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Teacher time out as a site for studying mathematical knowledge for teaching[☆]

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ABSTRACT

The special mathematical knowledge that is needed for teaching has been studied for decades but the methods for studying it have challenges. Some methods, such as measurement and cognitive interviews, are removed from the dynamics of teaching. Other methods, such as observation, are closer to practice but mostly involve an outsider perspective. Moreover, few methods tap into the tacit and often invisible demands that teachers encounter in teaching. This article develops an argument that teacher time outs in rehearsals and enactments might be a productive site for studying mathematical knowledge for teaching. Teacher time outs constitute a site for professional deliberation, which 1) preserves the complexity and gets inside the dynamics of teaching, where 2) tacit and implicit challenges and demands are made explicit, and where 3) insider and outsider perspectives are combined.

1. Introduction

For decades, researchers have agreed on the importance of mathematical knowledge for teaching, but their methods for studying it have changed. When reviewing early research on teachers' mathematical knowledge, Begle (1979, p. 28) noted that, "it seems to be taken for granted that it is important for a teacher to have a thorough understanding of the subject matter taught. The question is never raised in these studies as to how thorough the understanding needs to be." The main approach to studying teacher knowledge at the time was to consider the number of credits or courses taken in mathematics. This approach to studying mathematical knowledge for teaching was common within the process-product paradigm that dominated research on teaching at the time (Gage, 1963). For decades, research focused mostly on process variables and ignored the role of content in teaching, to the extent that Shulman (1986) identified this as a "missing paradigm" in research on teaching. As a response, Shulman proposed three types of content knowledge that matter for teaching, and pedagogical content knowledge gained particular attention as a special type of knowledge that integrates content and pedagogy. Whereas earlier research focused more on the *amount* of knowledge teachers have or need (Begle, 1979), studies following Shulman (1986) began to focus more on understanding what is special about mathematical knowledge for teaching and to explore new ways of investigating it (cf. Ball, 2017).

Numerous studies have incorporated the ideas of Shulman in mathematics education (for reviews, see Depaepe et al., 2013; Hoover et al., 2016), and the work of Deborah Ball and her colleagues has been particularly influential (Askew & Venkat, 2020; Depaepe et al.,

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2013). For a long time, efforts have been made to develop instruments for measuring mathematical knowledge for teaching (e.g., Hill et al., 2004), and such measures have provided useful operationalizations of the ideas about mathematical knowledge for teaching and served to test assumptions made (Hoover et al., 2016). For instance, measures have been used to demonstrate connections between mathematical knowledge for teaching and the quality of instruction (e.g., Hill et al., 2012), and to explore possible influence on student learning (e.g., Hill et al., 2005; Hill et al., 2011). Reflecting on decades of research on mathematical knowledge for teaching, however, Ball (2017) identified several challenges of measurement, and we highlight three of these challenges as starting points for our study. The first challenge lies in the complexity of teaching. Measures tend to simplify this complexity and are removed from the dynamics of teaching, and Ball calls for new methods that get more inside of the work of teaching. The second challenge is that many aspects of mathematics teaching are tacit and taken for granted, while the third challenge is that methods for studying mathematical knowledge for teaching often involve an outsider perspective on teaching, whereas Ball claims that methods should productively consider both insider and outsider perspectives. With these challenges in mind, we aim to contribute to the discussion of methods for studying mathematical knowledge for teaching—not as a substitute for existing methods, but as additional methods that might shed light on other dimensions of mathematical knowledge for teaching. We discuss how teacher time outs may constitute a productive site for studying mathematical knowledge for teaching. The teacher time out routine will be further unpacked in section three, but, in short, teacher time outs allow participants to pause instruction to think out loud in the moment of teaching and discuss how the teacher might respond to student contributions and determine where to direct the instruction (Gibbons et al., 2017; Wæge & Fauskanger, 2021, 2022).[1]

Below, we first elaborate on trends in research on mathematical knowledge for teaching and clarify our own conceptual foundation. We then elaborate on previous research on teacher time outs and identify the potential for using them to study mathematical knowledge for teaching. Finally, we use illustrative examples of teacher time outs in rehearsals and enactment as a foundation for discussing how teacher time outs might provide a productive site for studying mathematical knowledge for teaching.

2. Methods for studying mathematical knowledge for teaching

In their review of research on mathematical knowledge for teaching, Hoover et al. (2016) distinguish between studies that focus on the *nature* of mathematical knowledge for teaching, *development* of this knowledge, *influence* of this knowledge on teaching or learning, and studies with *other* focus—such as investigations into what teachers know. When considering more recent research on mathematical knowledge for teaching, we notice similar tendencies as the ones identified by Hoover et al. (2016). Some studies focus on exploring what mathematical knowledge for teaching is, whereas other studies explore connections between different aspects of mathematical knowledge for teaching, or with other variables. Many studies use different kinds of measures to investigate mathematical knowledge for teaching. For instance, Copur-Gencturk, Plowman et al. (2019) make use of the Learning Mathematics for Teaching (LMT) measures that were developed by Ball and colleagues to explore what aspects of professional development that influenced teachers' mathematical knowledge for teaching. Other studies have used measures from the Measures of Effective Teaching (MET) project (e.g., Copur-Gencturk, Tolar et al., 2019). Copur-Gecturk, Tolar, et al. (2019) write that "the design and formats of the MET items were almost identical to the items originally developed by the MKT developers" (p. 489). Other studies apply content measures from the Teacher Education and Development Study in Mathematics (TEDS-M) (e.g., Yang et al., 2020). Various measures of mathematical knowledge for teaching are often used in combination with other instruments that target other aspects of mathematical knowledge (e.g., Charalambous, Hill et al., 2020), or beliefs (e.g., Corkin et al., 2015). Whereas the above-mentioned measures often have a paper-and-pencil format, some studies apply cognitive interviews to investigate mathematical knowledge for teaching (e.g., Jacobson et al., 2018), sometimes in combination with measures or observations (e.g., Henderson Pinter et al., 2018). Yet others explore variations in the item format of measures (e.g., Fauskanger, 2015; Gómez-Torres et al., 2016).

Whereas most studies on the nature of mathematical knowledge for teaching apply measures or cognitive interviews, other studies explore alternative methods. A few studies apply vignettes or examples of student work. For instance, Fernández and Figueiras (2014) analyze a problem along with two examples of student work on this problem to explore the construct of horizon content knowledge. By using this approach, they unpack what kind of considerations a teacher might have when using this type of problem, and the mathematical entailments of dealing with different kinds of student approaches to working on such a problem. In another study, Hauk et al. (2014) also apply classroom vignettes in their study of pedagogical content knowledge. By applying these vignettes, which consist of a context description that includes the mathematical problems worked on, and transcripts, they situate their discussions in close proximity to the teaching of mathematics, but their underlying conception of mathematical knowledge for teaching differs significantly from that of Fernández and Figueiras (2014). Where Fernández and Figueiras explore types of situations that might occur and the *demands* of these situations, Hauk et al. (2014) use the vignettes to identify how a particular teacher *uses* their knowledge in teaching.

Many studies of teacher knowledge apply task-based interviews, and they often focus on evaluating participants' (lack of) understanding (e.g., Novikasari & Darhim, 2015). In her investigation of mathematical knowledge for teaching proof, Lesseig (2016) also observes teachers' work on tasks and their discussions of these tasks, but she makes a subtle distinction. The purpose of her analysis is "not to infer what individual teachers did or did not understand about proof or proving, but to explore the nature of teachers'

[1] Researchers use different terms, for example time outs (e.g., Gibbons et al., 2017, p. 28; Wæge & Fauskanger, 2021, p. 564), teacher educator/novice teacher exchanges (e.g., Lampert et al., 2013, p. 230), rehearsal debriefs (Munson et al., 2021), and pivotal teaching moments (Stockero & van Zoest, 2013, p. 127), to refer to similar phenomena.

mathematical activity" (p. 13). Where Novikasari and Darhim (2015) make claims about teachers and their level of understanding, Lesseig (2016) makes claims about the *nature* of mathematics teaching. Unlike Fernández and Figueiras (2014), who make an argument about the nature of knowledge in relation to the problems and demands of teaching, Lesseig (2016) uses analysis of proving activity as the basis for a theoretical claim that specialized knowledge "would be *useful for* and *useable in* teaching" (p. 22, emphasis added).

Mitchell et al. (2014) investigate the mathematical knowledge needed for teaching with representations. They analyze four teaching episodes with the aim of identifying "knowledge demands entailed in teaching with representations" (p. 38). In a first phase of the analysis, Mitchell and colleagues focus on what teachers do while teaching with representations, and how teachers and students interact. They then shift attention to identify "tasks entailed in teaching" and "infer the knowledge demands of this work" (p. 43). Instead of evaluating what teachers do, they aim to understand the requirements of the work of teaching. They refer to their work as developing a "teacher-knowledge conceptualization with respect to a certain teaching practice—that of using representations—-through close scrutiny of actual instruction" (p. 55). In this study, the researchers analyze videos of practice, along with teacher interviews, to investigate the entailments of practice. Their approach enables them to get close to the dynamics of teaching. However, even with the inclusion of teacher interviews, their analysis is limited to an outsider perspective on teaching and its demands.

The challenge of proximity to practice is often referred to, and this is one of the challenges that Charalambous (2020) aims to mitigate by using animated teaching simulations to study mathematical knowledge for teaching. Various efforts have already been made to implement simulations in research and teacher education contexts; Charalambous builds on the work of Herbst and Chazan (2006) to apply animated simulations that involve simple cartoon characters. Instead of asking participants to write down or explain how they would react to situations that occur in the simulation, the participants are asked to take the role of the cartoon teacher and interact with the animated students. Charalambous (2020) argues that this provides a more authentic environment and thus addresses "the challenge of *proximity* since it offers opportunities for capturing teacher knowledge that is proximal to the actual work of teaching" (p. 234, original emphasis).

In their discussion on how research on mathematical knowledge for teaching has developed, Hoover et al., (2016, p. 20) suggest that "a central problem for progress in the field is a lack of clearly understood and practicable methodology for the study and development of mathematical knowledge for teaching." Whereas early research in the area emphasized correlational studies, Hoover and colleagues notice that the field has later shifted toward use of interviews. These interview studies tend to focus on shortcomings in teachers' knowledge instead of identifying demands that are embedded in the mathematical work of teaching. They further argue that observational studies tend to be poorly specified. Hoover et al., (2016, p. 23) find it promising to "use sites where professional deliberation about teaching are taking place as sites where we might productively research the work of teaching and its mathematical demands." They suggest that development of measures might be one such productive site. Although development of measures might be productive, it is the end product that comes from the use of measures that often receives most attention, and an inherent challenge of measurement is that it might draw attention away from the dynamic and situated knowing and doing that is fundamental to the work of teaching (Ball, 2017).

In our efforts to explore methods for studying mathematical knowledge for teaching, we follow Ball et al. (2008) when our focus is on *teaching* rather than *teachers*. We consider mathematical knowledge for teaching to be concerned with "the tasks involved in teaching and the mathematical demands of these tasks" (Ball et al., 2008, p. 395), and we are inspired by Ball's (2017) efforts to unpack the mathematical demands that are entailed in the work of teaching while staying close to the dynamic and interactive nature of teaching. We agree with Hoover et al. (2014) that efforts to identify mathematical tasks of teaching may provide a foundation for studies of mathematical knowledge for teaching. Tasks of teaching constitute decompositions of the work of teaching—they are small slices of the complex work of teaching mathematics—and they have two defining features. First, tasks of teaching refer to "something teachers routinely do" (Ball et al., 2008, p. 400). One example is to select mathematical representations to be used for a particular purpose. This is a task that mathematics teachers are regularly faced with in their work. Second, tasks of teaching involve special mathematical demands (Ball et al., 2008). Selecting a mathematical representation that supports young students' developing understanding requires special mathematical understanding of the mathematical idea that is being represented as well as of strengths and limitations of different representations of this idea.

With this underlying conception, our focus is not on observing and trying to make sense of the actions that particular teachers make, nor to try to understand teachers' thinking or cognition, but instead to use observations of teaching as a starting point for identifying tasks of teaching that teachers might be routinely faced with and consider the mathematical demands of these tasks. We believe that teacher time outs constitute a particularly useful site for studying this. The following section provides another building block for this argument when we elaborate on teacher time outs and how they have been studied in previous research.

3. Research on teacher time outs

Teacher time outs are embedded within learning cycles of enactment and investigation (e.g., Kazemi et al., 2016), which constitutes a practice-based approach to teacher education and professional development. In learning cycles, participants engage in observation, analysis, preparation, rehearsals, and co-enactments. Since time outs are mainly used in rehearsals and co-enactments, we focus on these two phases of learning cycles in this article. In rehearsals, teachers lead an instructional activity with colleagues acting as students; in co-enactments, the instructional activity is tried out with real students. In both settings, participants can pause instruction by initiating a time out so they can collectively consider decision-making in the moment of teaching.[2] A time out is thus the point in time when teaching is paused so that the teachers and teacher educators can ask questions, think out loud together, and consider how the teacher might respond to the students' ideas and determine the direction of the further instruction. When appropriate, instruction continues (Gibbons et al., 2017; Wæge & Fauskanger, 2021, 2022). A teacher time out thus allows participants to make sense of their experiences by shifting positions between teacher and students in rehearsals and between teacher and observers in co-enactments. In both settings, time outs enable participants to discuss and try out core practices entailed in the work of teaching mathematics. Yet, there is a difference between the two settings. Time outs in rehearsals often involve longer and richer collective considerations on possible teacher strategies and moves, whereas time outs in co-enactments often consists of shorter exchanges and specific suggestions (Wæge & Fauskanger, 2022).

Although practice-based approaches to teacher education are generally acclaimed, they are not without challenges. As an example, the practices might become out-of-context, resulting in obscuring equity and justice (Dutro & Cartun, 2016; Philip et al., 2018). These critical perspectives are taken into consideration when resent research on learning cycles focus on teacher reasoning rather than practices alone, and, in teacher time outs, teachers are offered an opportunity to discuss and reason about the role that context plays in decision-making about mathematics teaching (Wæge & Fauskanger, 2021, 2022). Time outs in rehearsals are found to support teachers' learning of "how to respond to students' in-the-moment ideas while teaching and to develop a shared conceptual framework or shared repertoire that can enable them to use the practices and principles adaptively in new situations" (Wæge & Fauskanger, 2021, p. 583). Inclusion of co-enactments adds the student context and complements rehearsals to support teachers' engagement with the complexity of in-the-moment instructional decisions (Wæge & Fauskanger, 2022). Various aspects of teacher time outs in rehearsals and co-enactments have been investigated. Studies have explored how time outs provide opportunities for novices and experienced teachers to learn practices and principles of ambitious mathematics teaching and how the teacher educator supports their learning (e.g., Gibbons et al., 2017; Kavanagh et al., 2019; Kazemi et al., 2016, 2018; Lampert et al., 2013; Shaughnessy et al., 2019; Wæge & Fauskanger, 2021, 2022). Although rehearsals and co-enactments might be structured differently across research projects, findings from these studies indicate that time outs allow participants to engage deeply with the complexity of in-the-moment decision-making and offer opportunities to learn adaptive teaching while developing knowledge, skills, and identities. Moreover, findings indicate that time outs support teachers' engagement with pedagogical reasoning and opportunistic experimentations, and with connecting pedagogical reasoning to pedagogical actions and responsibilities.

When considering previous research, there is thus ample evidence that teacher time outs are productive in the process of *learning to teach* and in *developing teaching practice.* In this article, however, we suggest that teacher time outs may also constitute a productive site for studying mathematical knowledge for teaching. A couple of studies have already begun to explore how teacher time outs can stimulate development of mathematical knowledge for teaching. In the context of rehearsals, Ghousseini (2017) investigates the tasks of teaching that enlist mathematical knowledge for teaching in this setting and how a teacher educator supports this work. The initial step in her analysis is to identify situations when the rehearsals are paused—what we refer to as a time out—and she then identifies tasks of teaching that are elicited in the two instructional practices of 1) eliciting and responding to students' mathematical thinking, and 2) representing mathematical ideas. Her analysis indicates that there are frequent opportunities for novice teachers to participate in tasks of teaching that mobilize mathematical knowledge for teaching in rehearsals; posing questions to check for understanding is one example of such a task of teaching that the novice teachers are faced with. In another study, Ghousseini (2021) again investigates teacher time outs in rehearsals. The role of the teacher educator has particular emphasis in her study. Ghousseini points out how opportunities for learning hinge on the opportunities for reflection and action, and she argues that rehearsals provide novice teachers with opportunities to reorganize their mathematical knowledge for teaching and learn when it is most applicable. Both studies mainly focus on the development of mathematical knowledge for teaching in a teacher education context that involves rehearsals, but they also indicate a potential in using teacher time outs as a site for studying the nature of mathematical knowledge for teaching. In particular, both studies indicate that time out situations constitute vantage point for exploring tasks of teaching, and this taps into one of the challenges raised by Ball (2017).

4. Illustrative examples

In the following, we ground our discussion of how teacher time outs might constitute a productive site for studying mathematical knowledge for teaching on some illustrative examples—all of which focus on properties of multiplication and multiplicative properties, which has been highlighted as important in research (e.g., Verschaffel et al., 2007). We emphasize how time outs can be used as a starting point for identifying tasks of teaching and demands that teachers might have to face. These examples present time outs from rehearsals and co-enactments[3] that are embedded within a cycle of enactment and investigation, where a teacher leads an instructional activity while the other teachers and teacher educators (TE) provide suggestions on what the teacher might try to do in a particular situation. There is room for short and long discussions where the participants can consider different aspects of possible teacher strategies and consider reasons or rationales for possible teacher moves (Wæge & Fauskanger, 2021, 2022).

The first two examples below are from the rehearsal of a quick image activity. The mathematical learning goal for the students is to

[2] As learning cycles are developed for participants to learn to teach, teaching (instruction) is seen as included in both rehearsals and co-enactments.

[3] Data are from the *Mastering Ambitious Mathematics teaching* (MAM) project. See Wæge & Fauskanger (2021, 2022) for more about the project.

learn the distributive property of multiplication ($a \times (b + c) = a \times b + a \times c$). The students are shown a quick image (Fig. 1) for a few seconds, and they are then asked about how many dots they see and if they can explain how they found the number of dots.

We join the rehearsal shortly after the lead teacher (LT) has presented two student strategies on the board (Fig. 1). One of the teachers initiates a time out:

1Teacher 2:Should we ask the students to turn and talk before they offer their ideas?

….

8LT:They [the students] can talk a little bit before they start to tell me their ideas.

9Teacher 2:But then there's the risk that they [the students] might copy what someone else has said and won't offer their own ideas. They might feel that the others have found a way that is quicker and [unclear]. Then we might not get that $5 + 5 + 5$ [the sentence at the top of the board].

10LTPerhaps we get less variation now at the start.

11Teacher 4:Perhaps you [LT] should rather use turn and talk when you have some examples on the board and ask if they [the students] see any connections.

12LT:We did say that we wanted to use turn and talk when we wanted to have them work toward our goal [for the lesson], later in the lesson.

13Teacher 2:Write them underneath each other [vertically], or is it better to write it out horizontally and then add?

14Teacher 5: Write the expression.

15Teacher 2: Because then we get there faster.

16LT:That I write [erases what is on the board].

17Teacher 5:$3 \times 5 + 3 \times 4$.

18LT:Right away?

19The others:Yes. (Session three, quick image, group 2, rehearsal).

At the beginning of this time out, Teacher 2 suggests that the lead teacher could ask the students to turn and talk before she presents their strategies on the board (Line 1). This initiates a discussion on whether or not the lead teacher should use the turn and talk approach. One disadvantage they point out is that the students might copy ideas from each other, which may result in less variation in the student contributions (Lines 9 and 10). Teacher 3 suggests that the lead teacher could use turn and talk when she wants to highlight connections between strategies (Line 11). The lead teacher agrees and reminds them that in the planning phase they agreed to use turn and talk to help students to focus on the lesson goal (Line 12). The participants then switch to discussing how the teacher could represent the students' ideas on the board (Lines 13–19). They decide to represent each student contribution as one mathematical expression to help the students make connections between the expressions and to promote their understanding of the distributive property.

The example illustrates the complex work of *leading mathematical discussions* and dilemmas and challenges the teacher might face. In this time out the participants notice a constraint in the suggested turn-and-talk move by addressing a problem of practice that could result from letting the students discuss their strategies with each other at the beginning of the activity. Given that the purpose of the quick image activity is to allow students to share their thinking, asking the students to turn and talk in the beginning of the activity could lead to less variation in the strategies that students present, and consequently narrow the range of strategies and the richness of the mathematics they could work on. They then turn to discuss how the teacher could *represent students' mathematical thinking* on the board to highlight the distributive property of multiplication. This is another example of a mathematical task of teaching. When students share their strategies and thinking, teachers are routinely faced with the task of representing this. Representing students' mathematical thinking requires knowledge of the students and their development, but it also requires unique knowing of the mathematical content that involves the ability to select representations that are both mathematically correct and appropriate for students' mathematical development. In their discussion, the teachers attended to their knowledge of student thinking in relation to the mathematical goal of the activity. The complexity and demands of facilitating, structuring, and leading productive mathematical discussions becomes apparent in this time out. The participants' collective considerations are related to what Ball (2017) describes as "not-doings," which is the work of teaching that involves "to refrain purposefully at a given moment from doing something" (p. 30). The time out situation illustrates teacher decisions or acts that are often invisible to the observer. Similar to findings from Ghousseini (2021), this time out displays affordances and constraints of instructional moves in relation to mathematical knowledge for teaching.

Time outs—when asked for in the moment of teaching, which we consider to include the contexts of rehearsals—thus provide a productive site for studying mathematical knowledge for teaching that is not removed from the dynamics of teaching (Ball, 2017).

The second example is from yet another time out in the same rehearsal. Just prior to the time out, the lead teacher has represented different student strategies on the board, both by writing numerical expressions and by using the quick image (see Fig. 2). Note the two strategies of $3 \times 5 + 3 \times 4$ and 3×9 that are written on the board.[4] We join the rehearsal as the lead teacher (LT) represents the strategy "three times ten minus one" on the board as $3 \times 10 – 1$. The teacher educator pauses the instruction:

1TE:But then the idea of using parentheses is interesting [refers to the expression $3 \times 10 – 1$ on the board].

2Teacher 5:I was just thinking the same thing. I was thinking that this [the expression] with 3×9 [on the board] is an excellent opportunity to get them [the students] to see $3 \times (5 + 4)$. You [lead teacher] can ask them "Does anyone see 9 in another way?" Then we have 5 and 4, and then you [lead teacher] write down $3 \times (5 + 4)$.

[4] In Norway, "•" is used as the multiplication sign. Here we write "× ".

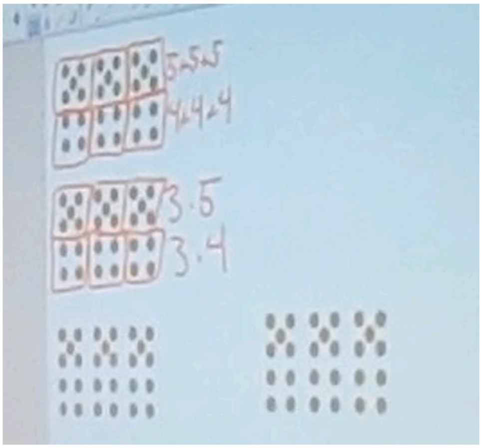

Fig. 1. Student strategies for a quick image represented on the board.

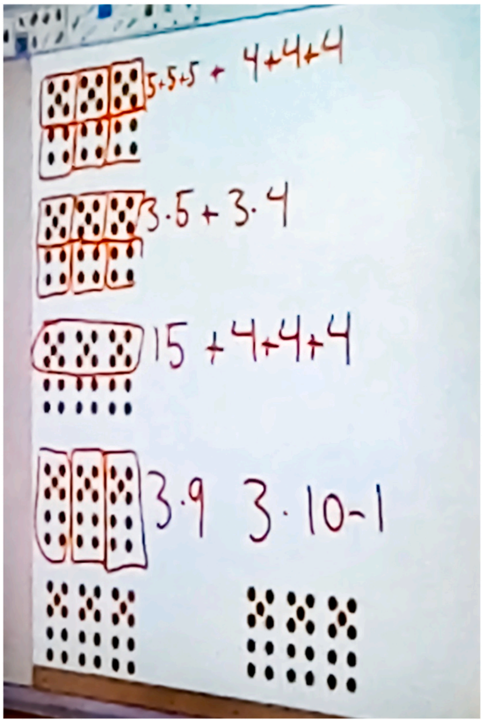

Fig. 2. Student strategies for finding the number of dots in a quick image.

3LT:Yes.

….

[The lead teacher writes 3 × (5 + 4) on the board].

25Teacher 5:And you could have drawn it [3 × (5 + 4)] too. On the [quick image with illustration of 3 × 9], the 4 and 5 together, like you did before. Most likely few see 4 plus 5 in the parenthesis. That's maybe difficult.

26LT:How would you draw it? [the discussion continues].

(Session three, quick image, group 2, rehearsal).

Teacher 5 points to the representation 3 × 9 as something to address. He suggests that the lead teacher could use 3 × 9 as a starting point for showing that 3 × (5 + 4) = 3 × 5 + 3 × 4, illustrating follow-up questions so they could focus on the distributive property (Line 2). This initiates a discussion about how the lead teacher could use the two strategies to highlight the distributive property (Lines 4–24). Following this discussion, Teacher 5 notes that it might be difficult for students to make the connection between 3 × 9 and 3 × (5 + 4) and suggests that the lead teacher should indicate where the 4 and the 5 are on the quick image representing 3 × 9 by drawing circles around them (Line 25). This leads to a discussion on how the lead teacher could use the quick image to help the students connect the two expressions (Line 26). During the same time out, the participants scrutinize the expression $3 \times 10 - 1$ (on the board) and discuss how the lead teacher could proceed and write a parenthesis $(3 \times (10 - 1))$ to make a mathematically correct representation of the student's idea. One teacher suggests that the lead teacher could write a parenthesis and then ask students if they could think about why she did this.

The example illustrates how the teachers share decision-making in the moment of teaching. The participants' rich considerations about what strategies the teacher might point out, difficulties the students might face, and courses of teacher actions that could support the students' understanding of the distributive property highlight the complexity and the demands that are involved in using students' thinking to *highlight key mathematical ideas*. For example, the participants pointed to and discussed the challenges that students might face in making connections between the numerical expressions 3×9 and $3 \times (5 + 4)$, and they discussed how the lead teacher could use the quick image to support the students in seeing the connection. The example also illustrates how the participants' reasoning and sensemaking might uncover pedagogical dilemmas of the work of teaching. In this situation, as in the previous example, the teachers approach the task of *representing students' mathematical thinking* whilst simultaneously considering the mathematical correctness of the representations. Thus, the example illustrates the complexity and demands of using, representing, and building on students' thinking to promote key mathematical ideas and to achieve the goal of the lesson.

Tasks of teaching are often left tacit and invisible. This example illustrates how time outs provide a site for making some of the complex intertwined moves and practices of leading productive mathematical discussions visible and explicit. In addition, this example illustrates how using time outs as a site for studying mathematical knowledge for teaching might also satisfy the call for developing methods that involve both insider and outsider perspectives on teaching (from Ball, 2017). In the time out, teaching is observed by the teacher educator, thus providing an outsider perspective on the demands of teaching in combination with the experienced demands faced by the lead teacher. In other time outs, the researcher also became involved in the discussion, and thereby entered a dual teacher-researcher role—a role that has been common in other studies (e.g., Ball, 2017; Lampert, 2001).

The final example is from a co-enactment of another quick image activity (see Fig. 3). The mathematical learning goal for the students was to learn the associative property of multiplication $((a \times b) \times c = a \times (b \times c))$.

After having discussed three student strategies for finding the number of dots, a fourth student, Lisa, presented her strategy as seeing "four six times." The lead teacher (LT) circled four dots six times in the image (see Fig. 4).

When invited to the board, Lisa said: "four times six" and wrote $4 \times 6 = 24$. The lead teacher built on Lisa's idea by asking the students how to split the 6 in 4×6 into "another multiplication task." Another student, Peter, answered, "two times three," and the lead teacher wrote $4 \times (2 \times 3)$ on the board. She asked Peter to explain his thinking, and another student, Anne, said: "He divided six by two and that equals three, and then he must have two three times to get six." The lead teacher responded by saying that she just realized that $4 \times (2 \times 3)$ could not be represented as the six fours in the image (Fig. 4). It appears from what follows that the instructor imagined 2×3 as six dots, as on the dice, and was confused. She then asked Elisabeth to draw six in a 'clean' quick image (Fig. 4).

The TE initiated a time out:

1TE:I think I see the six in the upper image [pointing at the board (Fig. 3)]. How do you see six in the upper image? What are there six of?

2Lisa:That are circled, or?

3TE:Yes, for example.

4LT:Axel.

5Axel:There are six [groups] of the fours.

6LT:Yes, precisely. There are six [groups] of the fours. Let us now go back and look at the ones [numerical expressions] on the board that I circle $[(4 \times 3) \times 2$, $4 \times (2 \times 3)$ and $(4 \times 2) \times 3]$, the ones with three factors (Session four, quick image, group 2, co-enactment).

The TE took a time out when the lead teacher appeared stuck and asked the students how they might see the six in the upper image

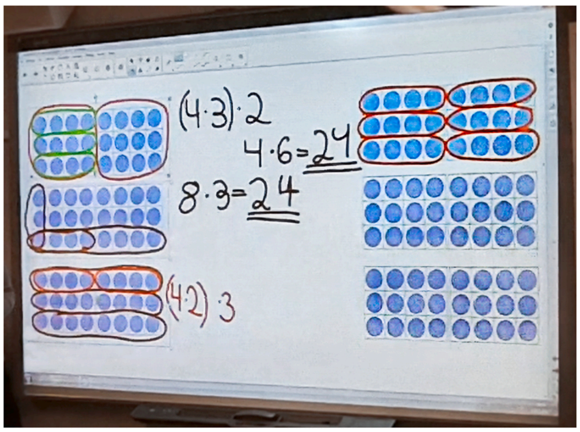

Fig. 3. Student strategies for a quick image represented on the board.

(Line 1). Instead of elaborating on Axel's answer (Line 5) and the fact that six in this quick image could be represented as six groups of four (Fig. 3) as well as six dots (Fig. 4), the lead teacher focused on the numerical expressions with three factors on the board (Line 6).

This example illustrates the complex work of *representing students' mathematical ideas* and challenges the teacher might face. The situation is an example of representing multiplier and multiplicand in a quick image, and the complexity of teaching commutative and associative properties of multiplication becomes apparent. In particular, it illustrates the complexity of truly understanding the connection between numerical and pictorial representations of decomposing the multiplier and the multiplicand. Simply exploring the mathematics of the situation outside of a classroom context of enactment could also provide some insight into the mathematical demands. However, enacting the problem with real students highlights the complexities of what it takes to *do* the work and make decisions on the fly. The time out and the lack of follow up from the lead teacher, emphasizes the complexities and demands even more to the researcher.

In this time out, the teacher educator intervened to support the lead teacher's work of representing students' thinking, and thus provided an outsider perspective in combination with the demands faced by the lead teacher in the interactive work of teaching (Ball, 2017).

5. Discussion

In their discussion of research on mathematical knowledge for teaching, Hoover et al., (2016, p. 23) found it promising to explore "sites where professional deliberation about teaching are taking place" as sites for studying the demands of the mathematical work of teaching. Taking up this proposal involves exploring contexts where teachers can come together to carefully consider or discuss teaching. It is our experience that teachers tend to complain about the lack of opportunity for professional deliberation about teaching; in most professional contexts where teachers get together there is little or no room for careful consideration of or discussion about teaching. Several sites in research and professional development efforts that involve collaboration between teachers and researchers are worth considering. We argue that learning cycles that use teacher time outs constitute a particularly productive site for studying mathematical knowledge for teaching. In the following, we discuss three aspects of teacher time outs that make them interesting for studying mathematical knowledge for teaching.

The first aspect to consider is that teacher time outs make explicit challenges and demands that are often left tacit and invisible. Although teachers often work in teams when it comes to planning of teaching, classroom teaching is normally a solitary practice where individual teachers face and solve challenges and demands that appear on the fly—often without having the opportunity to discuss them with colleagues. The challenges and demands of teaching, and what Ball et al. (2008) refer to as tasks of teaching, are thus often left tacit and invisible (cf. Ball, 2017). The examples in the previous section illustrate how these tacit and often invisible challenges become visible and explicit in teacher time outs. Charalambous (2020) used analysis of interaction with cartoon animation to make such challenges visible, and teacher time outs provide another site for making these tacit demands explicit. In another study, Ghousseini (2017) identified frequent opportunities for novice teachers to participate in tasks of teaching, such as posing questions to check for understanding and that mobilize mathematical knowledge for teaching. We argue that not only are there frequent opportunities to *participate* in tasks of teaching in teacher time outs, but time outs also provide a tool for teachers to *talk about* the experienced challenges and demands they are facing, and thus make them explicit and visible, right when they are facing them. This is a unique aspect of teacher time-out situations that makes them particularly well suited for studying mathematical knowledge for teaching.

The second aspect is the combination of insider and outsider perspectives. Several studies involve analysis of teaching practice to identify knowledge demands or mathematical tasks of teaching (e.g., Mitchell et al., 2014), and the idea of making the tacit challenges and demands of teaching visible is not new. However, such studies tend to mainly emphasize the researchers' outsider perspectives in the analysis of teaching performed by someone else. In some contexts, the researcher has a dual teacher-researcher role (e.g., Ball,

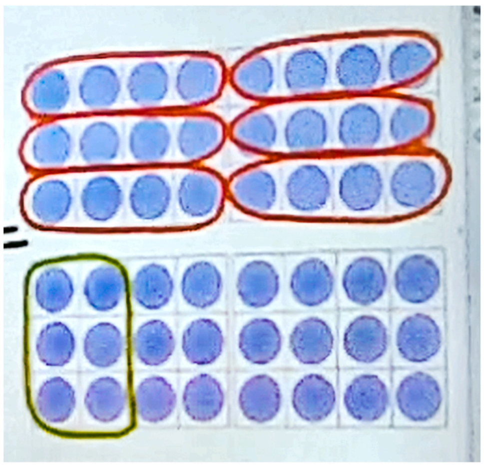

Fig. 4. Six dots as represented on dice.

2017; Lampert, 2001). Such a dual role has several benefits, for instance, it combines the insider and outsider perspectives on the demands of teaching, but it is not always feasible in practice. Teacher time-out situations are unique because they combine insider and outsider perspectives on the *experienced* and *observed* demands of teaching. Where other studies make the tacit challenges and demands of teaching visible by analyzing teaching practice to identify knowledge demands of teaching (e.g., Lesseig, 2016; Mitchell et al., 2014), the examples of teacher time outs above illustrate a more dynamic combination of insider and outsider perspectives. The participating teachers not only identify demands of someone else's teaching as passive observers, but they both attend to these demands from the perspective of observers and teachers who must act on them.

Finally, the third aspect that makes teacher time outs interesting is that they enable proximity to the dynamics of teaching. Many studies of mathematical knowledge for teaching use paper and pencil tests, which are taken when participants are withdrawn from the classroom. Some initiatives aim at closing this gap, but Ball (2017) argued that measurement work still has the disadvantage of shifting attention away from the dynamics of knowing and doing inside the work of teaching. The same can be said of interviews. Although carefully designed interview prompts or measures might be focused and specific, studying mathematical knowledge for teaching by using measures (e.g., Copur-Gencturk, Plowman et al., 2019; Copur-Gencturk, Tolar et al., 2019; Corkin et al., 2015; Fauskanger, 2015; Yang et al., 2020) or interviews (e.g., Henderson Pinter et al., 2018; Jacobson et al., 2018; Novikasari & Darhim, 2015) might have the disadvantage of shifting attention away from the dynamics of teaching. Where measures and interviews tend to simplify the complexity of teaching and are removed from the dynamics of teaching (Ball, 2017), the time-out situations presented above illustrate consideration of the demands of teaching—for instance considering the pros and cons of inviting students to turn and talk in the second example above—in close proximity to the dynamics of the work of teaching. Analysis of vignettes or examples of student work (e.g., Fernández & Figueiras, 2014) can be useful for unpacking the mathematical entailments of the work of teaching, but such analysis is also removed from the dynamics of teaching as compared to teacher time-out situations. Teacher time outs take place in practice while teachers are engaged in doing the work. Studying teacher time-out situations might therefore be useful since these situations are close to the dynamics of doing the work of teaching mathematics.

6. Conclusion

Mathematical knowledge for teaching, as we consider it, is a kind of knowledge or knowing that is entailed in teaching, which means that it is "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 399). Studying mathematical knowledge for teaching is thus fundamentally about investigating tasks of teaching and considering the mathematical demands of these tasks. Ball (2017) argues that methods for studying mathematical knowledge for teaching are challenged by the fact that tasks of teaching are often left tacit and invisible, methods for studying them rarely enable a combination of insider and outsider perspectives, and they are often removed from the dynamics of teaching. In this article, we have argued that teacher time outs may constitute a productive site for studying mathematical knowledge for teaching because they allow the researcher to overcome these challenges. In closing, we want to add that these are not the only challenges, and we are not trying to argue that teacher time outs can or should replace all other methods for studying mathematical knowledge for teaching. Development and use of measures, interview protocols, and other methods, have been and will continue to be important. With this article, however, we have point attention to another site where professional deliberation about teaching takes place (Hoover et al., 2016), and that we think may provide an additional productive method for studying mathematical knowledge for teaching. Time-out situations can be interesting to analyze in retrospect, as we have in this article, but they can also provide opportunities to investigate tasks of teaching in situ. We encourage future studies to explore the rich opportunities for investigating mathematical knowledge for teaching through teacher time outs.

Declaration of interest

The authors have no competing interests to declare.

Data Availability

Data will be made available on request.

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