European Journal of Control 68 (2022) 100670

Contents lists available at ScienceDirect

European Journal of Control

journal homepage: www.elsevier.com/locate/ejcon

# Towards a Taylor-Carleman bilinearization approach for the design of nonlinear state-feedback controllers

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A R T I C L E I N F O

*Article history:*
Received 8 April 2022
Accepted 6 June 2022
Available online 15 June 2022

Recommended by Prof. T Parisini

*Keywords:*
Carleman linearization
Linear matrix inequalities
Quadratic systems
Switched systems
Controllability

A B S T R A C T

The Carleman bilinearization is an approach that performs an exact conversion of a finite-dimensional nonlinear system into an infinite-dimensional bilinear system. A finite-dimensional system is later obtained through a truncation for analysis and control purposes. This paper investigates the linear matrix inequality (LMI)-based design of a switched state-feedback control law for the model obtained via Carleman bilinearization of a first-order nonlinear system. It is shown that in order to obtain feasible design conditions, the performance requirements must be relaxed in a neighborhood of the zero equilibrium point, so that problems arising from the uncontrollability of the linear part of the model can be avoided. The effectiveness of the proposed approach is shown using a numerical example and experimental results using a multi-input tank system.

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## 1. Introduction

Real-world systems are nonlinear [29], so designing a controller for nonlinear systems is a challenging yet fundamental problem of control theory. The simplest method to perform such design is through a first-order Taylor approximation of the nonlinear equations, thus obtaining a linear model that, in spite of being valid only locally about the linearization point, allows applying robust and optimal design techniques developed for linear systems [17].

An alternative approach that has attracted some interest by the control community is the Carleman linearization [10] (bilinearization in the case of non-autonomous systems), which converts a finite-dimensional nonlinear system into an infinite-dimensional linear system (bilinear in the case of non-autonomous systems) [22]. The conversion is exact, in the sense of perfect equivalence between the two systems, although a truncation is performed for analysis and control purposes [4].

In recent years, there have been several applications of the Carleman (bi)linearization to control problems. For instance, Rauh et al. [26] used this approach for the design of controllers and joint state and disturbance estimators for a linear axis driven by pneumatic muscle actuators. [31] designed a control law that ensured local asymptotic stability inside a domain of attraction that was shown to become bigger as the order of the truncation became higher. Mavelli and Palumbo [24] considered the optimal linear quadratic control of stochastic discrete-time systems affected by disturbances generated by a nonlinear stochastic exosystem and proposed a solution based on the Carleman approximation of the exosystem. The work [13] investigated the use of a bilinear Carleman approximation-based model predictive control. On the other hand, Bhatt and Sharma [8] proposed a novel estimation technique for noise influenced circuits that considers the Carleman linearisation with the Fokker-Planck equation for the Ito stochastic differential equations. Some recent literature has reported applications related to consensus problems [30], near-optimal control [5] and moving horizon estimation [14].

Another line of research that has been investigated in the last years is the control of nonlinear quadratic systems. Although the first results date back to the early 1980s, when Koditschek, Generio, and their coauthors investigated the stability of second-order systems containing quadratic terms [15,21], this field experienced a revival two decades later, when it was shown that the determination of the region of stability of the zero equilibrium point could be performed by solving a linear matrix inequalities (LMIs) feasibility problem [2]. Since then, nonlinear quadratic systems have been investigated in the context of state-feedback control [1], state estimation [3], parameter-varying techniques [28] and fault estimation [27]. The structure of the model arising from the Carleman bilinearization would make the techniques developed for nonlinear

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https://doi.org/10.1016/j.ejcon.2022.100670
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quadratic systems a natural fit. However, by looking at the scientific literature, this does not seem to be the case.

The class of systems considered in this paper is that of multi-input nonlinear systems affine to the control vector. It is known that for this class of systems, exact feedback linearization can be applied to transform the nonlinear systems into equivalent linear systems via a change of variables and a suitable control input [20]. However, such a transformation involves solving a partial differential equation to find a suitable diffeomorphism between the original and the transformed state coordinates, which in general is difficult to do. Moreover, drawbacks of the feedback linearization technique are that it is based on an exact cancellation of the nonlinearity, which can be achieved only under perfect model assumption, and that the controller design is performed in a nonlinearly transformed state-space, so that any physical meaning of the state variables is loss. Some of these issues are partially mitigated by approaches like those investigated in [16,19], where neural networks are trained to achieve an approximate linearization. However, when it comes to approaches that design a controller in the original state coordinates, the results in the literature are scarce. One can mention [11], which proposed the design of a controller via LMIs, under the assumption that the nonlinearities could be described via affine matrix functions of the state variables, or [7], that proposed a continuous switching adaptive controller which achieves robust stabilization with prescribed performance guarantees.

Motivated by the above discussion, the goal of this paper is to investigate the LMI-based design of a control law in the original state coordinates for a nonlinear first-order system affine to the control input, which is transformed into a suitable bilinear model via the Carleman approach. It is discussed that the uncontrollability of the linear part can lead to infeasibility of the design. For this reason, the performance requirements (in this paper, the rate of convergence of the Lyapunov function) are relaxed in a neighborhood of the zero equilibrium point, which can lead to the feasible design of a switched controller gain.

The paper is structured as follows. Section 2 recalls the main concepts related to the Taylor-Carleman bilinearization. Section 3 describes the proposed state-feedback controller design procedure. Section 4 shows some simulation results obtained using a nonlinear system for which the origin is an open-loop unstable equilibrium point. Section 5 shows experimental results using a multi-input tank system. Finally, Section 6 provides the conclusions and discusses possible future work.

*Notation:* The notation used in this paper is quite standard. $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{m\times n}, \mathbb{S}^{n\times n}$ denote the set of real numbers, real vectors of length $n$, real matrices with $m$ rows and $n$ columns, and symmetric matrices of order $n$, respectively. $\mathbb{R}_{>0}$ ($\mathbb{R}_{<0}$) denotes the set of positive (negative) real numbers. Given a matrix $A$, $\text{He}\{A\}$ is a shorthand notation for $A + A^T$.

## 2. Taylor-Carleman bilinearization

Let us consider the following first-order nonlinear system, assumed to be affine to the control input $u \in \mathbb{R}^m$:

$$\dot{x}(t) = h(x(t), u(t)) = f(x(t)) + g(x(t))^T u(t) \quad (1)$$

where $x \in \mathbb{R}$ is the state, $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^m$ are appropriate functions with $f(0) = 0$, so that $\bar{x} = 0$ is an equilibrium state when the input has a constant value $\bar{u} = 0$.

The Carleman bilinearization [10] allows converting (1) into a bilinear system through its expansion in the corresponding Taylor series [31]. In fact, under the assumption that $f(x)$ and $g(x)$ are analytic in $\mathbb{R}$, the following holds:

$$f(x) = \sum_{j=1}^{+\infty} \frac{1}{j!} \frac{d^j f}{dx^j}(0) x^j \quad (2)$$

$$g(x)^T u = g(0)^T u + \sum_{j=1}^{+\infty} \frac{1}{j!} \left[ \frac{d^j g}{dx^j}(0) \right]^T x^j u \quad (3)$$

By truncating the series at $j = k$, and computing $d\{x^2\}/dt$, $d\{x^3\}/dt$, ..., $d\{x^k\}/dt$ while neglecting terms of order larger than $k$, i.e., $x^{k+1}$, $x^{k+2}$, etc., the resulting approximated system can be reshaped into a bilinear form (if $k = 1$, the standard Taylor-based linearization is achieved):

$$\dot{z}(t) = Az(t) + [B + N(z(t))]u(t) \quad (4)$$

where:

$$z(t) = \begin{bmatrix} x(t) & x^2(t) & \cdots & x^k(t) \end{bmatrix}^T \in \mathbb{R}^k \quad (5)$$

The function $N(z)$ has the following structure:

$$N(z) = \begin{bmatrix} z^T N_1 \\ z^T N_2 \\ \vdots \\ z^T N_k \end{bmatrix} \quad (6)$$

and $A, B, N_1, N_2, \ldots, N_k$ are matrices of appropriate dimensions that can be computed from the Taylor series:

$$A = \begin{bmatrix} \frac{df}{dx}(0) & \frac{1}{2!}\frac{d^2 f}{dx^2}(0) & \frac{1}{3!}\frac{d^3 f}{dx^3}(0) & \cdots & \frac{1}{(k-1)!}\frac{d^{k-1} f}{dx^{k-1}}(0) & \frac{1}{k!}\frac{d^k f}{dx^k}(0) \\ 0 & 2\frac{df}{dx}(0) & \frac{2}{2!}\frac{d^2 f}{dx^2}(0) & \cdots & \frac{2}{(k-2)!}\frac{d^{k-2} f}{dx^{k-2}}(0) & \frac{2}{(k-1)!}\frac{d^{k-1} f}{dx^{k-1}}(0) \\ 0 & 0 & 3\frac{df}{dx}(0) & \cdots & \frac{3}{(k-3)!}\frac{d^{k-3} f}{dx^{k-3}}(0) & \frac{3}{(k-2)!}\frac{d^{k-2} f}{dx^{k-2}}(0) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (k-1)\frac{df}{dx}(0) & \frac{k-1}{2!}\frac{d^2 f}{dx^2}(0) \\ 0 & 0 & 0 & \cdots & 0 & k\frac{df}{dx}(0) \end{bmatrix} \quad (7)$$

$$B = \begin{bmatrix} g(0)^T \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad N_1 = \begin{bmatrix} \left[\frac{dg}{dx}(0)\right]^T \\ \frac{1}{2!}\left[\frac{d^2 g}{dx^2}(0)\right]^T \\ \frac{1}{3!}\left[\frac{d^3 g}{dx^3}(0)\right]^T \\ \vdots \\ \frac{1}{(k-1)!}\left[\frac{d^{k-1} g}{dx^{k-1}}(0)\right]^T \\ \frac{1}{k!}\left[\frac{d^k g}{dx^k}(0)\right]^T \end{bmatrix} \quad (8)$$

$$N_2 = \begin{bmatrix} 2g(0)^T \\ 2\left[\frac{dg}{dx}(0)\right]^T \\ \frac{2}{2!}\left[\frac{d^2 g}{dx^2}(0)\right]^T \\ \vdots \\ \frac{2}{(k-2)!}\left[\frac{d^{k-2} g}{dx^{k-2}}(0)\right]^T \\ \frac{2}{(k-1)!}\left[\frac{d^{k-1} g}{dx^{k-1}}(0)\right]^T \end{bmatrix} \cdots \quad N_k = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ kg(0)^T \\ k\left[\frac{dg}{dx}(0)\right]^T \end{bmatrix} \quad (8)$$

## 3. State-feedback controller design

First of all, let us recall a result from the literature [1] about the design of state-feedback controllers for nonlinear quadratic systems.

**Theorem 1.** *Given a system in the form (4) and a polytope* $\mathcal{P} \subset \mathbb{R}^k$*:*

$$\mathcal{P} = \text{conv}\{z_{(1)}, z_{(2)}, \dots, z_{(p)}\} = \{z \in \mathbb{R}^k : a_j^T z \leq 1, j = 1, \dots, q\} \quad (9)$$

*where p and q are suitable integers,* $z_{(i)}$ *denotes the i-th vertex of the polytope* $\mathcal{P}$*,* $a_j \in \mathbb{R}^k$ *and* $\text{conv}\{\cdot\}$ *denotes the operation of taking the convex hull of the argument, with* $0 \in \mathcal{P}$*, a linear state-feedback controller in the form:*

$$u(t) = Kz(t) \quad (10)$$

*with* $K \in \mathbb{R}^{m\times k}$ *exists such that the closed-loop system given by the interconnection of (4) and (10) is asymptotically stable and* $\mathcal{P}$ *belongs to the domain of attraction of the zero equilibrium point if there exist a scalar* $0 < \gamma < 1$*, a positive definite matrix* $P \in \mathbb{S}^{k\times k}$ *and a matrix* $L \in \mathbb{R}^{m\times k}$ *such that:*

$$\begin{bmatrix} 1 & \gamma a_j^T P \\ P a_j \gamma & P \end{bmatrix} \succeq 0 \quad (11)$$

$$\begin{bmatrix} 1 & z_{(i)}^T \\ z_{(i)} & P \end{bmatrix} \succeq 0 \quad (12)$$

$$\text{He}\left\{ \gamma(AP + BL) + \begin{bmatrix} z_{(i)}^T (N_1 L) \\ \cdots \\ z_{(i)}^T (N_k L) \end{bmatrix} \right\} \prec 0 \quad (13)$$

*for* $j = 1, 2, \dots, q$ *and* $i = 1, 2, \dots, p$*. In this case, the controller gain is given by* $K = LP^{-1}$*.*

**Proof.** See [1]. □

The above theorem ensures closed-loop stability of the zero equilibrium point, but it does not provide any information about the rate of convergence. In order to enforce such design specification, [1] proposed to guarantee that the poles of the linearized system belongs to the half-plane $\text{Re}(s) < -\alpha$, by adding the additional matrix inequality:

$$2\alpha P + \text{He}\{AP + BL\} \prec 0 \quad (14)$$

However, an issue with this idea is that enforcing the rate of convergence design specification by means of a pole constraint reasoning makes the specification valid only under linear assumption, i.e., when the effect of the quadratic term in (4) can be neglected. Constraining the rate of convergence of the Lyapunov function $V(z(t))$ which is used to prove stability, which means:

$$\dot{V}(z(t)) < -2\alpha V(z(t)) \quad (15)$$

would lead to a closed-loop system where the rate of convergence specification holds even when the quadratic term is non-negligible. In the following, we will say that (15) holds *in the strong sense* when it is indeed satisfied by the function $V(z(t))$, and *in the weak sense* when it is satisfied by an approximation of the function $V(z(t))$ instead.

Another issue that impedes the direct application of Theorem 1 for the design of a controller for the model arising from the Taylor-Carleman bilinearization, and described in the previous section, is the uncontrollability of the linear part of the system (4) with matrices as in (6)-(8). In fact, due to the structure of $A$ and $B$, the controllability matrix is given by:

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix} = \begin{bmatrix} g(0)^T & \frac{df}{dx}(0)g(0)^T & \cdots & \left[\frac{df(x)}{dx}(0)\right]^{k-1} g(0)^T \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad (16)$$

which is clearly not full rank, so that according to the algebraic controllability theorem, the pair $(A, B)$ is not controllable. A consequence of this fact is that if the matrix $A$ has unstable eigenvalues, then the matrix inequality (13) is necessarily infeasible, as it cannot hold for at least one point inside the polytope $\mathcal{P}$ (i.e., the point $0 \in \mathcal{P}$).

Motivated by the above discussion, the goal of this section is to provide a state-feedback controller design procedure that can be applied to a quadratic model obtained via the Taylor-Carleman bilinearization and that constrains the rate of convergence of the Lyapunov function used to assess stability of the zero equilibrium point.

**Theorem 2.** *Given a small enough* $\epsilon > 0$*, a scalar* $\tau >> 1$*, a desired rate of convergence* $\alpha > 0$ *and an interval* $\mathbb{I} \subset \mathbb{R}$*, define* $\mathbb{I}_0$*,* $\mathbb{I}_{0^+}$*,* $\mathbb{I}_{\epsilon_+}$*,* $\mathbb{I}_{\epsilon_-}$ *as the following intervals:*

$$\mathbb{I}_0 = [-\epsilon, \epsilon] \quad (17)$$

$$\mathbb{I}_{0^+} = [0, \epsilon] \quad (18)$$

$$\mathbb{I}_{\epsilon_+} = (\mathbb{I} \backslash \mathbb{I}_0) \cap \mathbb{R}_{>0} \quad (19)$$

$$\mathbb{I}_{\epsilon_-} = (\mathbb{I} \backslash \mathbb{I}_0) \cap \mathbb{R}_{<0} \quad (20)$$

*as* $\mathcal{P}_{\epsilon_+} \subset \mathbb{R}^k$ *and* $\mathcal{P}_{\epsilon_-} \subset \mathbb{R}^k$ *the polytopes containing the sets of all the values taken by* $z(t)$ *defined as in (5) when* $x(t) \in \mathbb{I}_{\epsilon_+}$ *and* $x(t) \in \mathbb{I}_{\epsilon_-}$*, respectively:*

$$\mathcal{P}_{\epsilon_+} = \text{conv}\left\{z_{(1)}^+, z_{(2)}^+, \dots, z_{(p)}^+\right\} \quad (21)$$

$$\mathcal{P}_{\epsilon_-} = \text{conv}\left\{z_{(1)}^-, z_{(2)}^-, \dots, z_{(p)}^-\right\} \quad (22)$$

*and as* $\mathcal{P}_0$ *the polytope containing the sets of all the values taken by* $\begin{bmatrix} x^2(t) & \cdots & x^k(t) \end{bmatrix}^T$ *when* $x(t) \in \mathbb{I}_{0^+}$*:*

$$\mathcal{P}_0 = \text{conv}\left\{z_{(1)}^0, z_{(2)}^0, \dots, z_{(p_0)}^0\right\} \quad (23)$$

*A switched linear state-feedback control law in the form:*

$$u(t) = \begin{cases} K^+ z(t) & \text{if}\, x(t) \geq 0 \\ K^- z(t) & \text{if}\, x(t) < 0 \end{cases} \quad (24)$$

*with* $K^+, K^- \in \mathbb{R}^{m\times k}$ *structured as follows:*

$$K^\pm = \begin{bmatrix} K_1^\pm & K_2^\pm & \cdots & K_k^\pm \end{bmatrix} \quad (25)$$

*with* $\pm \in \{+, -\}$ *existing such that:*

1. $x = 0$ *is an asymptotically stable equilibrium point for the closed-loop system given by the interconnection of (4) and (24)*
2. *there exists a Lyapunov function* $V(z(t))$ *for which (15) holds in the strong sense* $\forall x \in \mathbb{I}_{\epsilon_+} \cup \mathbb{I}_{\epsilon_-}$ *and in the weak sense* $\forall x \in \mathbb{I}_0$

*if there exist a matrix* $P \in \mathbb{S}^{k\times k}$ *and matrices* $L^+, L^- \in \mathbb{R}^{m\times k}$ *structured as follows:*

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_z \end{bmatrix} \quad (26)$$

$$L^\pm = \begin{bmatrix} L_1^\pm & L_z^\pm \end{bmatrix} \quad (27)$$

*with* $P_1 \in \mathbb{R}$*,* $P_z \in \mathbb{S}^{(k-1)\times(k-1)}$*,* $L_1^\pm \in \mathbb{R}^{m\times 1}$*,* $L_z^\pm \in \mathbb{R}^{m\times(k-1)}$*,* $\pm \in \{+, -\}$*, such that* $P \succ 0$ *and* $\forall i \in \{1, \dots, p\}$*,* $\forall j \in \{1, \dots, p_0\}$*,* $\forall \pm \in \{+, -\}$*:*

$$\text{He}\left\{ AP^\pm + BL^\pm + \begin{bmatrix} z_{(i)}^{\pm T} N_1 L^\pm \\ z_{(i)}^{\pm T} N_2 L^\pm \\ \vdots \\ z_{(i)}^{\pm T} N_n L^\pm \end{bmatrix} \right\} + 2\alpha P^\pm \prec 0 \quad (28)$$

$$\begin{bmatrix} z^0_{(j),1}P_1 & P_1(z^0_{(j)})^T \\ z^0_{(j)}P_1 & P_z/\tau \end{bmatrix} \succeq 0 \quad (29)$$

$$\text{He}\left\{ \frac{df}{dx}(0)P_1 + g(0)L_1^{\pm} \right\} + 2\alpha P_1 \prec 0 \quad (30)$$

*where $z^0_{(j),1}$ denotes the first element of $z^0_{(j)}$, $P^+ = P$ and $P^- = TPT$ with $T = \text{diag}\big((-1)^0, (-1)^1, (-1)^2, \ldots, (-1)^{k-1}\big)$. Then, the controller gains $K^+$ and $K^-$ to be used in (24) are given by $K^+ = L^+P^{-1}$ and $K^- = L^-TP^{-1}T$, respectively.*

**Proof.** The interconnection of (4) and (24) yields:

$$\dot{z}(t) = \big(A + BK^{\pm}\big)z(t) + \begin{bmatrix} z(t)^T N_1 K^{\pm} \\ z(t)^T N_2 K^{\pm} \\ \vdots \\ z(t)^T N_k K^{\pm} \end{bmatrix} z(t) \quad (31)$$

where ± is either + or - depending on the sign of $x(t)$.

We will consider a Lyapunov candidate function as follows:

$$V(x(t)) = \sum_{j=1}^{k} a_{2j} x(t)^{2j} + \sum_{j=1}^{k-2} a_{3+2j} |x(t)|^{3+2j} \quad (32)$$

where $a_2, a_4, a_5, \ldots, a_{2k}$ are coefficients to be determined. Given (5), the function $V(x(t))$ can be reshaped in terms of the variable $z(t)$ as follows:

$$V(z(t)) = \begin{cases} z(t)^T P^{-1} z(t) & \text{if } x(t) \geq 0 \\ z(t)^T T P^{-1} T z(t) & \text{if } x(t) < 0 \end{cases} \quad (33)$$

with $P$ structured as in (26).

Then, taking into account (31), the time derivative is computed as:

$$\dot{V}(z(t)) = \dot{z}(t)^T (P^{\pm})^{-1} z(t) + z(t)^T (P^{\pm})^{-1} \dot{z}(t) = \text{He}\left\{ z(t)^T (P^{\pm})^{-1} \left( A + BK^{\pm} + \begin{bmatrix} z(t)^T N_1 K^{\pm} \\ z(t)^T N_2 K^{\pm} \\ \vdots \\ z(t)^T N_k K^{\pm} \end{bmatrix} \right) z(t) \right\} \quad (34)$$

which means that (15) is satisfied if:

$$\text{He}\left\{ (P^{\pm})^{-1} \left( A + BK^{\pm} + \begin{bmatrix} z(t)^T N_1 K^{\pm} \\ z(t)^T N_2 K^{\pm} \\ \vdots \\ z(t)^T N_k K^{\pm} \end{bmatrix} \right) \right\} + 2\alpha (P^{\pm})^{-1} \prec 0 \quad (35)$$

By pre- and post-multiplying (35) by $P^{\pm}$, while defining $L^{\pm} = K^{\pm}P^{\pm}$, we obtain:

$$\text{He}\left\{ AP^{\pm} + BL^{\pm} + \begin{bmatrix} z(t)^T N_1 L^{\pm} \\ z(t)^T N_2 L^{\pm} \\ \vdots \\ z(t)^T N_k L^{\pm} \end{bmatrix} \right\} + 2\alpha P^{\pm} \prec 0 \quad (36)$$

Then, (28) is sufficient for (36) to hold $\forall z \in \mathcal{P}_{\epsilon_+} \cup \mathcal{P}_{\epsilon_-}$, and in turn $\forall x \in \mathbb{I}_{\epsilon_+} \cup \mathbb{I}_{\epsilon_-}$, since the matrix function appearing in (35) is an affine function of the state variables in $z(t)$, so that negative definiteness on the polytope vertices implies negative definiteness over the whole polytope [18].

Due to the structure of $P$ in (26), the Lyapunov function $V(z(t))$ can be approximated as:

$$V(z(t)) \approx V_1(x(t)) = P_1^{-1} x(t)^2 \quad (37)$$

when $x \in \mathbb{I}_0$ if the following holds:

$$P_1^{-1}x^2 \geq \tau \begin{bmatrix} x^2 & \cdots & x^k \end{bmatrix} P_z^{-1} \begin{bmatrix} x^2 \cdots & x^k \end{bmatrix}^T \quad (38)$$

with $\tau >> 1$. The condition (38) is equivalent via Schur complement to:

$$\begin{bmatrix} P_1^{-1}x^2 & \begin{bmatrix} x^2 & \cdots & x^k \end{bmatrix} \\ \begin{bmatrix} x^2 & \cdots & x^k \end{bmatrix}^T & P_z \end{bmatrix} \succeq 0 \quad (39)$$

which, by pre- and post-multiplication by $\text{diag}(P_1, I)$, is shown to hold if (29) holds.

Given a small enough $\epsilon$, (31) can be approximated by the following state equation:

$$\dot{x}(t) = \left( \frac{df}{dx}(0) + g(0)K_1^{\pm} \right) x(t) \quad (40)$$

which corresponds to controlling the first-order approximation of (1) by means of the switched linear state-feedback control law:

$$u(t) = \begin{cases} K_1^+ x(t) & \text{if } x(t) \geq 0 \\ K_1^- x(t) & \text{if } x(t) < 0 \end{cases} \quad (41)$$

Then, computing the time derivative of (37), taking into account (40):

$$\dot{V}(x(t)) = \dot{x}(t) P_1^{-1} x(t) + x(t)^T P_1^{-1} \dot{x}(t) = \text{He}\left\{ x(t)^T P_1^{-1} \left( \frac{df}{dx}(0) + g(0)K_1^{\pm} \right) x(t) \right\} \quad (42)$$

so that $\dot{V}(x(t)) < -2\alpha V(x(t))$ leads, after some manipulations, to (30). Since $\alpha > 0$, by virtue of Lyapunov's direct method, the inequality (30) enforces asymptotical stability of the equilibrium point $x = 0$, which completes the proof. □

**Remark 1.** The presence of the scalar $\gamma$ in Theorem 1 comes from the requirement that $\dot{V} < 0$ holds in an enlarged version of the polytope $\mathcal{P}$ that contains a level curve of the Lyapunov function that contains the polytope of interest $\mathcal{P}$. This is necessary to ensure convergence to the origin for any initial condition $x(0) \in \mathcal{P}$, otherwise it could happen that the state $x(t)$ reaches regions of the state-space which are outside $\mathcal{P}$, and where $\dot{V} > 0$. The reader might have noticed that such a reasoning is not included in the proof of Theorem 2, due to the special structure of the augmented state vector $z(t)$ (see Eq. (5)) that ensures that if $\dot{V}(z(t))$ holds, then each element of $z(t)$ will decrease in absolute value, thus moving simultaneously towards the origin, i.e., towards $x = 0$.

**Remark 2.** The statement of Theorem 2 contains the standard requirement that $P \succ 0$. However, given the interpretation of the Lyapunov function $V(z(t))$ in terms of $x(t)$, as given in (32), it is possible to replace $P \succ 0$ with the elementwise inequalities $P_1 > 0$ and $P_{z,ij} > 0$ $\forall i, j \in \{1, \ldots, k-1\}$, thus allowing for non-definite matrices $P$ for which negative quadratic forms $z^T P z < 0$ are obtained for values of $z$ that do not correspond to any value of $x$.

**Remark 3.** The proof of Theorem 2 relies on the approximation (37) holding true due to the *small* $\epsilon$ assumption. The value of $\epsilon$ should be selected small enough that the linear model (40) arising from the Taylor-series-based linearization describes the nonlinear dynamics well enough. To this end, nonlinearity measures such as the gap metric [9,12,25] can be used to decide whether a value of $\epsilon$ is acceptable or not.

## 4. Numerical example

Consider the following nonlinear system:

$$\dot{x}(t) = x(t)^3 + (x(t)+1)^2 u_1(t) + (x(t)-1)^2 u_2(t) \quad (43)$$

for which $\bar{u} = 0$ is an unstable equilibrium point when $\bar{u}_1 = \bar{u}_2 = 0$. By means of the procedure described in Section 2, we can define

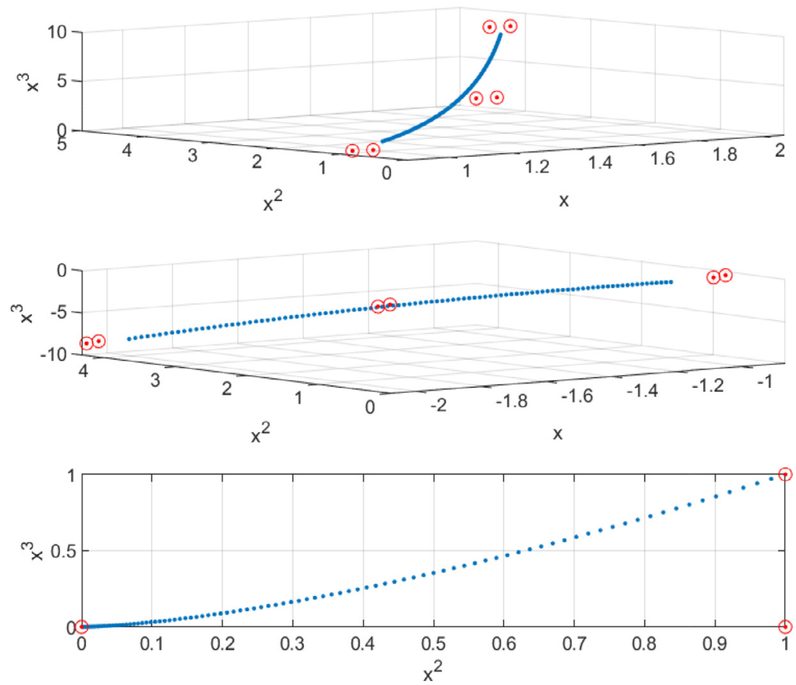


**Fig. 1.** Values of $z(t)$ (blue dots) and enclosing polytopes (red circles/dots). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

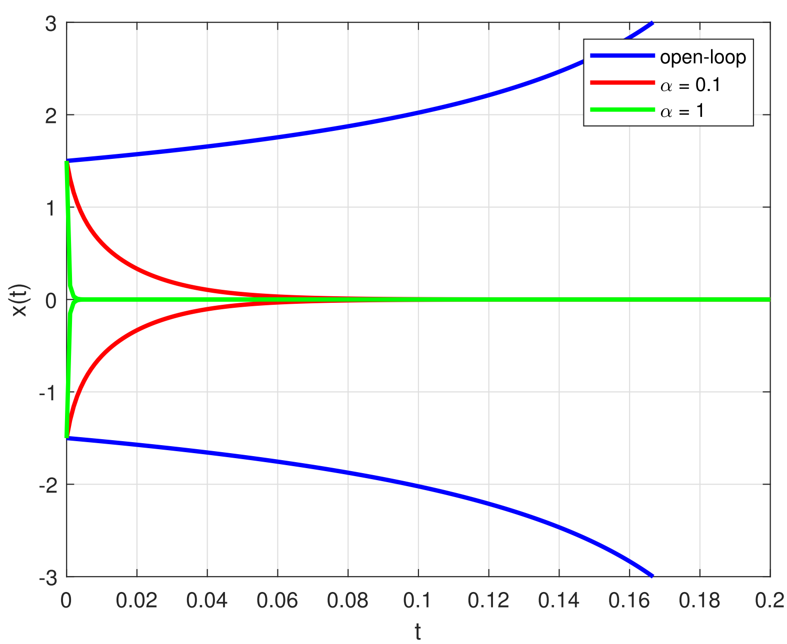


**Fig. 2.** Simulation results with initial conditions $x(0) = 1$ and $x(0) = -1$.

the augmented state vector $z(t) = [x(t), x(t)^2, x(t)^3]^T$ and describe the system in a structure as in (4) with matrices:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$N_1 = \begin{bmatrix} 2 & -2 \\ 2 & 2 \\ 0 & 0 \end{bmatrix} N_2 = \begin{bmatrix} 2 & 2 \\ 4 & -4 \\ 4 & 4 \end{bmatrix} N_3 = \begin{bmatrix} 0 & 0 \\ 3 & 3 \\ 6 & -6 \end{bmatrix}$$

For design purpose, the interval $\mathbb{I} = [-2, 2]$ with $\epsilon = 1$ will be considered. For the sake of computing $\mathcal{P}_{\epsilon^+}$, $\mathcal{P}_{\epsilon^-}$ and $\mathcal{P}_0$ in (21)-(23), the singular value decomposition (SVD) boxing [6] has been used, obtaining the enclosing polytopes shown in Fig. 1.

Then, using the YALMIP toolbox [23] with the SeDuMi solver [32], the LMIs in Theorem 2 have been solved for a required guaranteed convergence rate $\alpha = 0.1$, providing the following results:

$$P = \begin{bmatrix} 0.0006 & 0 & 0 \\ 0 & 0.5992 & 0.3309 \\ 0 & 0.3309 & 0.2124 \end{bmatrix}$$

$$K^+ = \begin{bmatrix} -23.329 & 0.0479 & -0.1493 \\ -37.546 & -0.8096 & 1.2848 \end{bmatrix}$$

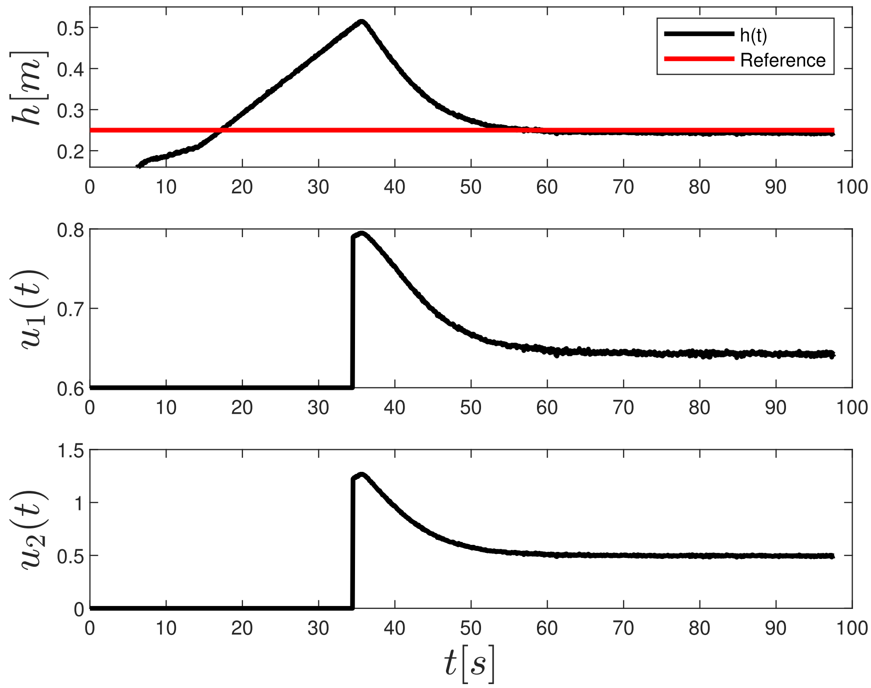


**Fig. 3.** Water level and input signals.

$$K^- = \begin{bmatrix} -37.546 & 0.8096 & 1.2848 \\ -23.329 & -0.0479 & -0.1493 \end{bmatrix}$$

Furthermore, a different solution with a bigger guaranteed convergence rate $\alpha = 1$ has been obtained as follows:

$$P = \begin{bmatrix} 9.4 \cdot 10^{-6} & 0 & 0 \\ 0 & 0.2106 & 0.1680 \\ 0 & 0.1680 & 0.1346 \end{bmatrix}$$

$$K^+ = \begin{bmatrix} -942.04 & 1.3866 & -1.951 \\ -464.81 & -17.115 & 21.389 \end{bmatrix}$$

$$K^- = \begin{bmatrix} -464.81 & 17.115 & 21.389 \\ -942.04 & -1.3866 & -1.951 \end{bmatrix}$$

In order to compare the designed controllers, the closed-loop responses from initial conditions $x(0) = 2$ and $x(0) = -2$ are plotted together with the corresponding open-loop responses (see Fig. 2). The simulation results show that the designed controllers stabilize the nonlinear system, and that the choice of the parameter $\alpha$ affects the convergence rate of the state variable $x(t)$.

## 5. Application to a multi-input tank system

The design procedure described in Section 3 has been tested using a process plant available at the University of Stavanger, consisting of a rectangular tank. The tank has a pump that takes water from a collection vessel (input signal $u_1(t)$) and a valve through which water can be drained out of the tank (input signal $u_2(t)$).

The nonlinear model of the tank can be obtained from a balance law as follows:

$$\frac{dh(t)}{dt} = \frac{1}{A}\left(f_1(u_1(t)) - \frac{K_v f_2(u_2(t))}{3600}\sqrt{\frac{\rho g(h(t)+h_v)}{100000}}\right) \tag{44}$$

where $h(t)$ denotes the water level in the tank, $A = 0.0096\,\mathrm{m}^2$ is the area of the tank, $K_v = 11.25\,\mathrm{m}^3/(\mathrm{hour}\cdot\sqrt{\mathrm{bar}})$ is the valve constant, $\rho = 1000\,\mathrm{kg/m}^3$ is the density of water, $g = 9.81\,\mathrm{m/s}^2$ is the gravity acceleration, $h_v = 0.05\,m$ is the height from the bottom of the tank down to the valve, and $f_1(\cdot)$, $f_2(\cdot)$ are two nonlinear functions representing the pump/valve characteristic curve, which are assumed to be known.

The nonlinear model (44) is approximated around the equilibrium point of interest ($\bar{h} = 0.25\,\mathrm{m}$, obtained by applying constant inputs $\bar{u}_1 = 0.65$ and $\bar{u}_2 = 0.52$), by performing the change of variables: $\delta h(t) \triangleq h(t) - 0.25$, $\delta u_1(t) \triangleq u_1(t) - 0.65$, and $\delta u_2(t) = u_2(t) - 0.52$. Then, a bilinear model in a structure as in (4) is obtained by defining: $x(t) \triangleq \delta h(t)$, $z(t) \triangleq \left[x(t), x(t)^2\right]^T$ and $u(t) \triangleq [\delta u_1(t), \delta u_2(t)]^T$, in which case the numerical values of the matrices $A$, $B$, $N_1$, $N_2$ are as follows:

$$A = \begin{bmatrix} -0.0297 & 0.0248 \\ 0 & -0.0594 \end{bmatrix} \quad B = \begin{bmatrix} -0.0525 & 0.052 \\ 0 & 0 \end{bmatrix}$$

$$N_1 = \begin{bmatrix} -0.0876 & 0 \\ 0.073 & 0 \end{bmatrix} \quad N_2 = \begin{bmatrix} -0.1051 & 0.1040 \\ -0.1751 & 0 \end{bmatrix}$$

For design purpose, the interval $\mathbb{I} = [-0.25, 0.25]$ with $\epsilon = 0.05$ has been considered, and the ensemble of possible values of the augmented state vector $z(t)$ has been bounded using triangles with vertices $(\pm\epsilon, \epsilon^2)$, $(\pm 0.25, \epsilon^2)$, $(\pm 0.25, 0.25^2)$. The LMIs have returned the following Lyapunov matrix and controller gains:

$$P = \begin{bmatrix} 0.0147 & 0 \\ 0 & 0.0835 \end{bmatrix}$$

$$K^+ = \begin{bmatrix} 3.0226 & 0.3199 \\ -1.0976 & -0.1744 \end{bmatrix} K^- = \begin{bmatrix} 2.5795 & -0.8682 \\ -0.7284 & -0.7877 \end{bmatrix}$$

Fig. 3 shows experimental results in which the tank has been driven first to an initial condition $x(t_0) = 0.5\,\mathrm{m}$ (in open-loop), which corresponds to one of the extremes of the interval $\mathbb{I}$. Then, starting at approx. $t = 35\,\mathrm{s}$, the designed augmented state-feedback controller has been activated. As shown in Fig. 4, the closed-loop operation of the system after the controller's activation generally satisfies the design constraint (15) during the transient (it is worth remarking that the presence of measurement noise affects the computed behavior of $\dot{V}(z(t))$).

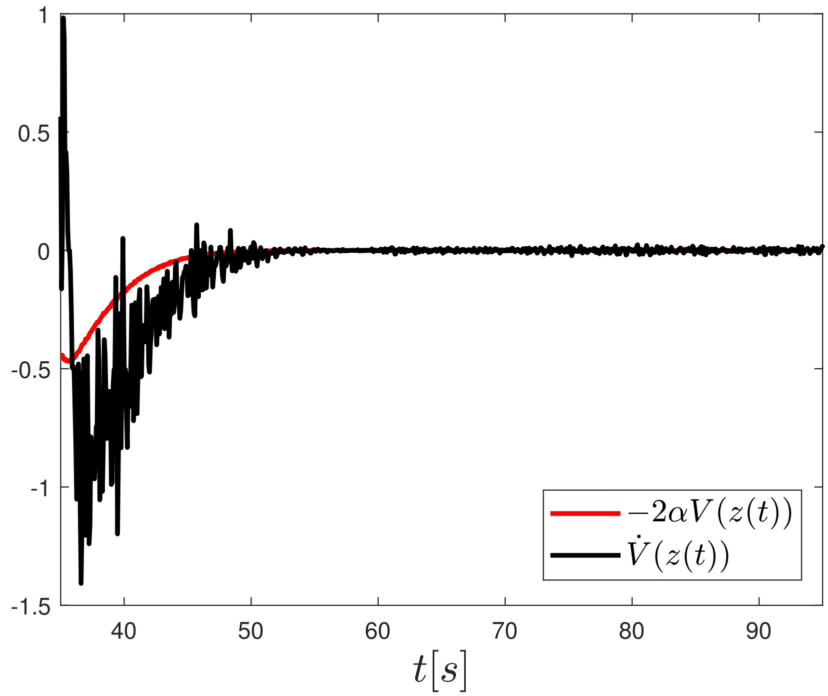


**Fig. 4.** Comparison between $-2\alpha V(z(t))$ and $\dot{V}(z(t))$.

## 6. Conclusions and future work

This paper has described an LMI-based approach to the design of nonlinear controllers for a class of systems which can been reshaped in a quadratic structure via the Taylor-Carleman bilinearization. Although restricted to first-order systems, thus somewhat limited in the scope of application, this paper has discussed several theoretical challenges and proposed a set of LMIs that can provide a feasible design. The simulation results obtained with an open-loop unstable nonlinear system and the experimental results obtained with a water tank have illustrated the application and performance of the proposed approach. Our hope is that the results contained in this paper will act as a foundation stone for a more general LMI-based design procedure that can be applied to higher-order systems, which will be the subject of future research in the area.

## Declaration of Competing Interest

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest, or non-financial interest in the subject matter or materials discussed in this manuscript.

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