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In this paper, we propose a methodological approach—the SCS cycle— suitable to develop prospective teachers Interpretative Knowledge (IK). In particular, this study focus on a group of 19 prospective secondary teachers attending a mathematics education course in Italy who were given an interpretative task involving measurement of the surface area of a rectangle. Their work on the task followed the SCS cycle and was video recorded and later transcribed. The analysis showed that the SCS cycle supported the development of the prospective teachers' IK, but further work is needed to evaluate the effectiveness of the cycle as a way for developing prospective teachers' IK.

Keywords: Interpretative knowledge, SCS cycle, measurement, surface area.

Introduction

Measurement (e.g., of length, area, volume, mass) is an integral part of the school mathematics curriculum (e.g., NCTM, 2000). However, how measurement should be taught for relational understanding (e.g., Skemp, 1976) is often neglected by teachers, as this topic is typically approached as a problem of "finding the correct number" by using a mathematical procedure, indicating that the main goal is finding the final result instead of developing relational understanding.

In order to enrich the understanding of a mathematical topic and use such understanding in the development of fruitful mathematical discussions with students, teachers need to possess a mathematical knowledge which is considered specialized to the practice of teaching mathematics. One of the teaching tasks entailed in the work of teaching is grounded in proposing and discussing tasks with students (Ball et al., 2008). We assume that developing students' relational understanding should be one of the goals of all mathematics teaching, and thus, it is critical to consider the starting point for a mathematical discussion on what the students know and how they know it. This requires that teachers can "listen to the students' thinking" and possess what we have termed as Interpretative Knowledge (IK), which is a specialized kind of knowledge that is not necessarily developed in teaching practice, and is thus an essential focus of teacher education (Mellone et al., 2020).

In our previous work, we discussed the nature and content of (prospective) teachers' IK on the topic of area measurement (e.g., Ribeiro et al., 2018) posed in a context aimed at giving meaning to the area formula for the rectangle. For that purpose, we developed an ad hoc interpretative task. In such a task, prospective teachers (PTs) are situated in a practice-based context where they have to mediate and give meaning to different students' reasonings and justifications for the formula for calculating the surface area of a rectangle. In particular, by proposing this task, we challenged PTs to give a mathematically meaningful justification for the area formula of the rectangle as a direct reading of multiplying the measurement of the length by the measurement of the width. In pursuing the goal of developing PTs' IK, an ad hoc way of implementing the task was developed. We call this method the Small group−Collective−Small group (SCS) cycle, as it is a modification of the Individual−Collective−Individual cycle proposed by Pacelli et al. (2020). In this paper, we focus on the following research questions: What kind of knowledge development can be recognized in PTs when experiencing an SCS cycle involving an interpretative task? In particular, can we recognize any development of IK in these PTs?

Literature review and theory

Understanding a mathematical topic goes beyond knowing the mathematical procedures associated with the topic. This also applies to measurement, considering that it is one of the core mathematical topics in the school curriculum since kindergarten (e.g., NCTM, 2000). For this reason, it is of fundamental importance to develop PTs' and students' understanding of the measurement process in general, and in particular in the context of surface area. Such an understanding can be described by the six principles defined by Clements and Stephan (2004) for the length measurement, which can also be adapted for other magnitudes.

In traditional school practice, the teaching of surface area measurement tends to focus on formulas without meaning, sometimes preceded or followed by teaching of the measurement process where only standardized measurement units are used (Policastro et al., 2017). Moreover, no opportunities are provided for exploring with students the differences and similarities concerning the measurement processes based on different magnitudes. For example, looking at similarities among the measurement processes, it is possible to recognize a common planned action of choosing a convenient unit of measurement to be compared to the quantity being measured, ensuring that both have the same magnitude, and counting how many times the unit of measure fits in the quantity to be measured (e.g., Clements & Stephan, 2004).

The Dynamic Measurement approach is an alternative method of surface area measurement (Parnorkou, 2020). The approach focuses on how space is measured by the lower-dimensional objects that generate it. An inductive approach to visualizing this generation of area (and volume) attributes involves moving objects in space (Parnorkou, 2020). By imagining that a line segment 'a' is swept in a perpendicular direction across a distance 'b', we generate a rectangle with area 'ab'. These two different approaches demonstrate the richness and complexity of a mathematical topic, such as measuring the surface area of an object that is usually considered a trivial task. From this perspective, we argue that teachers should possess a sound and broad mathematical knowledge that contributes to the development of students' mathematical knowledge. Such knowledge will allow them to take the students' own mathematical work, and the differences in the provided representations and argumentations as a starting point—including mathematical work that contains mathematical ambiguities, errors, and non-standard reasoning—assuming that they can be used in practice as learning opportunities (Borasi, 1996). In this sense, the notion of IK (Jakobsen et al., 2014) refers exactly to this deep and wide mathematical knowledge that enables teachers to support students in building their mathematical knowledge by starting from their own reasoning and productions (Di Martino et al., 2019). It includes the ability to expand one's own space of solutions by looking at situations from a wide range of points of view and the capacity for developing specific feedback based on the meaning ascribed to individual students' reasoning (Jakobsen et al., 2014).

Our aim is to develop a methodological approach to support mathematics teachers in the development of IK. Extant research indicates that having in/pre-service teachers working on interpretative tasks can be an effective tool for this purpose (Mellone et al., 2020). By interpretative task we mean a task that in which we ask teachers to solve a mathematical problem and then to interpret students' answers to the same problem (Jakobsen et al., 2014). However, even if PTs' individual work on the interpretative task can contribute to the development of new insights and awareness, it is insufficient for developing a sound IK. As a consequence, referring to the design study methodology (Cobb et al., 2009), we designed an ad hoc methodology for implementing an interpretative task based on the SCS cycle. As implied by the SCS, teachers first work on the interpretative task in small groups of two or three members, after which the task is discussed by all participants in a collective discussion mediated by the teacher educator. Finally, the same small groups of teachers are asked to work on the same interpretative task after one month. This methodology aims at disrupting the vicious cycle of teachers being passive listeners by prompting them to assume an active role in their learning—a strategy we expect they can transpose to their practice.

Moreover, it is widely established that, when teachers work and learn through collaboration, this can have a crucial positive effect on their practices (e.g., Jaworski et al., 2017). Thus, we used the collective mathematical discussion as a collaborative and knowledge-generating activity, in which students' productions are placed at the center of interpretation and feedback construction (e.g., Cobb et al., 2009). Perceiving knowledge as a social elaboration (e.g., Bartolini Bussi, 1996) and recognizing the crucial role of collective discussions in developing awareness about errors and nonstandard strategies (e.g., Levin, 1995) convinced us that we need to allow PTs to be active participants in the learning process and not passive listner, as "when they do talk they ask clarifying questions or acknowledge that they agree or understand" (Spillane, 2005, p. 394). By focusing on the collective discussions about students' productions related to a mathematical problem, our intention is to develop PTs' (teachers') IK from their mathematical social interactions with peers. The task for teacher education (Ribeiro et al., 2021)—an interpretative task in this case—is used both to measure the PTs' IK level and to stimulate subsequent peer discussions. Owing to its nature and structure, the interpretative task aims at prompting PTs to develop novel insights into the mathematical reasoning involved in students' productions. Consequently, IK development is transformed from an individual to a collective activity—a transformation characterized by the evolution of community's norms. This evolution is facilitated by the social setting, where the educator's knowledge is a crucial element for the development of PTs' IK. The collective discussions of PTs' diverse interpretations, reasoning, and reflections upon students' productions is the resource for the educator to orchestrate collective discussions, aimed at identifying mathematical and pedagogical insights and developing the IK. The ultimate goal of this strategy is an evolution from a group of PTs into a professional teaching community (e.g., Cobb et al., 2009), which requires a set of four types of norms pertaining respectively to: (a) general participation; (b) pedagogical reasoning; (c) mathematical reasoning; and (d) institutional reasoning. It is worth noting that the evolution of one type of norms creates conditions within the group for the evolution of norms of another type (Cobb et al., 2009).

Context and method

The context of this study is a Mathematics Education course held in the Autumn of 2020 as a part of a Master's Degree in Mathematics at an Italian University. The requirement for becoming a secondary mathematics teacher in Italy is to have Master's Degree in Mathematics (or Physics) and to successfully pass a public competition organized by Italian government for secondary school teacher recruitment. The students had already completed a Bachelor's Degree in Mathematics or in Physics and are considered to have a strong mathematical knowledge. The Mathematics Education course is a non-compulsory course, but it is typically chosen by students who intend to become secondary teachers, due to which all participants are considered to be PTs. The study participants are the 19 PTs who attended the course which was held online in a synchronous way—through Microsoft Teams—due to the restrictions imposed on mobility and gatherings to prevent the spread of COVID-19. The online teaching was recorded.

Task and activities

The interpretative task we discuss here was proposed to the PTs during the final part of the course. It consisted of three parts. First, the PTs were asked to answer a generic question on how to define the area of a figure in a plane. Then, there they were asked to find the area of a rectangle with sides measuring 3 cm and 4 cm, to provide an argument for their answer, and relate their answer to the first question. Finally, they were asked to interpret four 5th graders' productions to the area of the rectangle problem (Figure 1), focusing on "listening to the students' thinking and reasoning" to make sense of their solutions, and provide a constructive feedback to each one of those reasonings to support students' mathematical understanding (Ribeiro et al., 2018). One of the aims of the task was to discuss the meaning of the area formula for the rectangle and to refine the PTs' understanding and meaning attribution to the product of two lengths.

Consider the following students' productions to the question: Determine, and justify, the area of a rectangle with sides measuring 3 *cm* and 4 *cm*.

Caio: Multiplying the length by the width, we get $4\ cm \times 3\ cm = 12\ cm^2$.

Douglas: The area is a surface measurement and thus it has two dimensions (length and width) so we need to put the 2 in the exponent and we get $3 \times 4 = 12\ cm^2$.

Camila: We just need to count the number of square centimeters needed to cover the square, and thus we get $3\ cm^2 \times 4\ cm^2 = 12\ cm^2$ or, similarly, $4\ cm^2 \times 3\ cm^2 = 12\ cm^2$.

Fernanda: I think the area is $12\ cm^2$ as we have to do $4 \times 3\ cm^2 = 12\ cm^2$ or $3 \times 4\ cm^2 = 12\ cm^2$.

Figure 1: Students' productions included in the interpretative task

Although all students' numerical answers are correct (12 cm^2), their reasoning and argumentation differ and are associated with different interpretations of area, area units, and the meaning associated with the formula (A = length × width).

This interpretative task was implemented in three phases using the SCS cycle methodology. It was implemented in the middle of the master's course and the interpretative activity was proposed to the

PTs during an online class using the “Activity” function of the Microsoft Teams platform. Using the “Breakout rooms” function, the PTs were divided into small groups (of two or three members) for working on the task for one hour. At the end of this period, they had to deliver to the educator a shared written interpretation using the “Activity” function. Upon completion of this phase, the educator orchestrated a 90-minute collective discussion of the task for all the PTs using Microsoft Teams (phase two). In the final phase, the PTs were given Parnorkou’s (2020) paper as a reflection on the Dynamic Measurement approach for measurement of surface area. The PTs had one month to read the paper, reflect upon the task and co-write, using the same small working group as in phase one, a reflection on the lived experience and to hand it in to the educator. This phase was also part of our goal as educators to constitute a professional teaching community (Cobb et al., 2009).

When conducting the analysis of the PTs’ productions in the three phases of the SCS cycle, we focused on identifying the knowledge mobilized. In what follows, we present our analysis of one group’s work during the first and third phase, while attempting to trace the knowledge and awareness developed during the collective discussion (phase two).

Interpretative knowledge revealed and developed – some discussions

Danilo, Pietro, and Caterina, as a group, wrote in the first phase of task implementation (before the collective discussion) the following:

> Caio gave the standard definition and therefore it is not clear to us if he actually understood the meaning of the operation or if he simply applied a definition he had memorized. We would recommend a graphic approach in which the sides are divided into respectively 3 and 4 equal parts and from there we can see that each of these forms a square with a unitary area.

In some sense, we can see that these students are mature in giving feedback. In particular, in their suggestion to divide each side in equal segments—hence linking up to the idea of making a regular grid of squares with unit area—we can recognize their effort to help Caio to link his calculations with the meaning of covering the surface of the rectangle with unit squares.

The idea of making regular grid also emerged in the collective discussion:

Marco:	In my opinion, if we talk about units of measurement and therefore the symbolic expression, only Caio wrote well. The others have all made a mistake, either because they didn't write the unit of measure or because they added too many of them.
Rino:	In Fernanda’s case, I don't see formal errors.
Marco:	In my opinion, Fernanda is wrong because you can't write like this for units of measurement.
Rino:	The area is expressed in square centimeters, and she took four of them.
Pietro:	It is as if she had taken four strips three high, or three strips four long and covered the rectangle. It could be interpreted like this, obviously I don't know if that's what she thought.

We can see that there is a collective effort to give meaning to Fernando’s answer, and to create a link between Caio’s and Fernanda’s answers by making a regular grid covering the rectangle. After this, the educator prompted the PTs to choose between Caio and Fernanda:

Educator:	If you have to write on the blackboard which formula would you write?
Caterina:	If I had to explain the area, I would always use the covering technique. I would try to highlight this concept every time because it seems to me the central one in the measurement theory. But obviously I also like a unit of measurement written each time and, therefore, a dimensional analysis like Caio's works.
Danilo:	But in some way also Caio is a covering if you mean a covering of thin strips, as many strips that would be the height for how long the base is.
Caterina:	But in fact he did not make any mistake, he wrote well, if one really has to go to the bottom and ask what the area is, then I would like a child to understand that it is a covering with small cells at will.
Danilo:	But you must also go and explain what the elementary area with which he covers is.
Educator:	I believe that what Danilo tried to do by reading Caio's formula is to define this elementary square with a side of 1 cm and see it as generated by a segment of one cm which is repeated continuously a certain number of times. This is strange to say sometimes because it is a continuous movement, then cover it with very small strips in this case 1 high, which is precisely the fundamental theorem of integral calculus.
Danilo:	Yes, it was how I imagined it when I was as a child.
Educator:	So you already had a vision of the fundamental theorem of integral calculus and you didn't know it.

The educator's provocation was overcome by Caterina proposing to look at the area first as a covering process (referring to Fernanda's answer), but also expressing appreciation for the dimensional analysis present in Caio's answer. Caterina's comment created the opportunity to Danilo to present his crucial observation that gave new insight into Caio's answer. He proposed interpreting Caio's answer also as a covering process, but performed by using a "thin" linear segment to repeat for "how long the base is." We can recognize a link between this interpretation provided by Danilo and the Dynamic Measurement approach mentioned earlier that relies on visualizing the generation of a rectangular area by mentally sweeping a segment corresponding to one side of the rectangle along its perpendicular direction, in other words along the other side, for a distance corresponding of the length of the side (Parnorkou, 2020). It is important to underline that this interpretation of Caio's answer was not presented by Danilo in the previous phase in which the group comprising of Danilo, Pietro, and Caterina just expressed the grid covering perspective, and this represents an evolution of the mathematical reasoning norms (Cobb et al., 2009). It is also noteworthy that the evolution of one type of norms created conditions within the group for the evolution of norms of another type. In particular, the evolution in this mathematical reasoning norm also created an evolution in the pedagogical reasoning norms in the sense that the PTs were also expanding their space of solutions by looking at the situation from a wide range of different points of view, consistent with the IK approach.

Caterina's emphasis of the importance of considering the covering process as the underlying meaning of the area measurement shows that she has in fact failed to grasp Danilo's point of view. Still, the fact that Caterina does not understand gives Danilo a new opportunity to advocate for his point of view, challenging Caterina's suggestion for how to define the area of the square, used as unit of measure. The educator tried to respond to Danilo's challenge by proposing that PTs visualize generation of the area of the square (used as a unit of measure) using Danilo's dynamic way of looking at it, and by making an explicit reference to the Fundamental Theorem of integral calculus behind this vision.

One month after Parnorkou's (2020) paper was provided to all the PTs, Danilo, Pietro, and Caterina wrote a new interpretation, part of which is replicated below:

> Beyond some errors concerning the units of measurement, two different approaches emerge from the words of the students, already known in literature: that of covering and that of dynamic measurement. […] In any case, it is important to underline how the teacher must be able to recognize these two different approaches, consider them equally valid, but choose a starting one to present to the class, and then show equivalence with the alternative approach (perhaps following a discussion in the classroom from which this dichotomy may emerge).

In this excerpt, we appreciate the careful and effective summary made by this group of PTs of their new knowledge, which emerged from their participation in the SCS cycle and by working on this interpretative task. We stress that this final writing represents an important part of the cycle. By completing this written reflection, the PTs were able to organize their new IK. In particular, in the text provided by Danilo, Caterina, and Pietro, we see how their initial IK has been developed and enriched by the two approaches to surface measurement, and they are now able to present it in organized manner. The awareness of the possibility of approaching the measurement of the surface in two ways represents an evolution of the PTs' mathematical reasoning norms. This also prompted an evolution of their pedagogical reasoning norms (Cobb et al., 2009), as evident from this quote from one of the PTs: "the teacher must be able to recognize these two different approaches, consider them equally valid."

Some final comments

In this study, we have found that the adoption of the SCS cycle methodology has supported the IK development among the PTs. This is an initial study, and we propose that further research into the effect of the SCS cycle on developing the IK among PTs be conducted.

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