$:

**Lemma 2.4.11.** *With notation as in Lemma 2.4.9, let* $T = \operatorname{Spec} k[t]/(t^2)$ *and let* $T \subset S$ *be a closed embedding such that* $T \cap E$ *is the reduced point* $\{\xi\}$*. Let* $Y = \mathcal{Y}_\xi$ *and* $V = \mathcal{V}_\xi$*.*

*Define a subscheme* $Y' \subset Y$ *by the ideal*

$$(\mathcal{I} : t)/(t) \subset \mathcal{O}_V$$

*where* $\mathcal{I} \subset \mathcal{O}_{V\times T}$ *is the ideal of* $\mathcal{Y} \cap (V \times T)$*. Then there is a surjection*

$$\mathcal{I}_{\mathcal{Y}\cup\mathcal{V}} \otimes k(\xi) \twoheadrightarrow \mathcal{I}_{Y'/V}.$$

**Remark 2.4.12.** Since $t^2 = 0$ we trivially have $t \in (\mathcal{I} : t)$. Since also $\mathcal{I} \subset (\mathcal{I} : t)$ we furthermore have $Y' \subset Y$. If we extend $T$ to an actual one parameter family of objects $Y_t$, we may think of $Y'$ as the limit of $Y_t \cap V$ as $t \to 0$, in other words it is the part of $Y$ that remains in $V$ as we deform along our chosen direction.

*Proof.* Let $\mathcal{Y}_T \subset \mathbb{P}^3 \times T$ denote the restriction of $\mathcal{Y}$ to $T$. We *claim* that $\mathcal{I}_{Y'/V}$ is isomorphic to the relative ideal of $\mathcal{Y}_T \cup (V \times \{\xi\})$ in $\mathcal{Y}_T \cup (V \times T)$. Assuming this, there are surjections

$$\mathcal{I}_{\mathcal{Y}\cup\mathcal{V}} \otimes \mathcal{O}_T \twoheadrightarrow \mathcal{I}_{\mathcal{Y}_T\cup(V\times\{\xi\})} \twoheadrightarrow \mathcal{I}_{Y'/V}$$

(the middle term is the ideal of $(\mathcal{Y} \cup \mathcal{V})|_T = \mathcal{Y}_T \cup (V \times \{\xi\})$ as a subscheme of $\mathbb{P}^3 \times T$). Restriction to the fiber over $\xi$ gives the surjection in the statement.

To prove the claim, we first observe that for any two subschemes $A$ and $B$ of some ambient scheme, there is an isomorphism

$$\mathcal{I}_{(A\cap B)/A} \cong \mathcal{I}_{B/(A\cup B)}$$

between the relative ideal sheaves; this is the identity $(I+J)/I \cong J/(I\cap J)$ between quotients of ideals. Apply this to

$$A = V\times T, \quad B = \mathcal{Y}_T \cup (V\times\{\xi\})$$

so that

$$\begin{aligned}
&A\cup B = \mathcal{Y}_T \cup (V\times T)\\
&A\cap B = (\mathcal{Y}_T \cup (V\times\{\xi\}))\cap (V\times T) = (\mathcal{Y}\cap (V\times T))\cup (V\times\{\xi\}).
\end{aligned}$$

The *claim* as stated thus says that $\mathcal{I}_{Y'/V}$ is isomorphic to $\mathcal{I}_{B/A\cup B}$, and we are free to replace the latter by $\mathcal{I}_{(A\cap B)/A}$.

Next let Spec$R$ be an affine open subset in $V$ and $I\subset R[t]/(t^2)$ the ideal defining $\mathcal{Y}\cap(V\times T)$ there. Locally the ideal $\mathcal{I}_{(A\cap B)/A}$ is then $I\cap(t)\subset R[t]/(t^2)$. Now multiplication with $t$ is an isomorphism of $R[t]/(t^2)$-modules

$$(I:t)/(t)\cong I\cap(t).$$

The left hand side is precisely $\mathcal{I}_{Y'/V}$ in the open subset Spec$R$. $\square$

*Proof of Theorem 2.4.8* (i). By Lemma 2.4.9 there is a short exact sequence

$$0\to \mathcal{I}_{P/V}(-2)\to \mathcal{I}_{\mathcal{Y}\cup\mathcal{V}}\otimes k(\xi)\to \mathcal{O}_{\mathbb{P}^3}(-1)\to 0$$

and since $\mathcal{F}_{P,V}$ is the unique such nonsplit extension it is enough to show that the above extension is nonsplit. In view of Lemma 2.4.10 this follows once we can show the existence of a surjection

$$\mathcal{I}_{\mathcal{Z}\cup\mathcal{W}} \otimes k(\xi) \twoheadrightarrow \mathcal{O}_V(-2).$$

For this it suffices, in the notation of Lemma 2.4.11, to choose $T \subset \mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ such that the subscheme $Y' \subset V$ is a conic.

*Nondegenerate case.* First assume $Y = \mathcal{Y}_\xi$ is a disjoint union $Y = C \cup \{P\}$ of a conic $C \subset V$ and a point $P \in V$. Consider the one parameter family $Y_t = C \cup \{P_t\}$ in which the conic part $C$ is fixed while the point $P_t$ travels along a line intersecting $V$ in the point $P$. In suitable affine coordinates we may take $V$ to be the plane $z = 0$ in $\mathbb{A}^3 = \mathrm{Spec}k[x, y, z]$, the point $P$ to be the origin and $C$ to be given by some quadric $q = q(x, y)$ not vanishing at $P$. Let the one parameter family over $\mathrm{Spec}k[t]$ consist of the union of $C$ with the point $P_t = (0, 0, t)$. This is given by the ideal

$$(q, z) \cap (x, y, z - t) = (xq, yq, (z - t)q, xz, yz, (z - t)z).$$

Now restrict to $T = \mathrm{Spec}k[t]/(t^2)$ and intersect the family with $V \times T$. The resulting subscheme is defined by the ideal

$$I = (xq, yq, (z - t)q, xz, yz, (z - t)z) + (z) = (xq, yq, tq, z)$$

and $(I : t)/(t) = (q, z)$ which defines $C \subset V$. Thus $Y' = C$ and we are done.

*Embedded point with nonsingular support.* Suppose $Y$ is a conic $C \subset V$ with an embedded point supported at a point $P$ in which $C$ is nonsingular, where the normal direction corresponding to the embedded point is along $V$. Then take the one parameter family in which $C$ and the supporting point $P$ is fixed and the embedded structure varies in the $\mathbb{P}^1$ of normal directions. In suitable affine coordinates we may take $V$ to be the plane $z = 0$ in $\mathbb{A}^3 = \mathrm{Spec}k[x, y, z]$, $P$ to be the origin and $C$ given by a quadric $q = q(x, y)$ vanishing at $P$ and say tangent to the $x$-axis at $P$. Take the one parameter family of $C$ with an embedded point given by

$$(q,z) \cap (x, y^2, z-ty) = (xq, y^2q, (z-ty)q, xz, y^2z, (z-ty)z, z-tq)).$$

This gives

$$I = (xq, y^2q, tq, z)$$

and $(I:t)/(t) = (q,z)$. This is $C$.

*Embedded point at a singularity.* Let $C \subset V$ be the union of two distinct lines intersecting in $P$ and consider a planar embedded point at $P$. Despite the singularity, there is still a $\mathbb{P}^1$ of embedded points at $P$. We take this to be our one parameter family, i.e. we deform the embedded point structure away from the planar one.

In local coordinates we take $P$ to be the origin in $\mathbb{A}^3$ and $C$ to be the union of the $x$- and $y$-axes in the $xy$-plane $V$. Then

$$(xy, z)(x,y,z) + (z - txy) = (xy^2, x^2y, z-txy)$$

is our one parameter family of embedded points at the origin, with $t = 0$ corresponding to the planar embedded point. The intersection with $V \times T$ is given by

$$I = (xy^2, x^2y, z, txy)$$

and $(I:t)/(t) = (xy, z)$. This is $C$.

*Embedded point in a double line.* Let $C \subset V$ be a planar double line together with a planar embedded point at $P \in C$ and take the one parameter family of embedded points in $P$.

In local coordinates we take $P$ to be the origin in $\mathbb{A}^3$ and $C$ to be $V(z, y^2)$.

Then

$$(z, y^2)(x, y, z) + (z - ty^2) = (y^3, xy^2, z - ty^2)$$

is our one parameter family of embedded points at the origin, with $t = 0$ corresponding to the planar embedded point. The intersection with $V \times T$ is given by

$$I = (xy^2, y^3, z, ty^2)$$

and $(I : t)/(t) = (z, y^2)$. This is $C$. $\square$

*Proof of Theorem 2.4.8* (ii). The morphism $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3) \to \mathcal{M}^{\mathrm{II}}$ is clearly an isomorphism away from $E$, and it sends $\xi \in E$ (lying over $(P, V) \in I$) to $\mathcal{F}_{P,V}$, which determines and is uniquely determined by $(P, V)$. Moreover $\mathcal{M}^{\mathrm{II}}$ is smooth at these points by Proposition 2.3.7. The claim follows from uniqueness of normal contractions. $\square$

### 2.4.3 Moduli in chamber III

In this section we show that the moduli space $\mathcal{M}^{\mathrm{III}}$ is a contraction of $\mathcal{S}$. The argument parallels that for $\mathcal{M}^{\mathrm{II}}$ closely.

Let $F \subset \mathcal{S}$ be as in Notation 2.1.2. Thus an element $Y \in \mathcal{S}$ is either a pair of intersecting lines with a spatial embedded point at the intersection, or as degenerate cases, a planar double line with a spatial embedded point. It is in a natural way a $\mathbb{P}^2$-bundle over the incidence variety $I \subset \mathbb{P}^3 \times \check{\mathbb{P}}^3$ via the map

$$F \to I$$

that sends $Y$ to the pair $(P, V)$ consisting of the support $P \in Y$ of the embedded point and the plane $V$ containing $Y \setminus \{P\}$. Moreover, a fiber of $F$ over $(P, V) \in I$ is of the form $\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_V(-2), \mathcal{I}_P(-1)))$ and a standard computation shows that $\dim \mathrm{Ext}^1(\mathcal{O}_V(-2), \mathcal{I}_P(-1)) = 6$ for $P \notin V$, hence

$F \subset \mathcal{S}$ is a divisor. In parallel with Proposition 2.4.3 one may show that $\mathcal{O}_{\mathcal{S}}(F)$ restricts to $\mathcal{O}_{\mathbb{P}^2}(-1)$ in the fibers of $F/I$ and so there is a contraction

$$\psi \colon \mathcal{S} \to \mathcal{S}' \tag{2.4.6}$$

to a smooth algebraic space $\mathcal{S}'$, such that $\psi$ is an isomorphism away from $F$ and restricts to the $\mathbb{P}^2$-bundle $F \to I$. However, in this case we can be much more concrete thanks to the work of Chen–Coskun–Nollet [CCN11], where birational models for $\mathcal{S}$ are studied in detail (and in greater generality: moduli spaces for pairs of codimension two linear subspaces of projective spaces in arbitrary dimension). The following proposition is [CCN11, Theorem 1.6 (4)]; we sketch a simple and slightly different argument here.

**Proposition 2.4.13.** *There is a contraction as in (2.4.6) where $\mathcal{S}'$ is the Grassmannian $G(2,6)$ of lines in $\mathbb{P}^5$.*

*Proof.* First consider an arbitrary quadric $Q \subset \mathbb{P}^n$. Any finite subscheme in $Q$ of length 2, reduced or not, determines a line in $\mathbb{P}^n$. This defines a morphism

$$\mathrm{Hilb}^2(Q) \to G(2, n+1). \tag{2.4.7}$$

It is clearly an isomorphism away from the locus in $G(2, n+1)$ consisting of lines contained in $Q$. On the other hand, over every element of $G(2, n+1)$ defining a line contained in $Q$, the fiber is the $\mathbb{P}^2$ consisting of length two subschemes of that line.

Apply the above observation to the (Plücker) quadric $Q = G(2,4)$ in $\mathbb{P}^5$, so that $\mathcal{S} \cong \mathrm{Hilb}^2(Q)$ (see Section 2.2.1). For every plane $V \subset \mathbb{P}^3$ and every point $P \in V$, the pencil of lines in $V$ through $P$ defines a line in $Q = G(2,4)$ and in fact every line is of this form. The fiber of (2.4.7) above such an element of $G(2,5)$ consists of all pairs of lines in $V$ intersecting at $P$. It follows that (2.4.7) is the required contraction $\mathcal{S} \to \mathcal{S}'$. □

**Remark 2.4.14.** Chen–Coskun–Nollet furthermore shows that (2.4.6) is a $K$-negative extremal contraction in the sense of Mori theory. In fact, $\mathcal{S}$ is Fano and its Mori cone is spanned by two rays. Either ray is thus contractible; one contraction is (2.4.6) and the other is the natural map to the symmetric square of the Grassmannian of lines in $\mathbb{P}^3$. This statement is extracted from Theorem 1.3, Lemma 3.2 and Proposition 3.3 in *loc. cit.* Inspired by this work we return to the Mori cone of the conics-with-a-point component $\mathcal{C}$ in Section 2.5.

We proceed as for chamber II by modifying the universal family of pairs of lines in order to identify the moduli space $\mathcal{M}^{\mathrm{III}}$ with the contracted space $\mathcal{S}'$. Let

$$\mathcal{Y} \subset \mathbb{P}^3 \times \mathcal{S}$$

be the restriction of the universal family over $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ to the component $\mathcal{S}$. Moreover, there is a flat family over the incidence variety $I \subset \mathbb{P}^3 \times \check{\mathbb{P}}^3$ whose fiber over $(P, V)$ is the plane $V$ with an embedded point at $P$. Pull this back to $F$ to define a family

$$\mathcal{W} \subset \mathbb{P}^3 \times F.$$

We argue as in Section 2.4.2 but with the family of planes $\mathcal{V}$ replaced by the family of planes with an embedded point $\mathcal{W}$.

**Theorem 2.4.15.** *Let $\mathcal{Y}$ and $\mathcal{W}$ be as above.*

*(i) $\mathcal{I}_{\mathcal{Y}\cup\mathcal{W}}$ is flat as a coherent sheaf over $\mathcal{S}$. Its fibers $\mathcal{I}_{\mathcal{Y}\cup\mathcal{W}} \otimes k(\xi)$ over $\xi \in \mathcal{S}$ are stable objects for stability conditions in chamber III.*

*(ii) The morphism $\mathcal{S} \to \mathcal{M}^{\mathrm{III}}$ determined by $\mathcal{I}_{\mathcal{Y}\cup\mathcal{W}}$ induces an isomorphism $\mathcal{S}' \cong \mathcal{M}^{\mathrm{III}}$.*

For $\xi \in F$ lying over $(P, V)$ we have $\mathcal{I}_{\mathcal{Y}_\xi/\mathcal{W}_\xi} \cong \mathcal{O}_V(-2)$ and $\mathcal{I}_{\mathcal{W}_\xi} \cong \mathcal{I}_P(-1)$.

Thus Lemma 2.4.9 yields a short exact sequence

$$0 \to \mathcal{O}_V(-2) \to \mathcal{I}_{\mathcal{Y}\cup\mathcal{W}} \otimes k(\xi) \to \mathcal{I}_P(-1) \to 0$$

and we show that it is nonsplit by exhibiting a certain quotient sheaf of $\mathcal{I}_{\mathcal{Y}\cup\mathcal{W}} \otimes k(\xi)$. This time we use $\mathcal{I}_{Q/V}(-1)$ where $Q \in V$ is a point distinct from $P$.

**Lemma 2.4.16.** *Let $V \subset \mathbb{P}^3$ be a plane and $P, Q \in V$ two distinct points. There is no surjection from $\mathcal{O}_V(-2) \oplus \mathcal{I}_P(-1)$ to $\mathcal{I}_{Q/V}(-1)$.*

*Proof.* Every nonzero homomorphism

$$\mathcal{O}_V(-2) \to \mathcal{O}_V(-1)$$

has image of the form $\mathcal{I}_{L/V}(-1)$ where $L \subset V$ is a line, whereas every nonzero homomorphism

$$\mathcal{I}_P(-1) \to \mathcal{O}_V(-1)$$

has image $\mathcal{I}_{P/V}(-1)$. Thus any nonzero homomorphism from the direct sum of these two sheaves has image $\mathcal{I}(-1)$ where $\mathcal{I} \subset \mathcal{O}_V$ is one of $\mathcal{I}_{L/V}$, $\mathcal{I}_{P/V}$ or their sum

$$\mathcal{I}_{L/V} + \mathcal{I}_{P/V} = \begin{cases} \mathcal{I}_{P/V} & \text{if } P \in L \\ \mathcal{O}_V & \text{otherwise.} \end{cases}$$

Thus the image is never $\mathcal{I}_{Q/V}(-1)$ for $Q \neq P$. □

*Proof of Theorem 2.4.15.* The proof for Theorem 2.4.8 carries over; we only need to detail the construction of quotient sheaves via one parameter families. As before we write down families over $\mathbb{A}^1 = \operatorname{Spec} k[t]$ and then restrict to $T = \operatorname{Spec} k[t]/(t^2)$. We then apply Lemma 2.4.11, with $\mathcal{W}$ in the role of the family denoted $\mathcal{V}$ in the Lemma. The outcome of Lemma 2.4.11 will be a quotient sheaf of the form $\mathcal{I}_{Y'/W}$, where $W = \mathcal{W}_\xi$ is a plane $V$ with an embedded point at $P$. We end by intersecting with $V$ to produce a further quotient of the form $\mathcal{I}_{Y'\cap V/V}$. We shall choose one parameter families such that the latter is isomorphic to $\mathcal{I}_{Q/V}(-1)$ with $Q \neq P$.

*Distinct lines.* Let $C = L \cup L_0$ be a pair of distinct lines inside $V$ intersecting at $P$. Choose another plane $V'$ containing $L_0$ and a point $Q \in L_0$ distinct from $P$. The pencil of lines $L_t \subset V'$ through $Q$ yields a one parameter family

$$Z_t = L \cup L_t$$

of disjoint pairs of lines for $t \neq 0$, with flat limit $Z_0 \subset W$ being $C$ with a spatial embedded point at $P$.

In suitable affine coordinates $\mathbb{A}^3$ let $V$ be $V(z)$, let $P$ be the origin and let $C = V(z, xy)$. Then $W = V(xz, yz, z^2)$. Furthermore let $Q = (0, 1, 0)$ and $L_t = V(x, z - t(y-1))$. This leads to the family $Z$ defined by the ideal

$$(y, z) \cap (x, z - t(y-1)) = (xy, xz, (z - t(y-1))y, (z - t(y-1))z)$$

and the intersection with $W \times T$ is given by

$$I = (xz, yz, z^2, xy, ty(y-1), tz)$$

Thus $(I : t)/(t) = (z, xy, y(y-1))$, which defines the union of the $x$-axis and the point $Q$. This is $Y' \subset W$ and thus $\mathcal{I}_{Y'\cap V/V} = \mathcal{I}_{Y'/V}$ is isomorphic to $\mathcal{I}_{Q/V}(-1)$.

*Double lines.* Let $C \subset V$ be a double line inside the plane $V$ with $P \in C$. We shall define an explicit one parameter family with central fiber $Y_0 \subset W$ being $C$ with a spatial embedded point at $P$.

Geometrically, the family is this: let $L \subset V$ be the supporting line of $C$. Consider a line $M \subset V$ not through $P$ and let $Q$ be its intersection point with

$L$. Also let $M'$ be a line through $P$ and not contained in $V$. Let $R_t$ be a point on $M$ moving towards $Q$ as $t \to 0$, and let $R'_t$ be a point on $M'$ moving towards $P$, but much faster than $R_t$ moves (quadratic versus linearly). Then let $L_t$ be the line through $R_t$ and $R'_t$ and let $Y_t = L \cup L_t$ for $t \neq 0$.

Let $P$ be the origin in suitable affine coordinates $\mathbb{A}^3$, let $V$ be the $xy$-plane $V(z)$ and let $C \subset V$ be the double $x$-axis $V(y^2, z)$. Thus $Y_0$ corresponds to

$$(z, y^2) \cap (x, y, z)^2 = (xz, yz, z^2, y^2).$$

Now let $L = V(y, z)$, let $L_t$ be the line through $(1, t, 0)$ and $(0, 0, t^2)$, that is

$$L_t = V(tx - y, ty + z - t^2)$$

and take $Y_t = L \cup L_t$ for $t \neq 0$. This yields the family (the following identity requires a bit of fiddling)

$$(y, z) \cap (tx - y, ty + z - t^2) = ((tx - y)y, (tx - y)z, (ty + z - t^2)z, xz + ty(x - 1)).$$

Reducing this modulo $t$ gives the original $Y_0$. The intersection with $W \times T$ gives

$$I = (xz, yz, z^2, (tx - y)y, ty(x - 1))$$

and so $Y' \subset W$ is defined by

$$\begin{aligned}(I : t)/(t) &= (xz, yz, z^2, y^2, y(x - 1)) \\ &= (y, z) \cap (x - 1, y^2, z) \cap (x, y, z^2).\end{aligned}$$

This is the line $L$ with an embedded point at $Q$ (inside $V$) and another embedded point along the $z$-axis at $P$. Intersecting with $V$ removes the embedded point at $P$, leaving the line $L$ with an embedded point at $Q$. Thus $\mathcal{I}_{Y' \cap V/V} \cong \mathcal{I}_{Q/V}(-1)$.

This establishes part (i) precisely as in the proof of Theorem 2.4.8 and part (ii) then follows by smoothness of $\mathcal{M}^{\mathrm{III}}$ (from Proposition 2.3.9) and by uniqueness of normal (here smooth) contractions. □

## 2.5 The Mori cone of $\mathcal{C}$ and extremal contractions

In this final section we shall prove that $\mathcal{C} \to \mathcal{C}'$ is the contraction of a $K$-negative extremal ray in the Mori cone. It follows that the contracted space $\mathcal{C}'$ is projective.

To set the stage we recall the basic mechanism of $K$-negative extremal contractions. Let $X$ be a projective normal variety and $\alpha$ a curve class (modulo numerical equivalence) which spans an extremal ray in the Mori cone. If also the ray is $K$-negative, i.e. the intersection number between $\alpha$ and the canonical divisor $K_X$ is negative, then there exists a unique projective normal variety $Y$ and a birational morphism $f\colon X \to Y$ which contracts precisely the effective curves in the class $\alpha$.

### 2.5.1 Statement

We denote elements in $\mathcal{C}$ by the letter $Y$. It is the union of a (possibly degenerate) conic denoted $C$ and a point denoted $P$. If the point is embedded, $P \in C$ denotes its support. We also write $V \subset \mathbb{P}^3$ for the unique plane containing $C$.

Define four effective curve classes (modulo numerical equivalence) on $\mathcal{C}$. Each is described as a family $\{Y_t\}$, and we use a subscript $t$ to indicate a parameter on the piece that varies (all choices are to be made general, e.g. $C$ nonsingular unless stated otherwise, etc.):

$\delta$: fix a conic $C$ and a point $P \in C$. Let $Y_t$ be $C$ with an embedded point at $P$, varying in the $\mathbb{P}^1$ of normal directions.

$\epsilon$: fix a plane $V$, a conic $C \subset V$ and a line $L \subset V$. Let $P_t$ vary along $L$ and let $Y_t = C \cup \{P_t\}$.

$\zeta$: fix a plane $V$, a pencil of conics $C_t \subset V$ and a point $P \in V$. Let $Y_t = C_t \cup \{P\}$.

$\eta$: fix a line $L$ and a point $P \in L$. Let $V_t$ be the pencil of planes containing $L$ and let $C_t$ be the planar double structure on $L$ inside $V_t$. Then let $Y_t$ be $C_t$ with an embedded spatial point at $P$.

In $\epsilon$ there are implicitly two elements with an embedded point, namely where $L$ intersects $C$. Similarly there is one element in $\zeta$ with an embedded point, corresponding to the pencil member $C_t$ that contains $P$.

**Theorem 2.5.1.** *The Mori cone of $\mathcal{C}$ is the cone over a solid tetrahedron, with extremal rays spanned by the four curve classes $\delta, \epsilon, \zeta, \eta$. Of these, the first three are $K$-negative, whereas $\eta$ is $K$-positive. The contraction corresponding to $\zeta$ is $\mathcal{C} \to \mathcal{C}'$.*

**Corollary 2.5.2.** *$\mathcal{C}'$ is projective.*

**Remark 2.5.3.** The last claim in the theorem is clear: by contracting $\zeta$ we forget the conic part of $Y \subset V$, keeping only $V$ and the point $P \in V$. By uniqueness of (normal) contractions the contracted variety is $\mathcal{C}'$. Also, with reference to Diagram 2.4.1 (from Section 2.4.1), the contraction of $\delta$ is the blowing down $b$. The theorem furthermore reveals a third $K$-negative extremal ray spanned by $\epsilon$. The corresponding contraction has the effect of forgetting the point part of $Y \subset V$, keeping only the conic; thus the contracted locus in $\mathcal{C}$ is the same as for $\zeta$, but the contraction happens in a "different direction". We do not know if the contracted space has an interpretation as a moduli space for Bridgeland stable objects.

### 2.5.2 The canonical divisor

Use notation as in Diagram 2.4.1 and Lemma 2.4.2. We read off that the Picard group of $\mathcal{C}$ has rank 4 and is generated by the pullbacks of the following divisor classes:

| | |
|---|---|
| $H \subset \mathbb{P}^3$ | a plane, |
| $H' \subset \check{\mathbb{P}}^3$ | a plane in the dual space, |
| $A = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(1))$, | |
| $E' \subset \mathcal{C}$ | the exceptional divisor for the blowup $b$. |

Moreover numerical and linear equivalence of divisors coincide on $\mathcal{C}$. Here we only use that the Picard group of a projective bundle over some variety $X$ is $\mathrm{Pic}(X) \oplus \mathbb{Z}$, with the added summand generated by $\mathcal{O}(1)$, and the Picard group of a blowup of $X$ is $\mathrm{Pic}(X) \oplus \mathbb{Z}$, with the added summand generated by the exceptional divisor.

As long as confusion seems unlikely to occur we will continue to use the symbols $H$, $H'$ and $A$ for their pullbacks to $\mathcal{C}$, or to an intermediate variety such as $\mathbb{P}^3 \times \mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ in Diagram 2.4.1.

**Lemma 2.5.4.** *The canonical divisor class of $\mathcal{C}$ is*

$$K_{\mathcal{C}} = -4H - 8H' - 6A + E'.$$

*Proof.* This is a standard computation. First, for the blowup $b$, with center of codimension two, we have

$$K_{\mathcal{C}} = b^* K_{\mathbb{P}^3 \times \mathrm{Hilb}^{2m+1}(\mathbb{P}^3)} + E'$$

and for the product

$$K_{\mathbb{P}^3 \times \mathrm{Hilb}^{2m+1}(\mathbb{P}^3)} = pr_1^* K_{\mathbb{P}^3} + pr_2^* K_{\mathrm{Hilb}^{2m+1}(\mathbb{P}^3)}.$$

Now $K_{\mathbb{P}^3} = -4H$ and for the projective bundle $\mathrm{Hilb}^{2m+1}(\mathbb{P}^3) \cong \mathbb{P}(\mathcal{E}^\vee)$ we have

$$K_{\mathbb{P}(\mathcal{E}^\vee)} = \pi^* K_{\check{\mathbb{P}}^3} + c_1(\Omega^1_\pi).$$

Again $K_{\check{\mathbb{P}}^3} = -4H'$ and the short exact sequence

$$0 \to \Omega^1_\pi \to \pi^*(\mathcal{E}^\vee) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1) \to \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)} \to 0$$

gives

$$\begin{aligned} c_1(\Omega^1_\pi) &= c_1(\pi^*(\mathcal{E}^\vee) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1)) \\ &= \pi^* c_1(\mathcal{E}^\vee) + 6c_1(\mathcal{O}_{\mathcal{E}^\vee}(-1) \\ &= -\pi^* c_1(\mathcal{E}) - 6A. \end{aligned}$$

Putting this together, the stated expression for $K_{\mathcal{C}}$ follows once we have established that $c_1(\mathcal{E}) = 4H'$.

Recall that $\mathcal{E} = pr_{2*}(pr_1^* \mathcal{O}_{\mathbb{P}^3}(2)|_I)$. We compute its first Chern class by brute force: apply Grothendieck–Riemann–Roch to $pr_2 \colon \mathbb{P}^3 \times \check{\mathbb{P}}^3 \to \check{\mathbb{P}}^3$. Note that all higher direct images vanish, since $H^p(V, \mathcal{O}_V(2)) = 0$ for all $V \in \check{\mathbb{P}}^3$ and $p > 0$. Thus by Grothendieck–Riemann–Roch the class

$$c_1(pr_{2*}(pr_1^* \mathcal{O}_{\mathbb{P}^3}(2)|_I))$$

is the push forward in the sense of the Chow ring of the degree 4 homogeneous part of

$$\mathrm{ch}(pr_1^* \mathcal{O}_{\mathbb{P}^3}(2)|_I) pr_1^*(\mathrm{td}(\mathbb{P}^3)).$$

We have

$$\operatorname{td}(\mathbb{P}^3) = \left(\frac{H}{1-e^{-H}}\right)^4 = 1 + 2H + \tfrac{11}{6}H^2 + H^3. \tag{2.5.1}$$

Moreover $I \subset \mathbb{P}^3 \times \check{\mathbb{P}}^3$ is a divisor of bidegree $(1,1)$, so there is a short exact sequence

$$0 \to pr_1^*\mathcal{O}_{\mathbb{P}^3}(-1) \otimes pr_2^*\mathcal{O}_{\check{\mathbb{P}}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3\times\check{\mathbb{P}}^3} \to \mathcal{O}_I \to 0$$

from which we see (suppressing the explicit pullbacks $pr_i^*$ of cycles in the notation)

$$\operatorname{ch}(pr_1^*\mathcal{O}_{\mathbb{P}^3}(2)|_I)) = \exp(2H)(1 - \exp(-H)\exp(-H')). \tag{2.5.2}$$

Now multiply together (2.5.1) and (2.5.2) and observe that the $H^3H'$-coefficient is 4. Since the push forward $pr_{2*}$ of any degree 4 monomial $H^kH'^{4-k}$ equals $H'$ if $k = 3$ and 0 otherwise, this shows that $c_1(\mathcal{E}) = 4H'$. □

### 2.5.3 Basis for 1-cycles

We will need a few more effective curves, as before written as families $\{Y_t\}$:

$\alpha$ : fix a conic $C$ and a line $L$. Let the point $P_t$ vary along $L$ and let $Y_t = C \cup \{P_t\}$.

$\beta$ : fix a quadric surface $Q \subset \mathbb{P}^3$, a line $L$ and a point $P$. Let $V_t$ run through the pencil of planes containing $L$ and let $C_t = Q \cap V_t$. Then take $Y_t = C_t \cup \{P\}$.

$\gamma$ : fix a plane $V$ and a point $P$. Let $C_t \subset V$ run through a pencil of conics and let $Y_t = C_t \cup \{P\}$.

As before all choices are general, so that in the definition of $\alpha$, the line $L$ is disjoint from $C$, etc.

**Lemma 2.5.5.** *The dual basis to* $(H, H', A, E')$ *is* $(\alpha, \beta, \gamma, -\delta)$*.*

*Proof.* We need to compute all the intersection numbers and verify that we get 0 or 1 as appropriate. Here it is sometimes useful to explicitly write out the pullbacks to $\mathcal{C}$, e.g. writing $b^*(pr_1^*(H))$ rather than $H$. We view $\alpha, \beta, \gamma, \delta$ not just as equivalence classes, but as the effective curves defined above. Only the intersection numbers involving $\beta$ require some real work, and we will save this for last.

Intersections with $\alpha$: Since $pr_{1*}(b_*(\alpha)))$ is the line $L \subset \mathbb{P}^3$ defining $\alpha$ we have $b^*(pr_1^*(H)) \cdot \alpha = H \cdot L = 1$. Similarly $pr_{2*}(b_*(\alpha)) = 0$ shows that the intersections with $H'$ and $A$ vanish. Finally $\alpha$ has no elements with embedded points, so is disjoint from $E'$.

Intersections with $\gamma$: We have $A \cdot \gamma = 1$ because $\gamma$ is a line in a fiber of the projective bundle $\pi$, whereas $A$ restricts to a hyperplane in every fiber. The remaining intersection numbers vanish as we can pick disjoint effective representatives.

Intersections with $\delta$: We have $E' \cdot \delta = -1$ as $\delta$ is a fiber of the blowup $b$ and $E'$ is the exceptional divisor. The remaining divisors $H$, $H'$ and $A$ are all pullbacks, i.e. of the form $b^*(?)$ and then $b^*(?) \cdot \delta = (?) \cdot b_*(\delta) = 0$.

Intersections with $\beta$: We can choose $\beta$ to be disjoint from $H$ and $E'$. Moreover $\pi_*(pr_{1*}(b_*(\beta)))$ is the line $\check{L} \subset \check{\mathbb{P}}^3$ dual to the line $L$ defining $\beta$. This gives $b^*(pr_1^*(\pi^*(H'))) \cdot \beta = H' \cdot \check{L} = 1$.

It remains to verify $A \cdot \beta = 0$. The definition of $\beta$ can be understood as follows: choose a general section

$$\mathcal{O}_{\mathbb{P}^3} \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}^3}(2)$$

and apply $pr_{2*}(pr_1^*(-)|_I)$ to obtain a homomorphism

$$\mathcal{O}_{\check{\mathbb{P}}^3} \to \mathcal{E} \tag{2.5.3}$$

whose fiber over $V \in \check{\mathbb{P}}^3$ is exactly the restriction of $\sigma$ to $V$. This is nowhere zero, so (2.5.3) is a rank 1 subbundle and it defines a section

$$s_Q \colon \check{\mathbb{P}}^3 \to \mathbb{P}(\mathcal{E}^\vee) \cong \mathrm{Hilb}^{2m+1}(\mathbb{P}^3)$$

with $s_Q^*(\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1)) \cong \mathcal{O}_{\check{\mathbb{P}}^3}$ or in terms of divisors $s_Q^*(A) = 0$. If we let $Q \subset \mathbb{P}^3$ be the quadric defined by $\sigma$ then $s_Q(V) = Q \cap V$. Thus $pr_{2*}(b_*(\beta))) = s_{Q*}(\check{L})$ where $\check{L}$ is the dual to the line $L$ defining $\beta$. This gives

$$b^*(pr_2^*(A)) \cdot \beta = A \cdot pr_{2*}(b_*(\beta)) = A \cdot s_{Q*}(\check{L}) = s_Q^*(A) \cdot \check{L} = 0.$$

□

We also define the following three effective divisors, phrased as a condition on $Y \in \mathcal{C}$:

- $D$ : all $Y$ whose conic part $C$ intersects a fixed line $M \subset \mathbb{P}^3$.
- $D'$ : all $Y$ such that the line through $P$ and a fixed point $P_0 \in \mathbb{P}^3$ intersects the conic part $C$.
- $E$ : all planar $Y$ (as before).

Since $D$ is defined by a condition on $C$ only, it is the preimage by $pr_2 \circ b$ (see Diagram 2.4.1) of the similarly defined divisor in $\mathrm{Hilb}^{2m+1}(\mathbb{P}^3)$. Moreover $D'$ and $E$ are the *strict* transforms by $b$ of the similarly defined divisors on $\mathbb{P}^3 \times \mathrm{Hilb}^{2m+1}(\mathbb{P}^3)$.

We will need to control elements of $D'$ with an embedded point.

**Lemma 2.5.6.** *Fix $P_0$ so that $D'$ is defined as an effective divisor. Choose a plane $V$ not containing $P_0$, a possibly degenerate conic $C \subset V$ and a point $P \in C$. Then there is a unique $Y \in D'$ with conic part $C$ and an embedded point at $P$. More precisely:*

1. *If $C$ is nonsingular at $P$ then the embedded point structure is uniquely determined by the normal direction given by the line through $P_0$ and $P$.*
2. *If $C$ is a pair of lines intersecting at $P$ or a double line, then the embedded point is the spatial one, i.e. the scheme theoretic union of $C$ and the first order infinitesimal neighborhood of $P$ in $\mathbb{P}^3$.*

*Proof.* Let $Q$ be the cone over $C$ with vertex $P_0$. This is a quadratic cone in the usual sense when $C$ is nonsingular, otherwise $Q$ is either a pair of planes or a double plane. A disjoint union $C \cup \{P'\}$ with $P' \neq P_0$ is clearly in $D'$ if and only if it is a subscheme of $Q$.

On the one hand this shows that the subschemes $Y$ listed in (1) and (2) are indeed in $D'$, since they are obtained from $C \cup \{P'\}$ by letting $P'$ approach $P$ along the line joining $P_0$ and $P$.

On the other hand it follows that if $Y \in D'$ then $Y \subset Q$, since the latter is a closed condition on $Y$. In case (1) $Q$ is nonsingular at $P$ and so there is a unique embedded point structure at $P \in C$ which is contained in $Q$. In case (2) the following explicit computation gives the result: suppose in local affine coordinates that $C$ is the pair of lines $V(xy, z)$, the "vertex" $P_0$ is on the $z$-axis and $P$ is the origin. Then $Q$ is the pair of planes $V(xy)$. Any $C$ with an embedded point at $P$ has ideal of the form

$$(xy, z)(x, y, z) + (sxy + tz)$$

for $(s : t) \in \mathbb{P}^1$. This contains the defining equation $xy$ of $Q$ if and only if

$t = 0$, which defines the spatial embedded point. The case where $C$ is double line $V(x^2, z)$ is similar. □

**Lemma 2.5.7.** *We have*

$$D = 2H' + A,$$
$$D' = 2H + 2H' + A - E',$$
$$E = H + H' - E'.$$

*Proof.* The last equality was essentially established in the proof of Proposition 2.4.3: it follows from the observations (1) $E$ is the strict transform of $b(E)$, and (2) the latter is the pullback of the incidence variety $I \subset \mathbb{P}^3 \times \check{\mathbb{P}}^3$ which is linearly equivalent to $H + H'$.

The remaining two identities are verified by computing the intersection numbers with the curves in the basis from Lemma 2.5.5. All curves and divisors involved are concretely defined and it is easy to find and count the intersections directly. Some care is needed to rule out intersection multiplicities, and we often find it most efficient to resort to a computation in local coordinates. We limit ourselves to writing out only two cases.

The case $D \cdot \beta = 2$: As we noted $D$ is really a divisor on $\mathrm{Hilb}^{2m+1}(\mathbb{P}^3)$ and so we shall write it here as $b^*(pr_2^*(D))$. Then $D \subset \mathrm{Hilb}^{2m+1}(\mathbb{P}^3)$ consists of all conics intersecting a fixed line $M$. We have

$$b^*(pr_2^*(D)) \cdot \beta = D \cdot pr_{2*}(b_*(\beta))$$

and $pr_{2*}(b_*(\beta))$ is the family of conics $C_t = V_t \cap Q$ where $Q$ is a fixed quadric surface and $V_t$ runs through the pencil of planes containing a fixed line $L$. For general choices $M \cap Q$ consists of two points, and each point spans together with $L$ a plane. This yields exactly two planes $V_0$ and $V_1$ in the pencil for which $C_0 = V_0 \cap Q$ and $C_1 = V_1 \cap Q$ intersects $M$. It remains to rule out multiplicities.

In the local coordinates in Remark 2.4.1 let $M = V(x_0, x_1)$. Then the intersection between $M$ and the plane $x_3 = c_0x_0 + c_1x_1 + c_2x_2$ is the point $(0 : 0 : 1 : c_2)$. Now $D$ is the condition that this point is on $C$, i.e. it satisfies equation (2.4.3); this gives that $D$ is $s_{22} = 0$. On the other hand, $pr_{2*}(b_*(\beta))$ is a one parameter family in which $c_i$ and $s_{ij}$ are functions of degree at most 2 in the parameter. To stay concrete, let $Q$ be $\sum_i x_i^2 = 0$ and let $V_t$ be $x_3 = tx_2$. Substitute $x_3 = tx_2$ in the equation for $Q$ to find $C_t = Q \cap V_t$. This gives in particular $s_{22} = 1 + t^2$ and so the intersection with $D$ is indeed two distinct points, each of multiplicity 1.

The case $D' \cdot \delta = 1$: This is essentially Lemma 2.5.6, but to ascertain there is no intersection multiplicity to account for we argue differently. $D'$ is the strict transform of the divisor $b(D') \subset \mathbb{P}^3 \times \mathrm{Hilb}^{2m+1}(\mathbb{P}^3)$, which contains the center of the blowup. Since $b(D')$ is nonsingular (pick $P_0 = (0 : 0 : 0 : 1)$ in the definition of $D'$, then in the local coordinates of Remark 2.4.1 it is simply given by the equation (2.4.3)) we have $D' = b^*(b(D')) - E'$. Thus

$$D' \cdot \delta = b(D') \cdot b_*(\delta) - E' \cdot \delta = 0 - (-1)$$

and we are done.

The remaining cases are either similar to these or easier. $\square$

### 2.5.4 Nef and Mori cones

It is clear that $H$, $H'$ and $D$ are base point free, hence nef. For instance, consider $D$: given $Y \in \mathcal{C}$, choose a line $M \subset \mathbb{P}^3$ disjoint from $Y \subset \mathbb{P}^3$. This defines an effective representative for $D$ not containing $Y$.

**Lemma 2.5.8.** *The divisor $D' + H'$ is nef.*

*Proof.* We begin by narrowing down the base locus of $D'$. First consider an element $Y \in \mathcal{C}$ without embedded point, that is a disjoint union $Y = C \cup \{P\}$. Then choose $P_0$ such that the line through $P_0$ and $P$ is disjoint from $C$. This defines a representative for $D'$ not containing $Y$, so $Y$ is not in the base locus.

Next let $Y$ be a conic $C$ with an embedded point at a point $P \in C$ where $C$ is nonsingular. The tangent to $C$ at $P$ together with the normal direction given by the embedded point determines a plane. Pick $P_0$ such that the line through $P$ and $P_0$ defines a normal direction to $C$ which is distinct from that defined by the embedded point. This determines a representative for $D'$ which by Lemma 2.5.6(i) does not contain $Y$, so $Y$ is not in the base locus.

The remaining possibility is that $Y$ is either a pair of intersecting lines with an embedded point at the singularity, or a double line with an embedded point. Pick a representative for $D'$ by choosing $P_0$ outside the plane containing the degenerate conic. If the embedded point is not spatial, then Lemma 2.5.6(ii) shows that $Y$ is not in $D'$. So $Y$ is not in the base locus unless the embedded point is spatial.

Thus let $B \subset \mathcal{C}$ be the locus of intersecting lines with a spatial embedded point at the origin, together with double lines with a spatial embedded point. By the above $B$ contains the base locus of $D'$, so if $T \subset \mathcal{C}$ is an irreducible curve not contained in $B$ then

$$(D' + H') \cdot T = D' \cdot T + H' \cdot T \geq 0$$

as both terms are nonnegative. If on the other hand $T \subset B$ we observe that $T \cdot E = 0$: in fact $B$ and $E$ are disjoint, since every element in $B$ has a spatial embedded point, whereas all elements in $E$ are planar. By the relations in Lemma 2.5.7

$$D' + H' = H + D + E$$

and so, using that $H$ and $D$ are nef,

$$(D' + H') \cdot T = (H + D + E) \cdot T = \underbrace{H \cdot T + D \cdot T}_{\geq 0} + \underbrace{E \cdot T}_{0} \geq 0.$$

□

**Lemma 2.5.9.** *The dual basis to $(H, H', D, D' + H')$ is $(\epsilon, \eta, \zeta, \delta)$.*

*Proof.* Lemma 2.5.5 together with the relations in Lemma 2.5.7 implies that $(D' + H') \cdot \delta = 1$ and the other tree intersection numbers with $\delta$ vanish.

Of the remaining intersection numbers only those involving $\eta$ requires some care and we shall write out only those.

A representative for $\eta$ is obtained by fixing a line $L$ and a point $P \in L$ and letting the plane $V_t$ vary in the pencil of planes containing $V_t$. Then $C_t$ is the double $L$ inside $V_t$ and $Y_t$ is $C_t$ together with an embedded spatial point at $P$. Then:

- Intersecting with $H$ imposes the condition that $P$ is contained in a fixed but arbitrary plane, but $P$ is fixed, so $H \cdot \eta = 0$.
- Intersecting with $H'$ imposes the condition that $V_t$ contains a fixed but arbitrary point $P_0$, this gives $H' \cdot \eta = 1$. (In fact this can be identified with the intersection number $H' \cdot \check{L} = 1$ in $\check{\mathbb{P}}^3$, where $\check{L}$ is the dual line to $L$, so there is no subtlety regarding transversality of the intersection.)
- Intersecting with $D$ imposes the condition that $C_t$ intersects a fixed but arbitrary line $M$, but $C_t$ has fixed support $L$, so $D \cdot \eta = 0$.

As $\eta$ is contained in the base locus of $D'$ we cannot find $D' \cdot \eta$ directly. As in the proof of Lemma 2.5.8 we instead rewrite $D' + H'$ as $H + D + E$ and take advantage of $\eta$ being disjoint from $E$. This gives

$$(D' + H') \cdot \eta = (H + D + E) \cdot \eta = 0.$$

□

We wish to point out that the computation in the very last paragraph, showing $D' \cdot \eta = -1$, is what made us realize that the addition of $H'$ is necessary to produce a nef divisor.

*Proof of Theorem 2.5.1.* The four divisors in Lemma 2.5.9 are nef (the first three are base point free, and the fourth is treated in Lemma 2.5.8) and the four curves are effective by definition. Hence they span the nef and Mori cones of $\mathcal{C}$, respectively. Finally by Lemmas 2.5.4 and 2.5.7 we have

$$K = -2H + 5H' - 5D - (D' + H')$$

so in view of the dual bases in Lemma 2.5.9 we read off that $K$ is negative on $\epsilon$, $\zeta$, $\delta$ and positive on $\eta$. □

# Chapter 3

# Appendix

In the 2019 version of our thesis, we computed the dimension of a certain Ext space of two families of sheaves using a number of interesting arguments and tools, including Leray spectral sequences. This was done as part of our strategy at the time for constructing a family of sheaves as in Chapter 2, now abandoned since it unfortunately does not yield the desired outcome. The purpose of this appendix is to showcase this non trivial computation of the Ext space's dimension and flesh out the details of its proof. In Section 3.1, we briefly recall the spectral sequence machinery, then in Section 3.2 we state and prove the result on the Ext space.

## 3.1 Spectral sequences

The following is extracted from [GM96] and [Mur06].

**Definition 3.1.1.** Let $\mathcal{A}$ be an abelian category and $a \geq 0$. A spectral sequence $E = (E_r^{p,q}, E^n)$ starting from page $r \geq a$ consists of the following:

a) Objects $E_r^{p,q}$ of $\mathcal{A}$ for every $p, q \in \mathbb{Z}$ and $r \geq a$.

b) A family $\{E^n\}_{n\in\mathbb{Z}}$ of objects of $\mathcal{A}$, each with a decreasing filtration by subobjects

$$\ldots \subset F^{p+1}E^n \subset F^pE^n \subset F^{p-1}E^n \subset \ldots$$

such that $\bigcap\limits_{p\in\mathbb{Z}} F^pE^n = 0$ and $\bigcup\limits_{p\in\mathbb{Z}} F^pE^n = E^n$.

Before we give the rest of the axioms, we briefly talk about a helpful way to visualize this data: imagine that for each page $r \geq a$ there is a square grid where each point is given by coordinates $(p,q) \in \mathbb{Z}^2$. Then, an object $E_r^{p,q}$ is assumed to sit at the point $(p,q)$ on the $r$-th page. An object $E^n$ sits at the last page at "infinity" where it is "spread out" all over the diagonal $n = p+q$, for every $p,q \in \mathbb{Z}$.

c) Morphisms $d_r^{p,q} \colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ for all $p,q \in \mathbb{Z}$ and $r \geq a$ satisfying $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$. The cohomology of the $r$-th page is defined as

$$H^{p,q}(E_r) = \operatorname{Ker}(d_r^{p,q}) / \operatorname{Img}(d_r^{p+r,q-r+1})$$

d) Isomorphisms $\alpha_r^{p,q} \colon H^{p,q}(E_r) \longrightarrow E_{r+1}^{p,q}$ for all $p,q \in \mathbb{Z}$ and $r \geq a$.

e) For any pair $(p,q)$ there exists $r_0 \geq a$ such that $d_r^{p,q} = 0$, $d_r^{p+r,q-r+1} = 0$ for $r \geq r_0$ (this means $E_r^{p,q} \cong E_{r+1}^{p,q}$ for all $p,q \in \mathbb{Z}$ and $r \geq r_0$). We set $E_\infty^{p,q} = E_r^{p,q}$ for all $p,q \in \mathbb{Z}$ and $r \geq r_0$.

f) Isomorphisms

$$\beta^{p,q} \colon E_\infty^{p,q} \longrightarrow F^p(E^{p+q})/F^{p+1}(E^{p+q})$$

for each pair $(p,q) \in \mathbb{Z}^2$.

If these conditions are satisfied, we say that $\{E^n\}_{n\in\mathbb{Z}}$ is the limit of the spectral sequence $E = (E_r^{p,q}, E^n)$, or that the spectral sequence converges to $\{E^n\}_{n\in\mathbb{Z}}$, which we indicate by the notation $E_r^{p,q} \Rightarrow E^{p+q}$.

We say that $(E_r^{p,q}, E^n)$ is a *first quadrant* spectral sequence if $E_r^{p,q} = 0$ unless $p \geq 0$ and $q \geq 0$. For such a spectral sequence, the filtrations $\{F^pE^n\}_{n\in\mathbb{Z}}$ are finite for every $n$ with $F^pE^n = 0$ for all $p > n$ and $F^pE^n = E^n$ for all $p \leq 0$.

**Theorem 3.1.2** (Grothendieck spectral sequence)**.** *Let $F\colon \mathcal{A} \to \mathcal{A}'$ and $G\colon \mathcal{A}' \to \mathcal{A}''$ be additive functors between abelian categories where $\mathcal{A}$, $\mathcal{A}'$ have enough injectives and all small colimits exist in $\mathcal{A}''$, and suppose that $F$ sends injectives to $G$-acyclics. Then for any object $A \in \mathcal{A}$ there is a first quadrant spectral sequence $E$ starting on page zero, such that*

$$E_2^{p,q} = R^pG \circ R^qF(A) \Rightarrow R^{p+q}(G \circ F)(A)$$

*Proof.* See [Mur06, Theorem 10]. □

**Corollary 3.1.3** (Leray spectral sequence)**.** *Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ a sheaf of abelian groups on $X$. Then there is a first quadrant spectral sequence*

$$E_2^{p,q} = H^p(Y, R^qf_*(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$

This result follows immediately from Theorem 3.1.2 by taking $\mathcal{A}$ to be the category of sheaves over $X$, $\mathcal{A}'$ the category of sheaves over $Y$, $\mathcal{A}''$ the category of abelian groups, $F = f_*$ and $G = \Gamma(X, -)$ (with the latter being the functor of global sections of $X$).

## 3.2 Ext space

First we set up some notation: recall from Section 2.4 in Chapter 2 that $\mathcal{C} \subset \mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ is the component parametrizing plane conics union a point, $E \subset \mathcal{C}$ is the locus of all planar $Y \in \mathcal{C}$, and $\phi\colon \mathcal{C} \to \mathcal{C}'$ is the contraction from Corollary 2.4.4 which maps $E$ onto the incidence variety $I = \{(p, V) \in \mathbb{P}^3 \times \check{\mathbb{P}}^3 \mid p \in V\}$. Also, $\mathcal{Y} \subset \mathcal{C} \times \mathbb{P}^3$ is the (the pullback of) the universal family over the Hilbert scheme, and $\mathcal{V} \subset E \times \mathbb{P}^3$ is the pullback of the universal family of hyperplanes. Now let $\mathcal{Y}' = \phi \times \mathrm{id}_{\mathbb{P}^3}(\mathcal{Y})$ be the image of the family $\mathcal{Y}$ over the contracted space $\mathcal{C}'$, and let $\mathcal{V}' \subset I \times \mathbb{P}^3$ be the pullback of the universal family of hyperplanes[1]. We view $\mathcal{V}'$ and $I \times \mathbb{P}^3$ as closed subschemes in $\mathcal{C}' \times \mathbb{P}^3$.

Take the sheaf $p_2^*\mathcal{O}_{\mathbb{P}^3}(-1)$ over $I \times \mathbb{P}^3$. Its fibers over $I$ are $\mathcal{O}_{\mathbb{P}^3}(-1)$. We construct an $I$-flat family of sheaves $\mathcal{T}$ over $I \times \mathbb{P}^3$ whose fibers over $I$ are $\mathcal{I}_{p/V}(-2)$, then we prove:

**Proposition 3.2.1.** *The vector space* $\mathrm{Ext}^1_{I \times \mathbb{P}^3}(p_2^*\mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{T})$ *is one dimensional.*

In the 2019 version of our thesis, we wanted to construct the universal family over $\mathcal{C}'$ by taking the generator of the above Ext space and lifting it to $\mathrm{Ext}^1_{\mathcal{C}' \times \mathbb{P}^3}\left(\mathcal{I}_{\mathcal{Y}' \cup \mathcal{V}'}, \mathcal{T}\right)$ via the morphism $\mathcal{I}_{\mathcal{Y}' \cup \mathcal{V}'} \to \mathcal{I}_{\mathcal{V}'/I \times \mathbb{P}^3}$ — which is the composition of the inclusion $\mathcal{I}_{\mathcal{Y}' \cup \mathcal{V}'} \hookrightarrow \mathcal{I}_{\mathcal{V}'}$ that comes from $\mathcal{V}' \subset \mathcal{Y}' \cup \mathcal{V}'$ and the surjection $\mathcal{I}_{\mathcal{V}'} \twoheadrightarrow \mathcal{I}_{\mathcal{V}'/I \times \mathbb{P}^3}$ from $\mathcal{V}' \subset I \times \mathbb{P}^3$. This construction does not give the wanted fibers over $I$, because $\mathcal{I}_{\mathcal{Y}' \cup \mathcal{V}'} \otimes k(\xi) \neq \mathcal{I}_{\mathcal{Y}' \cup \mathcal{V}'|\xi}$ for all $\xi \in I$ since $\mathcal{Y}' \cup \mathcal{V}'$ is not flat over $I$.

To prove Proposition 3.2.1, we first show how to build $\mathcal{T}$ then introduce a couple of lemmas (mostly standard computations). The interesting aspect of the proof is a light utilization of Leray spectral sequences and infinity pages.

Consider the diagonal $\Delta \subset \mathbb{P}^3 \times \mathbb{P}^3$ as the universal family of points over $\mathbb{P}^3$ and the incidence variety $I \subset \check{\mathbb{P}}^3 \times \mathbb{P}^3$ as the universal family over $\check{\mathbb{P}}^3$ of hyperplanes in $\mathbb{P}^3$. Take their inverse images in $I \times \mathbb{P}^3$ of these universal

[1] Pulling $\mathcal{V}'$ further back by the $\mathbb{P}^5$-bundle $E \times \mathbb{P}^3 \to I \times \mathbb{P}^3$ yields the family $\mathcal{V}$ from Section 2.4.2.

families

$$\mathcal{P} = pr_{13}^{-1}(\Delta) \cap (I \times \mathbb{P}^3) = \{(p, V, q) \in \mathbb{P}^3 \times \check{\mathbb{P}}^3 \times \mathbb{P}^3 \mid p \in V,\ p = q\}$$
$$\mathcal{V}' = pr_{23}^{-1}(I) \cap (I \times \mathbb{P}^3) = \{(p, V, q) \in \mathbb{P}^3 \times \check{\mathbb{P}}^3 \times \mathbb{P}^3 \mid p,\ q \in V\}$$

and we see that $\mathcal{P} \subset \mathcal{V}'$. Since both $\mathcal{P}$ and $\mathcal{V}'$ are $I$-flat, we get $\mathcal{I}_{\mathcal{P}/\mathcal{V}'} \otimes k(\xi) = \mathcal{I}_{p/V}$ where $\xi = (p, V) \in I$.

The restriction of $p_1$ on $\mathcal{P}$

$$p_1|_{\mathcal{P}} \colon \mathcal{P} \xrightarrow{\sim} I$$
$$(p, V, p) \longmapsto (p, V)$$

is an isomorphism, so we construct the morphism $\pi\colon I \longrightarrow \mathbb{P}^3$ as the composition $p_2|_{\mathcal{P}} \circ p_1^{-1}|_{\mathcal{P}}$.

$$\begin{tikzcd} \mathcal{P} \arrow[r, "p_2|_{\mathcal{P}}"] & \mathbb{P}^3 \\ I \arrow[u, "p_1^{-1}|_{\mathcal{P}}"] \arrow[ur, "\pi"'] & \end{tikzcd} \tag{3.2.1}$$

For every line bundle $\mathcal{L}$ over $I$, the sheaf

$$\mathcal{I}_{\mathcal{P}/\mathcal{V}'} \otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-2) \otimes p_{1*}\mathcal{L} \tag{3.2.2}$$

has the right fibers over $I$, i.e. $\mathcal{I}_{p/V}(-2)$. We fix $\mathcal{L} = (\pi^*\mathcal{O}_{\mathbb{P}^3}(-1))^{-1}$ and set $\mathcal{T}$ to be the sheaf over $I \times \mathbb{P}^3$ of the type (3.2.2) for this choice[2] of $\mathcal{L}$.

Thus, we have

$$\mathrm{Ext}^1_{I\times\mathbb{P}^3}(p_2^*\mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{T}) \cong \mathrm{H}^1(I \times \mathbb{P}^3, \mathcal{F}) \tag{3.2.3}$$

where $\mathcal{F} = \Big(\mathcal{T} \otimes \big(p_2^*\mathcal{O}_{\mathbb{P}^3}(-1)\big)^{-1} \otimes p_1^*\mathcal{L}\Big) = \Big(\mathcal{I}_{\mathcal{P}/\mathcal{V}'} \otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-1) \otimes p_1^*\mathcal{L}\Big)$.

**Lemma 3.2.2.** *For a line bundle $\mathcal{L}$ over $I$ and for $0 \le i \le 2$, we have*

[2] the reason for picking such a line bundle will be clear in the proof of Lemma 3.2.3

*1.* $R^i p_{1*}\big(\mathcal{O}_{I\times\mathbb{P}^3}(-\mathcal{V}')\otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-1)\otimes p_1^*\mathcal{L}\big)=0,$ *and*

*2.* $R^i p_{1*}\big(p_2^*\mathcal{O}_{\mathbb{P}^3}(-1)\otimes p_1^*\mathcal{L}\big)=0$

*where* $R^i p_{1*}$ *is the* $i$*-th derived functor of the pushforward* $p_{1*}$*,* $i\in\mathbb{Z}_{\geq 0}$*.*

*Proof.* Let $f\colon I\times\mathbb{P}^3\longrightarrow\check{\mathbb{P}}^3\times\mathbb{P}^3$ be the composition of the inclusion $I\times\mathbb{P}^3\hookrightarrow\mathbb{P}^3\times\check{\mathbb{P}}^3\times\mathbb{P}^3$ and $pr_{23}$, and consider the following commutative diagram

$$\begin{tikzcd}
I\times\mathbb{P}^3 \arrow[r, "f"] \arrow[d, "p_1"'] & \check{\mathbb{P}}^3\times\mathbb{P}^3 \arrow[r, "q_2"] \arrow[d, "q_1"'] & \mathbb{P}^3 \arrow[d, "r_1"] \\
I \arrow[r, "g"] & \check{\mathbb{P}}^3 \arrow[r, "r_2"] & \mathrm{Spec}(k)
\end{tikzcd}$$

Since $\mathcal{V}'\subset I\times\mathbb{P}^3$ is a divisor we have $\mathcal{O}_{I\times\mathbb{P}^3}(-\mathcal{V}')=f^*\mathcal{O}_{\check{\mathbb{P}}^3\times\mathbb{P}^3}(-1,-1)$. By definition of the box product and the fact that $p_2=q_2\circ f$ we have

$$\begin{aligned}
R^i p_{1*}\big(\mathcal{O}_{I\times\mathbb{P}^3}(-\mathcal{V}')\otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-1)\otimes p_1^*\mathcal{L}\big)&\cong\\
R^i p_{1*}\big(f^*q_1^*\mathcal{O}_{\check{\mathbb{P}}^3}(-1)\otimes f^*q_2^*\mathcal{O}_{\mathbb{P}^3}(-2)\otimes p_1^*\mathcal{L}\big)&\cong\\
R^i p_{1*}\big(f^*\mathcal{O}_{\check{\mathbb{P}}^3\times\mathbb{P}^3}(-1,-2)\otimes p_1^*\mathcal{L}\big)&\cong\\
R^i p_{1*}\big(f^*\mathcal{O}_{\check{\mathbb{P}}^3\times\mathbb{P}^3}(-1,-2)\big)\otimes\mathcal{L}&
\end{aligned}$$

where the last line is obtained by the projection formula. Thus, it is enough to prove $R^i p_{1*}\big(f^*\mathcal{O}_{\check{\mathbb{P}}^3\times\mathbb{P}^3}(-1,-2)\big)=0$ in order to establish the first part of the lemma.

By Theorem (III.5.1) in [Har77] and since $\mathcal{O}_{\mathbb{P}^3}(-2)$ has no global sections, the cohomology group $\mathrm{H}^i(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(-2))$ vanishes for $0\leq i\leq 2$, and in view of $r_{1*}=\Gamma(\mathbb{P}^3,-)$ we therefore get

$$(R^i r_{1*})\mathcal{O}_{\mathbb{P}^3}(-2)=\mathrm{H}^i(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(-2))=0$$

and thus $r_2^*\big(R^i r_{1*}\mathcal{O}_{\mathbb{P}^3}(-2)\big)$ is also zero. Moreover, via the rightmost square of the diagram we see that the sheaf $r_2^*\big(R^i r_{1*}\mathcal{O}_{\mathbb{P}^3}(-2)\big)$ is equivalent to pulling back $\mathcal{O}_{\mathbb{P}^3}(-2)$ via $q_2$ then pushing forward via $q_1$, hence

$$R^i q_{1*}\big(q_2^*\mathcal{O}_{\mathbb{P}^3}(-2)\big) = 0 \tag{3.2.4}$$

Furthermore, using the expression of $\mathcal{O}_{\check{\mathbb{P}}^3\times\mathbb{P}^3}(-1,-2)$ as a box product and then using the projection formula we get

$$\begin{aligned} R^i q_{1*}\big(\mathcal{O}_{\check{\mathbb{P}}^3\times\mathbb{P}^3}(-1,-2)\big) &\cong R^i q_{1*}\big(q_1^*\mathcal{O}_{\check{\mathbb{P}}^3}(1)\otimes q_2^*\mathcal{O}_{\mathbb{P}^3}(-2)\big) \\ &\cong \mathcal{O}_{\check{\mathbb{P}}^3}(-1)\otimes \underbrace{R^i q_{1*}\big(q_2^*\mathcal{O}_{\mathbb{P}^3}(-2)\big)}_{0 \text{ by } (3.2.4)} \\ &= 0 \end{aligned}$$

On the other hand, from the leftmost square, we see that pulling back $\mathcal{O}_{\check{\mathbb{P}}^3\times\mathbb{P}^3}(-1,-2)$ by $f$ then pushing forward by $p_1$ is the same as first pushing forward via $q_1$ then pulling back via $g$, therefore

$$R^i p_{1*}\big(f^*\mathcal{O}_{\check{\mathbb{P}}^3\times\mathbb{P}^3}(-1,-2)\big) \cong g^*\big(R^i q_{1*}\mathcal{O}(-1,-2))\big) = 0$$

as claimed.

Next, by the same reasons as above we have $\mathrm{H}^i(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(-1)) = 0$, thus with the help of the commutative diagram

$$\begin{tikzcd} I\times\mathbb{P}^3 \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathbb{P}^3 \arrow[d, "s_1"] \\ I \arrow[r, "s_2"] & \mathrm{Spec}(k) \end{tikzcd}$$

we see that

$$R^i s_{1*}\mathcal{O}_{\mathbb{P}^3}(-1) \cong \mathrm{H}^i(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(-1)) = 0$$

as well as

$$R^i p_{1*}\big(p_2^*\mathcal{O}_{\mathbb{P}^3}(-1)\big) \cong s_2^*\big(\underbrace{R^i s_{1*}\mathcal{O}_{\mathbb{P}^3}(-1)}_{0}\big) = 0$$

and thus

$$R^i p_{1*}\big(p_2^*\mathcal{O}_{\mathbb{P}^3}(-1) \otimes p_1^*\mathcal{L}\big) \cong \underbrace{R^i p_{1*}\big(p_2^*\mathcal{O}_{\mathbb{P}^3}(-1)\big)}_{0} \otimes \mathcal{L} = 0$$

finishing the proof.

□

**Lemma 3.2.3.** *For* $\mathcal{L} = \big(\pi^*\mathcal{O}_{\mathbb{P}^3}(-1)\big)^{-1}$, *we have*

$$p_{1*}\mathcal{F} = 0 \;\text{ and }\; R^1 p_{1*}\mathcal{F} = \mathcal{O}_I$$

*where* $\pi\colon I \to \mathbb{P}^3$ *is the morphism from Diagram 3.2.1 and* $\mathcal{F} = \mathcal{I}_{\mathcal{P}/\mathcal{V}'} \otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-1) \otimes p_1^*\mathcal{L}$.

*Proof.* Since $\mathcal{V}' \subset I \times \mathbb{P}^3$ is a divisor, we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_{I\times\mathbb{P}^3}(-\mathcal{V}') \longrightarrow \mathcal{O}_{I\times\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathcal{V}'} \longrightarrow 0.$$

Twist by $\big(p_2^*\mathcal{O}_{\mathbb{P}^3}(-1) \otimes p_1^*\mathcal{L}\big)$ and apply the left exact pushforward functor $p_{1*}$. Using the results of Lemma 3.2.2 on the obtained long exact sequence of higher pushforward sheaves, we read off

$$R^i p_{1*}\big(\mathcal{O}_{\mathcal{V}'} \otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-1) \otimes p_1^*\mathcal{L}\big) = 0 \tag{3.2.5}$$

for $0 \le i \le 1$. Now consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathcal{P}/\mathcal{V}'} \longrightarrow \mathcal{O}_{\mathcal{V}'} \longrightarrow \mathcal{O}_{\mathcal{P}} \longrightarrow 0.$$

Twisting by $\big(p_2^*\mathcal{O}_{\mathbb{P}^3}(-1) \otimes p_1^*\mathcal{L}\big)$ and applying the functor $p_{1*}$, we obtain the following long exact sequence

$$
\begin{aligned}
0 \to p_{1*}\mathcal{F} \to & \underbrace{p_{1*}\big(\mathcal{O}_{\mathcal{V}'} \otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-1) \otimes p_1^*\mathcal{L}\big)}_{0} \\
& \to p_{1*}\big(\mathcal{O}_{\mathcal{P}} \otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-1) \otimes p_1^*\mathcal{L}\big) \xrightarrow{\sim} R^1p_{1*}\mathcal{F} \\
& \to \underbrace{R^1p_{1*}(\mathcal{O}_{\mathcal{V}'} \otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-1) \otimes p_1^*\mathcal{L})}_{0} \to \dots
\end{aligned}
$$

where the underlined sheaves are zero by (3.2.5). Thus it remains to show that $p_{1*}\big(\mathcal{O}_{\mathcal{P}} \otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-1) \otimes p_1^*\mathcal{L}\big)$ is trivial: using the projection formula we get $p_{1*}\big(\mathcal{O}_{\mathcal{P}} \otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-1)\big) \otimes \mathcal{L}$, where $\mathcal{L} = \big(\pi^*\mathcal{O}_{\mathbb{P}^3}(-1)\big)^{-1}$ by assumption. The sheaf $p_{1*}\big(\mathcal{O}_{\mathcal{P}} \otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-1)\big)$ is read off as pulling back $\mathcal{O}_{\mathbb{P}^3}(-1)$ via $p_2$, restricting to $\mathcal{P} \subset I \times \mathbb{P}^3$ and then pushing forward by $p_1$, which, in view of Diagram 3.2.1, is equivalent to taking the pullback of $\mathcal{O}_{\mathbb{P}^3}(-1)$ by $\pi$, i.e.

$$p_{1*}\big(\mathcal{O}_{\mathcal{P}} \otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-1)\big) = \pi^*\mathcal{O}_{\mathbb{P}^3}(-1)$$

and we are done.

$\square$

*Proof of Proposition 3.2.1.* Using Corollary 3.1.3 on the morphism

$$I \times \mathbb{P}^3 \xrightarrow{p_1} I$$

we get a Leray spectral sequence $E = (E_r^{p,q}, E^n)$ such that

I. $E_2^{0,1} = \mathrm{H}^0(I, R^1p_{1*}\mathcal{F})$ and $E_2^{1,0} = \mathrm{H}^1(I, p_{1*}\mathcal{F})$,

II. $E_\infty^{p,q} = E_r^{p,q}$ for all $r \geq 3$,

III. $E$ converges to $E^1 = \mathrm{H}^1(I \times \mathbb{P}^3, \mathcal{F})$,

IV. $E^1$ comes with a finite resolution

$$E^1 = F^0E^1 \supset F^1E^1 \supset F^2E^1 = 0$$

such that $E_\infty^{0,1} \cong E^1/F^1E^1$ and $E_\infty^{1,0} \cong F^1E^1$ (by Definition 3.1.1 (f)). The resolution is finite because $E$ is a first quadrant spectral sequence.

Thus, by point (IV) we can compute $E^1 = \mathrm{H}^1(I \times \mathbb{P}^3, \mathcal{F})$ via the short exact sequence

$$0 \to E_\infty^{1,0} \to E^1 \to E_\infty^{0,1} \to 0. \tag{3.2.6}$$

Before proceeding, we remind the reader the classical fact that $R^ip_{1*} = 0$ for all $i < 0$. By point (II) and Definition 3.1.1 (c) we have

$$\begin{aligned} E_\infty^{1,0} = H^{1,0}(E_2) &= \frac{\mathrm{Ker}\big(\mathrm{H}^1(I, p_{1*}\mathcal{F}) \to \mathrm{H}^3(I, R^{-1}p_{1*}\mathcal{F})\big)}{\mathrm{Img}\,\big(\mathrm{H}^3(I, R^{-1}p_{1*}\mathcal{F}) \to \mathrm{H}^5(I, R^{-2}p_{1*}\mathcal{F})\big)} \\ &= \mathrm{H}^1(I, p_{1*}\mathcal{F}) = 0 \end{aligned}$$

since $p_{1*}\mathcal{F} = 0$ from Lemma 3.2.3. Similarly, we have

$$\begin{aligned} E_\infty^{0,1} = H^{0,1}(E_2) &= \frac{\mathrm{Ker}\big(\mathrm{H}^0(I, R^1p_{1*}\mathcal{F}) \to \mathrm{H}^2(I, p_{1*}\mathcal{F})\big)}{\mathrm{Img}\,\big(\mathrm{H}^2(I, p_{1*}\mathcal{F}) \to \mathrm{H}^4(I, R^{-1}p_{1*}\mathcal{F})\big)} \\ &= \mathrm{Ker}\big(\mathrm{H}^0(I, R^1p_{1*}\mathcal{F}) \to \mathrm{H}^2(I, \underbrace{p_{1*}\mathcal{F}}_{0})\big) \\ &= \mathrm{H}^0(I, R^1p_{1*}\mathcal{F}) \end{aligned}$$

But since by Lemma 3.2.3 $R^1p_{1*}\mathcal{F}$ is trivial we get

$$E_\infty^{0,1} = \mathrm{H}^0(I, O_I) \cong k.$$

Plugging these values of the infinity pages in (3.2.6), we obtain $\mathrm{H}^1(I \times \mathbb{P}^3, \mathcal{F}) \cong k$, which finishes the proof in light of (3.2.3). $\square$

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