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Connecting analysis, algebra, and topology; Generalizing Maxwell's equations

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Abstract

This thesis explores the mathematical concepts of differential forms and their applications in higher dimensional geometries, known as manifolds. We will see how the topological invariants of a geometry are related to whether a differential form can be solved or not. We will study some examples to gain an understanding of how the number of solutions to Maxwell's differential equations is related to cohomology groups.

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Contents

1 Introduction

As long as humans has walked the earth, we have had a urge to understand and describe real life experiences. Historians finds mathematical writings far back, and as we have discovered new phenomenons, mathematics has developed in step with the discoveries. Mathematics has been and still is in many ways a language used to explain and model the reality. New discoveries are made every day, and mathematicians and physicists are working together to develop new tools to describe these new discoveries.

Some physical theories presuppose that the mathematics tools to describe these are able to be applied to higher dimensions. The goal for this thesis is exactly to benefit from the connection between some mathematical branches, so it would be easier to work in higher dimensions:

Analysis / Solutions to diff. eq.	$\longleftrightarrow$	Algebra / Cohomology	$\longleftrightarrow$	Topology / Geometry

Differential equations is a very useful part of mathematics and physics because it gives us the opportunity to describe the relationship between the function and the change of the the function. We remember that in order to solve a differential equation there must be specified some boundary conditions, which says something about the geometry of the space we a solving the differential equations on. In higher dimensions, we often use differential forms to describe our differential equations.

Before we will see the connection between differential forms and geometries in higher dimension, which we call *manifolds*, we have to introduce some group theory, which will give us the opportunity to characterize and sort topology of a manifold in terms of differential forms.
Topology is a another part of mathematics where geometric objects are studied and classified based on their ability to preserve its structure under continues deforming. A good way to describe topology is to see the classical example of the torus and the cup:

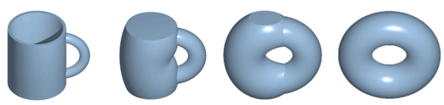


Figure 1:
Original figure form [9]. All structure is preserved under deformation. Torus and the cup has the same topology.

Generally, the topology of a geometry has a lot to say if the differential equation can be solved or not in these particular geometries. For example, it is possible to find the solutions to Maxwell's equations in vacuum on the 2-sphere, but not on the 3-sphere. The number of solutions is related to topological invariant of the geometry, known as the *cohomology groups*. It is often easier to find the cohomology groups than the explicit solutions to the differential equation. Said with other words, we can count number of solutions, even though we do not have a clue how they would look.
This shows a close connection between different parts of mathematics; analysis (differential equations), topology, and algebra (number theory, groups, etc.). This connection is often exploited in modern mathematics, and is for example used to prove that there is a spot on the earth where it is no wind. This would not be the case if the earth would have another topology, such as a torus.

This thesis is organised as follows: Section 2 introduces differential forms, including the wedge product and the exterior derivative. In section 3, we delve deeper into algebra, and present necessary prior knowledge about group theory in order to take advantage of factor groups. In section 4, we explore manifolds and how differential forms are relevant for defining geometric invariants. Section 5 connects the dots between the previous chapters. Firstly, we see how cohomology groups can give us the number of solutions to Maxwell's differential equations. Secondly, we generalise Maxwell's equations to higher dimensions. And finally, we look at the connection between boundary value problem and topology.

2 Differential forms

We follow chapter 17 in [2].
Differential forms is a part of calculus that provides us the opportunity to explore and describe physics and other sciences. Differential forms is a tool for multivariable calculus which is independent of coordinates. Differential forms can be used on curves, surfaces and higher-dimensions, and is therefore a very meaningful tool in calculus. In the following section we will look at the wedge product and exterior derivative, and further on in the thesis we will see what a differential form on a manifold means. We will then see how they can be used to extract geometric invariants of topologies known as cohomology groups.

2.1 Differential k-forms

Definition 2.1. An arbitrary k-from on $\mathbb{R}^n$ is an element of the vector space of differential forms, denoted as $\Omega^k(\mathbb{R}^n)$. Let $\kappa \in \Omega^1(\mathbb{R}^n)$. For $1 \leq i \leq n$. dx_i will be basis that assign κ to its i'th component κ_i. With Einsteins summation notation, a differential k-form can be written as:

$$\kappa = \frac{1}{k!}\kappa_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},\ \in \Omega^k\left(\mathbb{R}^n\right)$$

Although all this may seem very abstract, we are already quite familiar with 0-form, 1-form, 2-from and 3-form in $\mathbb{R}^3$. These are more commonly known as functions or scalars, vectors or line elements, surface elements and volume elements. In vector calculus we normally denote these elements with dx, dy and dz, but in differential calculus we use the notation dx_1, dx_2 and dx_3.

Example From chapter 17 in [2]

a) 0-form: Any arbitrary function $f\ \in \Omega^0\left(\mathbb{R}^3\right)$

b) 1-form: $\alpha = \alpha_1 dx_1 + \alpha_2 dx_2 + \alpha_3 dx_3,\ \in \Omega^1\left(\mathbb{R}^3\right)$

c) 2-form: $\beta = \beta_1 dx_2 \wedge dx_3 + \beta_2 dx_3 \wedge dx_1 + \beta_3 dx_1 \wedge dx_2,\ \in \Omega^2\left(\mathbb{R}^3\right)$

d) 3-form: $\Gamma = \gamma dx_1 \wedge dx_2 \wedge dx_3,\ \in \Omega^3\left(\mathbb{R}^3\right)$

Notice: for $n \geq 4$, $\Omega^4\left(\mathbb{R}^3\right)$ is not defined in the domain.

2.2 Wedge Product

The wedge product, also known as exterior product, is a operator used in differential forms. The wedge product has the following properties [6]:

1. Anti-symmetri:

$$dx \wedge dy = -dy \wedge dx$$

Corollary 2.1. *The wedge product of a 1-form with itself will always be 0*

Proof.

$$\begin{aligned} &dx \wedge dx = -dx \wedge dx \\ &\Rightarrow dx \wedge dx + dx \wedge dx = 0 \\ &\Rightarrow 2dx \wedge dx = 0 \\ &\Rightarrow dx \wedge dx = 0 \end{aligned}$$

□

2. Associative:

$$(dz \wedge dx) \wedge dy = dz \wedge (dx \wedge dy)$$

In a very practical point of view $dx \wedge dy$ will form a square in $\mathbb{R}^3$, and give the normal-vector as an output. It follows from this that the wedge product of dx and dy will point in z direction.

Corollary 2.2. *Given α and β, where $\alpha,\ \beta \in \Omega^1\ (\mathbb{R}^3)$. The cross product of α and β is equivalent to the wedge product of α and β.*

$$\begin{aligned}
&\alpha = \alpha^1 x_1 + \alpha^2 x_2 + \alpha^3 x_3 \\
&\beta = \beta^1 x_1 + \beta^2 x_2 + \beta^3 x_3 \\
&\alpha \times \beta \simeq \alpha \wedge \beta \ \in \mathbb{R}^3
\end{aligned}$$

Proof.

$$\alpha \times \beta =(\alpha^2\beta^3 - \alpha^3\beta^2)\hat{x}_1 - (\alpha^1\beta^3 - \alpha^3\beta^1)\hat{x}_2 + (\alpha^1\beta^2 - \alpha^2\beta^1)\hat{x}_3$$

$$\begin{aligned}
\alpha \wedge \beta =& \alpha^1\beta^1 dx_1 \wedge dx_1 + \alpha^1\beta^2 dx_1 \wedge dx_2 + \alpha^1\beta^3 dx_1 \wedge dx_3 \\
&+ \alpha^2\beta^1 dx_2 \wedge dx_1 + \alpha^2\beta^2 dx_2 \wedge dx_2 + \alpha^2\beta^3 dx_2 \wedge dx_3 \\
&+ \alpha^3\beta^1 dx_3 \wedge dx_1 + \alpha^3\beta^2 dx_3 \wedge dx_2 + \alpha^3\beta^3 dx_3 \wedge dx_3 \\
\alpha \wedge \beta =& \alpha^1\beta^2 dx_1 \wedge dx_2 + \alpha^1\beta^3 dx_1 \wedge dx_3 + \alpha^2\beta^1 dx_2 \wedge dx_1 \\
&+ \alpha^2\beta^3 dx_2 \wedge dx_3 + \alpha^3\beta^1 dx_3 \wedge dx_1 + \alpha^3\beta^2 dx_3 \wedge dx_2 \\
\alpha \wedge \beta =& (\alpha^1\beta^2 - \alpha^2\beta^1) dx_1 \wedge dx_2 + (\alpha^2\beta^3 - \alpha^3\beta^2) dx_2 \wedge dx_3 + (\alpha^3\beta^1 - \alpha^1\beta^3) dx_3 \wedge dx_1
\end{aligned}$$

As mentioned above, $dx_1 \wedge dx_2$ points in the z-direction. Hence $dx_1 \wedge dx_2 \simeq \hat{x}_3$

$$\alpha \wedge \beta \simeq (\alpha^2\beta^3 - \alpha^3\beta^2)\hat{x}_1 + (\alpha^3\beta^1 - \alpha^1\beta^3)\hat{x}_2 + (\alpha^1\beta^2 - \alpha^2\beta^1)\hat{x}_3 = \alpha \times \beta$$

□

It is important to mention that this is only true for $\mathbb{R}^3$, precisely because this is where the cross-product is defined.

2.3 Exterior derivative

We have already established that a 0-form is something we already know as a function in $\mathbb{R}^3$. If we would take the derivative of function $f(x, y, z)$ on $\mathbb{R}^3$ we would get $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$. Notice how by taking the derivative on f, the outcome describes each term in terms of the differential direction dx_1, dx_2 and dx_3.

Definition 2.2. The exterior derivative is a differential operator d which maps a k-form on $\mathbb{R}^n$ to a $(k+1)$-form on $\mathbb{R}^n$:

$$d\ :\quad \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)\ .$$

Explicitly, from definition 2.1 we have $\kappa \in \Omega^k(\mathbb{R}^n)$, d acts as;

$$d\kappa = \tfrac{1}{k!}\partial_j \kappa_{i_1 i_2 \ldots i_k} dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}\ .$$

Example: Applying the operator $d = dx^i\partial_i$ on $V = dx^m V_m \in \Omega^1(\mathbb{R}^3)$, we get:

$$dV = \partial_i V_m dx^i \wedge dx^m \quad = \quad \frac{\partial V_m}{\partial x^i} dx^i \wedge dx^m$$

The exterior derivative has these three properties [6]:

1. The operator d is linear from Ω^k into Ω^{k+1}.
   Let α and $\beta \in \Omega^k(\mathbb{R}^n)$
   $$d(\alpha + \beta) = d(\alpha) + d(\beta)$$
2. If α is a k-form, and γ is a p-form, by applying the product rule:
   $$d(\alpha \wedge \gamma) = (d\alpha) \wedge \gamma + (-1)^k \alpha \wedge (d\gamma)$$
3. $d(dV) = 0$

Proof. We follow the proof of [2]

$$d(dV) = 0$$

$$d(dV) = \left(\partial_j \partial_i V_m dx^j \wedge dx^i\right) \wedge dx^m \;=\; \left(\frac{\partial^2 V_m}{\partial x^j \partial x^i} dx^j \wedge dx^i\right) \wedge dx^m$$

It is already been shown that if $j = i$, then $dx^j \wedge dx^i = 0$. So we set $j \neq i$. And since the $\partial_i\partial_j$ operator is symmetric, and $dx^i \wedge dx^j$ is anti-symmetric, the terms inside the brackets will cancel each other out:

$$\begin{aligned} &\frac{\partial^2 V_m}{\partial x^j \partial x^i} dx^j \wedge dx^i \;+\; \frac{\partial^2 V_m}{\partial x^i \partial x^j} dx^i \wedge dx^j \\ =&\frac{\partial^2 V_m}{\partial x^j \partial x^i} dx^j \wedge dx^i \;+\; \frac{\partial^2 V_m}{\partial x^j \partial x^i} \left(-dx^j \wedge dx^i\right) \;= 0 \end{aligned}$$

□

From vector calculus we are already familiar with the identities $\nabla\cdot(\nabla\times\vec{V}) = 0$, and $\nabla\times(\nabla\,\vec{V} = 0)$. From the proof above we can connect vector calculus with exterior derivative. Lets consider following example:

Example: Apply the exterior derivative on $V \in \Omega^1(\mathbb{R}^3)$

$$V = V_1(x^1, x^2, x^3)dx^1 + V_2(x^1, x^2, x^3)dx^2 + V_3(x^1, x^2, x^3)dx^3$$

$$dV = \left(\frac{\partial V_2}{\partial x^1} - \frac{\partial V_1}{\partial x^2}\right) dx^1 \wedge dx^2 + \left(\frac{\partial V_3}{\partial x^2} - \frac{\partial V_2}{\partial x^3}\right) dx^2 \wedge dx^3 + \left(\frac{\partial V_1}{\partial x^3} - \frac{\partial V_3}{\partial x^1}\right) dx^3 \wedge dx^1$$

Let us take the curl of V, and see how the exterior derivative is related to the curl in $\mathbb{R}^3$.

$$\nabla \times V = \left(\frac{\partial V_3}{\partial x^2} - \frac{\partial V_2}{\partial x^3}\right) dx^1 + \left(\frac{\partial V_1}{\partial x^3} - \frac{\partial V_3}{\partial x^1}\right) dx^2 + \left(\frac{\partial V_2}{\partial x^1} - \frac{\partial V_1}{\partial x^2}\right) dx^3 \tag{1}$$

We remember how $dx^2 \wedge dx^3$ points in the direction of dx^1, so the first term in $\nabla \times V$ is:

$$\left(\frac{\partial V_3}{\partial x^2} - \frac{\partial V_2}{\partial x^3}\right) dx^2 \wedge dx^3$$

Which is exactly the same as we find in the second term in equation 1. We say that the exterior derivative is equivalent to the curl in $\mathbb{R}^3$. From here it is obvious that since dV is equivalent to $\nabla \times V$, there is also and equivalence between $d(dV) = 0$ and $\nabla \cdot (\nabla \times V) = 0$.

3 Group Theory

Before we move to study curved manifolds, we need to introduce some concepts from group theory that we will need when we come to define geometric invariants for manifolds known as cohomology groups.

We follow chapter 0 in [3].
A collection of elements is called a set, and we use the notation: $A = \{a, b, c, \dots\}$. A partition of a set A is a collection of non-trivial subsets of A, such that each element $a \in A$ exist in one and only one subset. The subsets in the partition are called cells or coset, both terms will be used. We use the notation $\bar{x}$ or $[x]$ for the cell containing x, $x \in \bar{x}$. An equivalence relation on a set A is one that satisfies these three properties:

1. $x \in A \Rightarrow x \sim x$ for every $x \in A$ *(Reflexive).*
2. $x \sim y \Rightarrow y \sim x$ for every $x, y \in A$ *(Symmetric).*
3. $x \sim y$ and $y \sim z \Rightarrow x \sim z$ for every $x, y, z \in A$ *(Transitive).*

An equivalence relation gives rise to a partition for A, and visa versa.

Example: Look at $\sim$ on $\mathbb{Z}$ such that $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$.

$$a \sim b \text{ if } a = b + 2n \text{ for } n \in \mathbb{Z} \tag{2}$$

$$\begin{aligned} \text{Let us look at } \bar{1} &= \{x \in \mathbb{Z} \mid x \sim 1 \text{ by (2), so there exist a } n \in \mathbb{Z} \text{ such that } x + 2n = 1\} \\ &= \{\text{all odd numbers}\} \\ \bar{2} &= \{y \in \mathbb{Z} \mid y \sim 2 \text{ by (2), so there exist a } n \in \mathbb{Z} \text{ such that } y + 2n = 2\} \\ &= \{\text{all even numbers}\} \end{aligned}$$

$\mathbb{Z} = \bar{1} \cup \bar{2}$

3.1 Binary operation

We follow chapter 1 in [3].
Let S and S be two sets. Then the set $S \times S = \{(a, b) \mid a \in S \textit{ and } b \in S\}$ is the direct product of S and S. A binary operation or product $\star$ on a set S is a function mapping $S \times S \to S$. For each $(a, b) \in S \times S$ we write: $\star((a, b)) = a \star b$.

Example of different binary operations on numbers: Addition, subtraction, multiplication, division,...

Remark: The operation has to be defined for every $(a, b) \in S$.

Definition 3.1. Let $\star$ be a binary operation on S. $\star$ is commutative if $a \star b = b \star a$

Definition 3.2. $\star$ is associative if $a \star (b \star c) = (a \star b) \star c$

Example: Addition and multiplication of integers are both commutative and associative binary operations.

Example: Subtraction and division for integers are either commutative or associative binary operations.

3.2 Group

We follow chapter 1 in [3].
We have looked at sets and binary operation and some associated properties. A set G together with a binary operation $\star$ is called a group if and only if $(G, \star)$ satisfies the group axioms:

1. $\star$ is associative: $a \star (b \star c) = (a \star b) \star c$.
2. $(G, \star)$ has a identity element $e \in G$:
$$e \star a = a \star e = a \; \forall \; a \in G.$$
3. For all $a \in G$ there exist an $a^{-1} \in G$ such that:
$$a \star a^{-1} = e = a^{-1} \star a.$$

Example: Look at $(\mathbb{Z}, +)$ and determine if it is a group:

1. Addition is associative.
2. The identity element $e = 0 \in \mathbb{Z} \qquad x \in \mathbb{Z} \Rightarrow x + 0 = 0 + x = x.$
3. Given $a \in \mathbb{Z}. \qquad a \star a^{-1} = e \;\Rightarrow\; a + (-a) = 0 \;\rightarrow a^{-1} = -a$

$(\mathbb{Z}, +)$ satisfies the group axioms, and is a group.

Definition 3.3. A group $(G, \star)$ is called abelian if and only if $\star$ is commutative:
$$a \star b = b \star a, \;\; a, b \in G.$$

Definition 3.4. A mapping Φ from a group $(G_1, \star_1)$ to a group $(G_2, \star_2)$ which preserve the product is called a homomorphisan. For $a, b \in G$, then following must be true for a homomorphism:
$$\Phi(a \star_1 b) = \Phi(a) \star_2 \Phi(b).$$

Definition 3.5. Let $\Phi : G_1 \rightarrow G_2$ be a homomorphisan. If Φ has an inverse $\Phi^{-1} : G_2 \rightarrow G_1$, then Φ is a isomorphism. We write $G_1 \cong G_2$.

3.3 Subgroup

We follow chapter 1 in [3].
In the beginning of this topic we talked about how sets could have subsets. This occurs for groups as well, and are called subgroups, however with some conditions: Given $H \subseteq G$.

1. $a, b \in H \Rightarrow a \star b \in H$
H is closed under $\star$.
2. The identity element $e \in H$.
3. For $a \in H$, then $a^{-1} \in H$

If $H \subseteq G$ satisfies the subgroup axioms then we write $H \leq G$.

Example: Is $(\mathbb{Z}, +)$ a subgroup of $(\mathbb{Q}, +)$?

1. $a, b \in \mathbb{Z} : \; a + b \in \mathbb{Z} \;\Rightarrow\; \mathbb{Z}$ is closed under $+$.
2. $e = 0 \Rightarrow e \in \mathbb{Z}$.
3. $a \in \mathbb{Z} \;\Rightarrow\; a^{-1} = -a \in \mathbb{Z}$.

$\Rightarrow \; (\mathbb{Z}, +) \leq (\mathbb{Q}, +)$

3.4 Factor group

Remember how a set is divided into cells, and the collection of all of these cells is called a partition. Two elements in a set have an equivalence relation if and only if these two elements are in the same cell. Factor groups is a way of representing a group by aggregating elements that have an equivalence relation and preserve some of the group structure [4]. The cohomology group we will define later are factor groups.

Definition 3.6. See the book [3], chapter 3.
Let $N \leq G$. For $a \in G$, then Na will give us the coset of the right partition of G under N. In the case where the right and left cosets are equal; $aN = Na$, we say that N is a normal subgroup of G. A factor group is defined as:

$$G/N \cong H$$

and reads "G modulo N". The group product of the factor group is the set of all cosets of N in G, equipped with the group operation defined by $(a_1N) \star (a_2N) = (a_1a_2)N$ for all a_1, $a_2 \in G$.

Example: Let us consider example 3, and let $\sim$ be the equivalence relation on $\mathbb{Z}$. The quotient group to $\sim$:

$$\mathbb{Z} \,/ \sim = \{\bar{1}\ ,\ \bar{2}\} \cong \mathbb{Z}_2,$$

where $\mathbb{Z}_2$ is the only group of order 2.

The factor group gives us a lot of essential information, and I would argue that it is one of the key elements for the essence of this thesis. As we will see, the factor group characterise topology of a manifold in terms of the properties of differential forms [5]. But we need some more theory, so we are going to introduce the fundamental homomorphism theorem, but for that we have to take a recap of some basic knowledge from linear algebra:

Definition 3.7. The kernel of a homomorphism $\Phi : G_1 \to G_2$ is all the elements in G_1 mapped to the identity in G_2

$$ker(\Phi) = \{x \in G_1 \mid \Phi(x) = e_2 \in G_2\} \subseteq G_1$$

Note that $ker(\Phi) \leq G_1$ as a normal subgroup.

Theorem 3.1. *The fundamental homomorphism theorem*
Given two groups G and H and a group homomorphism $\Phi : G \to H$. Let N be a normal subgroup of G and $Ker(\Phi) = N$ Then.

$$\frac{G}{Ker(\Phi)} \cong \Phi(G) \leq H,$$

where the factor group is a representation of G in H under Φ, and is called $\Phi(G)$.

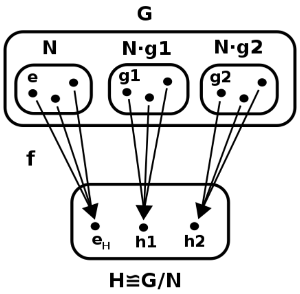


Figure 2: Original figure from [10]. Here we assume Φ is onto, so $\Phi(G) = H$.

Proof. See the book [3] p.136 □

The factor group preserve some of the structure of G, and it is precisely this characteristic which will prove to be particularly important as we will look further at differential equations on curved geometries.

4 Manifold

The mathematics that describes curves and surfaces in our 3-dimensional world is something we start to learn already at the elementary school. But the more you study mathematics and physics, the greater is the need to look at curves and surfaces in higher dimensions. These curves and surfaces in higher dimensions are called manifolds.

Let us consider mother earth, which is a sphere. If you look out on the horizon form a mountain top, or just from you window, it looks like we are living in a 2 dimensional space. But we all know we are living on a sphere, which is a space in $\mathbb{R}^3$. The same analogy is used for manifolds in higher dimensions. Look at a manifold $M \subseteq \mathbb{R}^m$. If you take a point and look at the area around this point it may look like a space in $\mathbb{R}^n$. A manifold is locally homeomorphic to $\mathbb{R}^n$, but may be different form $\mathbb{R}^n$ globally. Because of this homeomorphism we can divide the manifold into local patches with local coordinates. With a coordinate function Φ, patches can be mapped to space in $\mathbb{R}^n$.

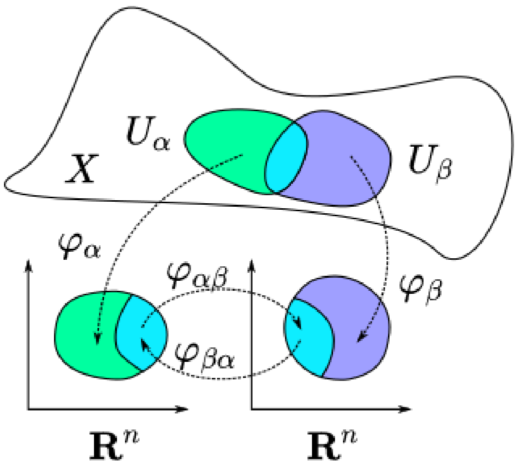


Figure 3: Original figure form [7]. A manifold $X \subseteq \mathbb{R}^m$ with two local patches mapped to $\mathbb{R}^n$

.

Definition 4.1. For us, a manifold M will be a designation for geometric forms embedded $\mathbb{R}^m$ with dimension $\leq m$ [2]. A patch (U, Φ), also called a local surface, is a differentiable mapping $\Phi : U \to \mathbb{R}^n$, where $U \subseteq M$. The function Φ is differentiable, and can be taken to be bijection without loss of generality.
Explicitly, a set of local patches (U_i, Φ_i) is called an atlas if the manifold M can be written as

$$M = \cup_i U_i \, .$$

In figure 3 there is an overlap between the two patches on the manifold. In this case the blue region on the manifold will be represented in both maps. Let the green map to left be given by x-coordinates, and the pruple map to the right be given by y-coordinates. Then will the function $\varphi_{\alpha\beta}$ be a coordinate transformation of the blue space form x-coordinates to y-coordinates. The local coordinate maps are differentiable, and we use this to define differential forms on a manifold.

4.1 Differential forms on a manifold

Differential forms is primarily the theory which makes it possible to integrate over a manifold, but this is not something that will be covered in this thesis. Nevertheless, differential forms do play a bigger part on manifolds than just integration. In particular, they are relevant for defining the geometric invariants known as cohomology groups.

Let us continue with the blue region in figure 3. Lets say we have a 0-form, a function f on our manifold X. So let us say this function gives us the amount of rainfall at some coordinates. Our manifold X is built up by patches, for example lets say $(U_\alpha, \varphi_\alpha)$ is the map including Oslo, and f_α the function which tells us the amount of rainfall in Oslo. And similarly, we say that (U_β, φ_β) is the map including Berlin, and f_β gives us the amount of rainfall. There is a overlapping in the blue region, and in this region $f_\alpha = f_\beta$, which means the amount of rainfall in Oslo and Berlin is the same. The function $\varphi_{\alpha\beta}$ could be interpreted as a translator between Norwegian and German.

Note that it is only in the overlap region $(U_\alpha \cap U_\beta)$ we can have a coordinate transformation. We will introduce this in the next section. Finally, the set of all k-forms on a manifold M is denoted as $\Omega^p(M)$. The highest p-form we can have on a manifold is dimension of the manifold itself.

4.2 Change of coordinates

Given the differential $\beta = \beta_\mu dx^\mu \in \Omega^1(\mathbb{R}^2)$. The coordinate transformation for the differentials are given by:

Coordinate x^μ	Coordinate y^ν
dx^μ	$\frac{\partial x^\mu}{\partial y^\nu} dy^\nu$

However the form β never transforms. No matter which coordinate system, a n-form will stay the same.

Example: Write the components in $\beta = \beta_\mu dx^\mu \in \Omega^1(\mathbb{R}^2)$ in temrs of y-coordinates.

$$\begin{aligned} \beta &= \beta_\mu dx^\mu \\ &= \beta_\mu \frac{\partial x^\mu}{\partial y^\nu} dy^\nu \\ &= \hat{\beta}_\nu dy^\nu \end{aligned}$$

$\beta_\mu \frac{\partial x^\mu}{\partial y^\nu} := \hat{\beta}_\nu$, $\hat{\beta}_\nu$: components in β in y-coordinates,
β_μ: components in β in x-coordinates

So the component functions β_μ of β transforms, but β itself does not. In the following example will we consider the 2-sphere to illustrate how a manifold is built up by patches mapped into the xy-plane.

Example: *Stereographic projection*

As we already have covered, we live on a 2-dimensional surface. But if we would map this surface into the xy-plane there would be spots that wouldn't be covered. So we need to divide the sphere into two patches, respectively northern hemisphere and southern hemisphere and map each of them into the xy-plane. We will find a coordinate transformation for each hemisphere to the xy-plane, and finally find the coordinate transformation in the intersection between these two maps.

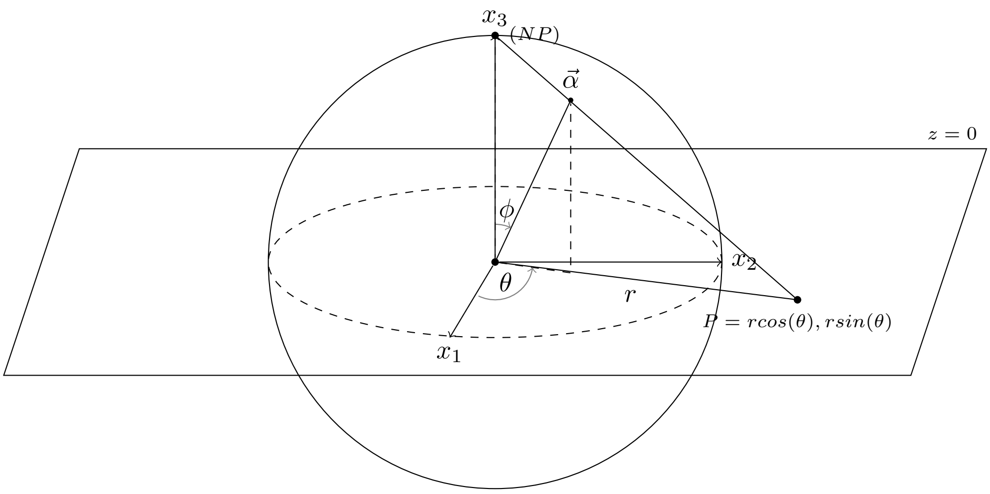


Figure 4: Figure based on [11]. Projection from the north-pole to the xy-plane

Let $0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi$. The point P is function of both r and the angle θ. The value of r depends on the angle ϕ. If $\phi \mapsto 0°$, then $r \mapsto \infty$.

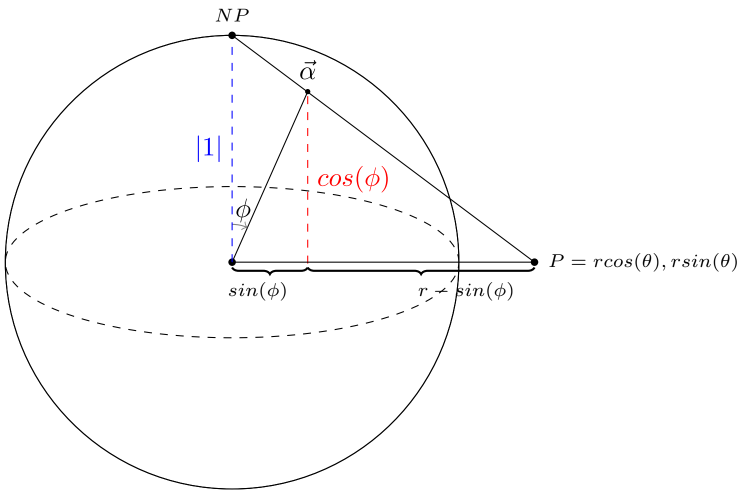


Figure 5: Figure based on [12]. Projection form the north-pole to the xy-plane with fixed θ.

To find the coordinate transformation from the north pole to the xy-plane we have to find $r(\phi)$. Some straight forward trigonometry gives

$$r = \frac{sin(\phi)}{1 - cos(\phi)}.$$

Note that:

- If $\phi \to 0°$, then $r \to \infty$ *North.*
- If $\phi \to \frac{\pi}{2}$, then $r \to 1$ *Equator.*
- If $\phi \to \pi$, then $r \to 0$ *South.*

Note thar this coordinate transformation does not define r at the north pole. To get a coordinate patch including the north pole we make a stereographic projection from south pole to the xy-plane.

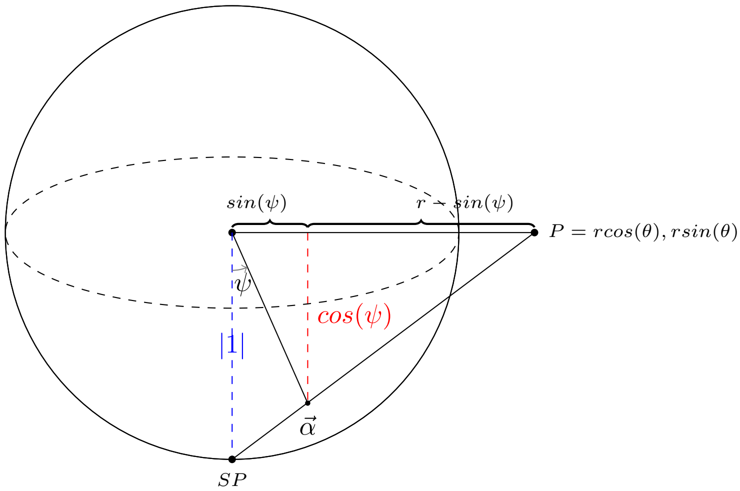


Figure 6: Figure based on [12]. Projection form the south-pole to the xy-plane with fixed θ.

$$r = \frac{sin(\psi)}{1 - cos(\psi)}$$

- If $\psi \to \pi$, then $r \to 0$ *North.*
- If $\psi \to \frac{\pi}{2}$, then $r \to 1$ *Equator.*
- If $\psi \to 0°$, then $r \to \infty$ *South.*

See Figure 8 for visual representation of the atlas.
Let's define $U_1(\theta, \phi)$: Chart for $S^2 - NP$, and $U_2(\theta, \psi)$: Chart for $S^2 - SP$. Then $U_1 \cup U_2 = S^2$.
Let Φ be the transition function: $\Phi : U_1 \to U_2$.

$$\Phi(\theta, \phi) = (\theta, \underbrace{(\pi - \phi)}_{\psi})$$

Notice how $\psi = \pi - \phi$ is a linear transformation, which is quite special. We do not find the same linearity if we insist on using xy-coordinates $(x, y) = (rcos(\theta), rsin(\theta))$.

The example above illustrates that local patches, such as $M = S^2$, can be mapped to coordinates in $\mathbb{R}^n$. We may as well use local $\mathbb{R}^n$ coordinates to build up our geometries.
In general, for a manifold $M \subseteq \mathbb{R}^m$ whose $dim(M) = n$, and $\alpha \in \Omega^p(M)$. Locally on $U_i \subseteq M$:

$$\alpha = \frac{1}{p!}\alpha_{i_1}, \ldots, _{ip}\,(x \in U_i)dx^{i_1} \wedge \cdots \wedge dx^{ip}. \tag{3}$$

where $dx^{i_1} \wedge \cdots \wedge dx^{ip}$ is the basis for $\alpha_{i_1}, \ldots, _{ip}$ on U_i. We have the differentiable mapping:
$\Phi : U_i \to \mathbb{R}^n$
$\Phi_i^{-1} : \mathbb{R}^n \to U_i$, and by mapping U_i to $\mathbb{R}^n$ we get:

$$\alpha = \frac{1}{p!}\alpha_{i_1}, \ldots, _{i_p}\,(\Phi^{-1}(y) \mid y \in \mathbb{R}^n)d\Phi^{-1}(y)^{i_1} \wedge \cdots \wedge d\Phi^{-1}(y)^{i_p}$$

$$= \frac{1}{p!}\alpha_{i_1} \cdots _{i_p} \circ \Phi^{-1}(y)\frac{\partial \Phi^{-1}(y)^{i_1}}{\partial y^{j_1}}dy^{j_1} \wedge \cdots \wedge \frac{\partial \Phi^{-1}(y)^{i_p}}{\partial y^{j_p}}dy^{j_p}$$

$$= \frac{1}{p!}\alpha_{i_1}, \ldots, _{ip}\,(\Phi^{-1}(y))\frac{\partial {\Phi^{-1}}^{i_1}}{\partial y^{j_1}} \cdot \frac{\partial {\Phi^{-1}}^{i_2}}{\partial y^{j_2}} \cdots \frac{\partial {\Phi^{-1}}^{ip}}{\partial y^{j_p}}dy^{j_1} \wedge \cdots \wedge dy^{j_p}$$

Let $\tilde{\alpha}_{j_1 \ldots j_p}(y) := \frac{1}{p!}\alpha_{i_1}, \ldots, _{i_p}\,(\Phi^{-1}(y))\frac{\partial {\Phi^{-1}}^{i_1}}{\partial y^{j_1}} \cdot \frac{\partial {\Phi^{-1}}^{i_2}}{\partial y^{j_2}} \cdots \frac{\partial {\Phi^{-1}}^{i_p}}{\partial y^{j_p}}$, where $dy^{j_1} \wedge \cdots \wedge dy^{j_p}$ is the basis for $\tilde{\alpha}_{j_1 \ldots j_p}$ on $\mathbb{R}^n$. Form this it follows that α is now locally given as $\alpha \in \Omega^p(\mathbb{R}^n)$, where $y \in \mathbb{R}^n$:

$$\alpha = \frac{1}{p!}\tilde{\alpha}_{j_1}, \ldots, _{j_p}\,(y)dy^{j_1} \wedge \cdots \wedge dy^{j_p} \tag{4}$$

4.3 Closed and Exact form

Definition 4.2. $\alpha \in \Omega^p(M)$ is said to be closed on a domain M if $d\alpha = 0$. The set of closed p-forms on M is denoted $Z^p(M)$

Definition 4.3. $\beta \in \Omega^p(M)$ is globally exact if $\beta = d\gamma$ for a global $\gamma \in \Omega^{p-1}(M)$. The set of exact p-forms on M is denoted $B^p(M)$.

Example:

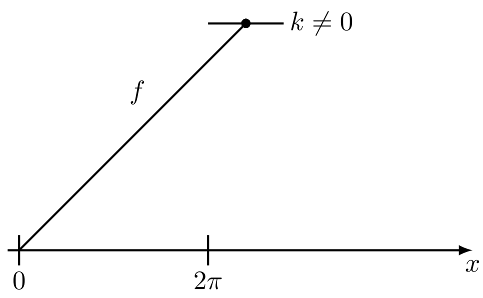


Figure 7: S^1 : A circle represented as a line segment where $0 \sim 2\pi$.

Consider the form $\alpha = df$. The slope of f is constant, so $df = cdx$. But f is not well-defined on the hole domain S^1. It is clear that f do not have the same value on the point $x = 0 = 2\pi$. In such cases df is said to be *local* exact, not globally exact $\Rightarrow \quad B^1(S^1) \subsetneq Z^1(S^1)$.

Proposition 4.1. *The closed forms and the exact forms satisfies the group axiom listed at 3.2 with addition as binary operation. Without further proof, the closed forms and exact forms are a group;* $(Z^p(M), +)$ *and* $(B^p(M), +)$. *In addition they are both a abelian group, due to the commutative property of* $+$.

Proposition 4.2. *An exact p-form* $B^p(M) = \{\alpha \in \Omega^p(M) \mid \alpha = d\gamma, \gamma \in \Omega^{p-1}(M)\}$ *is a subgroup of the closed p-forms,* $Z^p(M) = \{\alpha \in \Omega^p \mid d\alpha = 0\}$. *Explicitly;* $B^p(M) \leq Z^p(M)$.

Proof.

$$\begin{aligned} \alpha \in B^p(M) &\Rightarrow \alpha = d\gamma \\ &\Rightarrow d(\alpha) = d^2(\gamma) = 0 \\ &\Rightarrow \alpha \in Z^p(M). \end{aligned}$$

Without further proof, the remaining subgroup axioms listed at 3.3 are satisfied, and it follows that $B^p(M) \leq Z^p(M)$. $\square$

Let us take the example of the two-sphere a little further, and consider a function p that describes pressure. As we know, the derivative of something gives you the rate of change. So if we take the derivative of some pressure bar on our sphere, we will get the rate of change in pressure at this point. Wind is a product of pressure difference, and it blows orthogonal on the pressure bars from high pressure to low pressure. So if p represents a pressure-function, and the derivative of this gives us the wind-field, we get:

$$v = -dp, \;\; v \in \Omega^1(M).$$

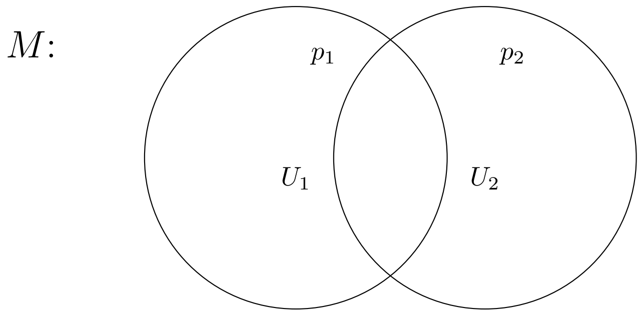


Figure 8: The two patches builds up M, with the respective pressure function

Using stereographic projection, we have $S^2 = M = U_1 \cup U_2$. We will call the pressure function in U_1 for p_1, and p_2 for U_2. In the intersection between U_1 and U_2, we will have the case where $\underbrace{dp_1}_{v_1} = \underbrace{dp_2}_{v_2}$ as $v_1 = v_2 = v$. It follows that $d(p_1 - p_2) = 0 \Longleftrightarrow p_1 - p_2 = c_{12}$. In the intersection the difference in pressure will be equal to a constant. We know that there exist a local p so the wind field is locally exact. In order to say that the wind field is globally exact, there has to exist a global pressure function $p \in \Omega^0(M)$. Notice how the pressure must then have a maximum and minimum somewhere on a closed and bounded manifold M, and at this position $x_0 \in M$ we know that:

$$v = -dp(x_0) = 0$$

Hence, there exist a point x_0 on M where there is no wind.

It is already established that any exact form is also an closed form. Must a closed form also be an exact form? For one-forms on S^2 the answer to this question is yes, and is actually the explanation to why there is a windless point on the sphere. I can already reveal that this is not the case for a torus, which we will look at later.

On the 2-sphere

To prove that there is always a point on the sphere where there is no wind, we have to show that any closed one-form on the sphere is also an exact form.

$$B^1(S^2) = Z^1(S^2)$$

It follows from the definition of closed form that:

$$d\alpha = 0$$

Lemma 4.3. *On S^2, for $\alpha \in \Omega^1(S^2)$. If $d\alpha = 0 \Rightarrow \alpha = df$, for some function f*

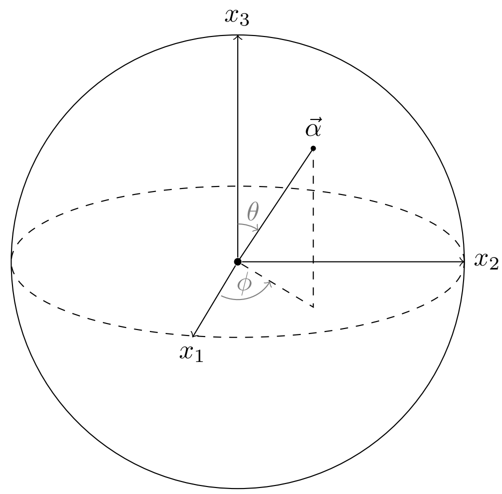


Figure 9: Figure from [12]

For simplicity, we will continuity to work with polar-coordinates. (ϕ, θ) is a point on the surface of the sphere. α is a 1-form who is a function of (ϕ, θ):

$$\alpha = A(\phi, \theta)d\phi + B(\phi, \theta)d\theta, \ \alpha \in \Omega^1(S^2)$$

Where $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \pi$.

Proof. $d\alpha = 0$ gives:

$$
\begin{aligned}
\frac{\partial A}{\partial \theta} &= \frac{\partial B}{\partial \phi} \\
\Rightarrow \int_0^{\tilde{\theta}} \frac{\partial A}{\partial \theta}\, d\theta &= \int_0^{\tilde{\theta}} \frac{\partial B}{\partial \phi}\, d\theta \\
\Rightarrow A(\tilde{\theta}, \phi) - A(0, \phi) &= \frac{\partial}{\partial \phi} \int_0^{\tilde{\theta}} B(\theta, \phi)\, d\theta \\
\Rightarrow \qquad A(\tilde{\theta}, \phi) &= \frac{\partial}{\partial \phi} \int_0^{\tilde{\theta}} B(\theta, \phi)\, d\theta + A(0, \phi) \\
\Rightarrow \qquad A(\tilde{\theta}, \phi) &= \frac{\partial}{\partial \phi} \left[\underbrace{\int_0^{\tilde{\theta}} B(\theta, \phi) d\theta + \int_0^{\phi} A(0, \tilde{\phi}) d\tilde{\phi}}_{f(\tilde{\theta}, \phi)} \right]
\end{aligned}
$$

We also have:

$$
\begin{aligned}
B(\tilde{\theta}, \phi) &= \frac{\partial}{\partial \tilde{\theta}} \int_0^{\tilde{\theta}} B(\theta, \phi) d\theta \\
B(\tilde{\theta}, \phi) &= \frac{\partial}{\partial \tilde{\theta}} \left[\underbrace{\int_0^{\tilde{\theta}} B(\theta, \phi) d\theta + \int_0^{\phi} A(0, \tilde{\phi}) d\tilde{\phi}}_{f(\tilde{\theta}, \phi)} \right]
\end{aligned}
$$

It follows that:

$$
\left.
\begin{aligned}
A(\theta, \phi) &= \frac{\partial}{\partial \phi} f(\theta, \phi) \\
B(\theta, \phi) &= \frac{\partial}{\partial \theta} f(\theta, \phi)
\end{aligned}
\right\} \alpha = \frac{\partial f}{\partial \phi} d\phi + \frac{\partial f}{\partial \theta} d\theta := df
$$

Hence the wind field is described as $\alpha = \frac{\partial f}{\partial \phi} d\phi + \frac{\partial f}{\partial \theta} d\theta$, where $\alpha \in \Omega^1(S^2)$ and $f \in \Omega^0(S^2)$. □

To sum it up; for every closed one-form, $dv = 0 \in \Omega^1(S^2)$, there is a $p \in \Omega^0(S^2)$ so that $v = -dp \in \Omega^1(M)$. Any closed 1-form is also exact.

4.4 Boundry

The boundary of a area or surface is simply the edge [6], and this easy explanation applies for higher dimension as well. The symbol used for boundary of a region M is ∂M. If $\dim(M) = n$, then $\dim(\partial M) = n - 1$.

Example: Let us consider the unit disc D in $\mathbb{R}^2$. The boundary ∂D is the unit circle.

Some regions does not have a boundary, and the sphere is one of these regions. We denote the boundary for these region as $\partial S = \emptyset$, where $\emptyset$ is the empty set. It must be mentioned that the orientation of the boundary is impotent to take into consideration. Yet this is not something that will be addressed further due to the thesis' limitation.

Stokes theorem

In conjunction with boundary it will be natural to talk about Stokes theorem. This is a theorem which is really fundamental and useful for both mathematicians and physicists.

Theorem 4.4. *Stokes theorem:*
Let Σ be a surface in $\mathbb{R}^3$, and $\partial\Sigma$ be the respectively boundary. If $\vec{v}$ is defined on the surface Σ, then according to Stokes theorem

$$\int_{\Sigma} \nabla \times \vec{v} = \int_{\partial\Sigma} \vec{v}$$

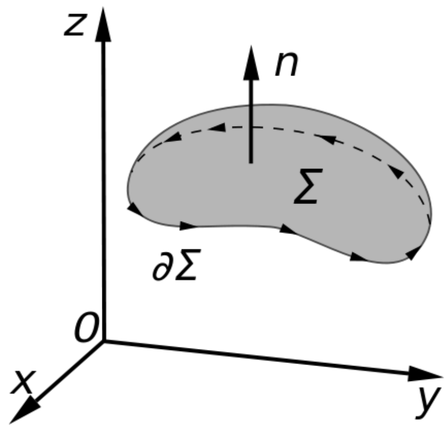


Figure 10: Original figure from [8]. Visuel representation of Stokes theorem.

Stokes theorem can be generalized to apply for any dimension. Let us consider M which is a oriented manifold on $\mathbb{R}^m$ of dimension n. If $\omega \in M$ is a differentiable $(n-1)$-form [2], then

$$\int_{\partial M} \omega = \int_{M} d\omega.$$

We will use Stokes theorem as a tool in this paper, and see how this theorem simplifies many problems.

5 Connecting the dots

So far, we have looked at several examples of differential forms, and how they can be used as a meaningful tool on a manifold. We saw how the 2-sphere has a point where there is no wind, and it has something to do with the geometry of the sphere. We will see that the cohomology factor group characterises topology on a manifold in terms of differential forms. Based on all of the examples and knowledge we have retrieved so far, we should have enough information to put it all together;

Analysis / Solutions to diff. eq.	$\longleftrightarrow$	Algebra / Cohomology	$\longleftrightarrow$	Topology / Geometry

We will see more practical applications to build on understanding of this connection, further on in this section.

5.1 Cohomology

It has already been shown that both the exact and closed forms satisfies the group axioms, and that they are an abelian group. The p-th cohomology factor group for any arbitrary manifold $M \in \mathbb{R}^m$ that is built up by local patches is given by:

$$H^p(M) := \frac{Z^p(M)}{B^p(M)}.$$

We take all the closed p-forms, and factor out the exact ones. The elements in $H^p(M)$ are the equivalence classes of closed forms which differ by exact forms [5] *p. 101.* The p-th *cohomology group* of M describes the solution space to generalisations of Maxwell's differential equations on the manifold, which are to be defined below. In other words, if we manage to find the cohomology group, we know how many solutions to look for, without even knowing how they look like! The factor group is topologically independent of coordinate choices. It is a topological invariant! The coordinate transformation done at the end of section 4.2 represented in equation 3, where $\alpha \in \Omega^p(M)$ to 4, where $\alpha \in \Omega^p(\mathbb{R}^n)$ can be done for any local patch on M. So no matter what coordinate representation we choose for α, the factor group is the same.

We are now going to look further to our example of the sphere, in order to get a practical view of the connection between cohomology group and differential forms.

Cohomology group for 2-sphere

From the proof of lemma 4.3 we know that in order for there to be a place on the sphere where there is no wind, the exact 1-forms has to be the same as the closed 1-forms. The cohomology group for the differential 1-forms over the 2-sphere is given by:

$$H^1(S^2) := \frac{Z^1(S^2)}{B^1(S^1)}$$

$H^1(S^2)$ contains only elements of closed forms that differs form the exact forms, but since $Z^1(S^2) = B^1(S^2)$, there is no difference. So the factor group has no essential information about the structure of $Z^1(S^2)$ that is not present in $B^1(S^2)$, so we get:

$$H^1(S^2) := \frac{Z^1(S^2)}{B^1(S^1)} = \{0\}$$

There is no solution to Maxwell's differential equations for one-forms, exactly as anticipated.

Let us look at the 0-forms on the sphere:

$$H^0(S^2) := \frac{Z^0(S^2)}{B^0(S^2)}.$$

$Z^0(S^2)$: $d\alpha = 0 \;\Rightarrow\; \alpha$ has to be a constant; $\alpha = c$. $B^0(S^2)$: $\alpha = d\gamma,$ where $\gamma \in \Omega^{-1}(M)$. But $\Omega^{-1}(M) = \{0\}$ as there are no forms of negative degree. Hence

$$H^0(S^2) = \{\alpha \mid d\alpha = 0\} \cong \mathbb{R}^1$$

$Z^0(S^2)$ consists of constants, and there exist as many constants as real numbers, hence $H^0(S^2) \cong \mathbb{R}^1$. Form finding the cohomology group of 0-forms on S^2, we know that Maxwell's quations for 0-forms has one solution. This is quite remarkable!

Regarding the 2-forms, we are going to prove that $H^2(S^2) \cong \mathbb{R}^1$ is true:

$$H^2(S^2) := \frac{Z^2(S^2)}{B^2(S^2)} = \frac{\{\alpha \in \Omega^2(S^2) \mid d\alpha = 0\}}{\{\alpha = d\alpha \mid \gamma \in |\Omega^1(S^2)\}} = \frac{\{\Omega^2(S^2)\}}{\{\alpha = d\gamma \mid \ \gamma \in \Omega^1(S^2)\}} \cong \mathbb{R}^1 \tag{5}$$

In order for equation 5 to be true we will use theorem 3.1 to proceed. Let us define the homomorphism:
$\Phi :\ \Omega^2(S^2) \to (\mathbb{R}^1, +)$
$\alpha \in \Omega^2(S^2); \quad \Phi(\alpha) = \int\limits_{S^2} \alpha.$
Express α in polar coordinates for simplicity; $\quad \alpha = \alpha(\phi,\theta) d\phi \wedge d\theta$. Consider $\alpha = c\ d\phi \wedge d\theta$. Then the integral becomes a constant $c \times$ area of the sphere. This shows that Φ is surjective.
It follows from theorem 3.1:

$$\frac{\Omega^2(S^2)}{ker(\Phi)} \cong \mathbb{R}^1.$$

For an abelian group the identity element is $0 \to ker(\Phi) = \{\alpha(\phi,\theta) \mid \int\limits_{S^2} \alpha(\phi,\theta) = 0\}$. Hence if we can show that $B^2(S^2) = ker(\Phi)$ we are done.

$B^2(S^2) \subseteq ker(\Phi)$: $\quad \alpha \in B^2(S^2)$,

$$\begin{aligned} \alpha = d\gamma &= (d\phi\partial\phi + d\theta\partial\theta)\gamma \\ &= (d\phi\partial\phi + d\theta\partial\theta)(\gamma_\phi d\phi + \gamma_\theta d\theta) \\ &= (\partial_\phi\gamma_\theta - \partial_\theta\gamma_\phi) d\phi \wedge d\theta \end{aligned}$$

$$\Phi(\alpha) = \Phi(d\gamma) = \int\limits_{S^2} (\partial_\phi\gamma_\theta - \partial_\theta\gamma_\phi) d\phi d\theta \tag{6}$$

But $\partial_\phi\gamma_\theta - \partial_\theta\gamma_\phi \sim \nabla \times \gamma \ (\gamma \in \Omega^1)$. This implies we can use the Stoke´s theorem 4.4. Since the 2-sphere does not have any boundary, $\partial S^2 = 0$, it follows that $\Phi(\alpha) = 0$, hence $B^2(S^2) \subseteq ker(\Phi)$.

$ker(\Phi) \subseteq B^2(S^2)$: We have the vector space $\Omega^2(S^2)$. Let us choose two elements $\alpha, \beta \in \Omega^2(S^2)$. The inner product written as (α, β); $\ \Omega^2(S^2) \times \Omega^2(S^2) \to \mathbb{R}^1$, is given by:

$$(\alpha, \beta) := \int\limits_{S^2} \alpha(\phi,\theta)\beta(\phi,\theta) d\phi d\theta \quad \in \mathbb{R}^1.$$

We will define the sets $\mathscr{H} = \{\alpha \in \Omega^2(S^2) \mid \alpha = c d\phi \wedge d\theta,\ \ c \in \mathbb{R}^1\}$, $B^2(S^2) = \{\alpha \in \Omega^2(S^2) \mid \alpha = d\gamma\}$.
Let $\alpha \in \mathscr{H}^2(S^2), \quad \beta = d\gamma$. Then

$$(\alpha, \beta) = \int\limits_{S^2} \alpha(\phi,\theta)\beta(\phi,\theta) d\phi d\theta \ = c \int\limits_{S^2} \beta(\phi,\theta) d\phi d\theta \ = c\ \Phi(\beta) = 0.$$

So α and β have to be orthogonal, since the inner product is $0 \Rightarrow \ \mathscr{H}^2 \perp B^2(S^2)$.
So far we have considered 2-forms included $\mathscr{H}^2(S^2)$ and $B^2(S^2)$, but there could be more to take into

consideration. Therefore all the 2-forms will be expressed as $\Omega^2(S^2) = \mathscr{H}^2(S^2) \times B^2(S^2) \times \mathcal{R}(S^2)$, where $\mathcal{R}(S^2) \perp \mathscr{H}(S^2)$ and $\mathcal{R}(S^2) \perp B^2(S^2)$. Choose a element $\kappa \in \mathcal{R}(S^2)$, $\quad \kappa = \kappa(\theta,\phi)d\theta \wedge d\phi$. We will define: $d^\dagger\kappa := \partial_\theta\kappa(\theta,\phi)d\phi - \partial_\phi\kappa(\theta,\phi)d\theta \quad \in \Omega^1(S^2)$. Then

$$\begin{aligned} d(d^\dagger\kappa) &= (d\theta\partial_\theta + d\phi\partial_\phi)(\partial_\theta\kappa d\phi - \partial_\phi\kappa d\theta) \\ &= (\partial_\theta^2\kappa + \partial_\phi^2\kappa)d\theta \wedge d\phi \\ &= \Delta\kappa(\theta,\phi)d\theta \wedge d\phi, \end{aligned}$$

where Δ is the laplace operator.
Now lets look at: $\int\limits_{S^2} \big((\partial_\theta\kappa)^2 + (\partial_\phi\kappa)\big)d\theta d\phi$. First note that

$\nabla\kappa(\theta,\phi) = (\partial_\theta\kappa, \partial_\phi\kappa) \quad \rightarrow \quad \kappa(\theta,\phi)\nabla\kappa(\theta,\phi) = (\kappa\partial_\theta\kappa, \kappa\partial_\phi\kappa)$.
Consider the integral of the divergence, which will be zero by Stokes theorem.

$$\begin{aligned} 0 &= \int\limits_{S^2} \nabla \cdot \Big(\kappa(\theta,\phi)\nabla\kappa(\theta,\phi)\Big)d\theta d\phi \\ 0 &= \int\limits_{S^2} \underbrace{(\partial_\theta\kappa)^2 + (\partial_\phi\kappa)^2}_{*} d\theta d\phi + \int\limits_{S^2} \underbrace{\kappa\Delta\kappa d\theta d\phi}_{**} \end{aligned}$$

$**$: This is the inner product $(\kappa, dd^\dagger\kappa)$, which is equal to zero due to the assumption that $\kappa \in \mathcal{R}^2(S^2)$ is orthogonal on $\mathscr{H}^2(S^2)$ and $B^2(S^2)$. The first term $*$ then gives
$\implies \partial_\phi\kappa = \partial_\theta\kappa = 0$.
$\implies \kappa = \text{constant} = 0$ due to $\kappa \perp \mathscr{H}^2(S^2)$. So there exists no such $\kappa \in \mathcal{R}(S^2)$ since they are already represented in $\mathscr{H}^2(S^2)$. So we can exclude $\mathcal{R}^2(S^2)$, and conclude $\Omega^2(S^2) = \mathscr{H}(S^2) \times B^2(S^2)$.
Let $\alpha = \alpha_h + d\gamma$, $\quad \alpha_h = cd\theta \wedge d\phi \in \mathscr{H}^2(S^2)$. Assume that $\alpha \in ker(\Phi)$

$$\begin{aligned} 0 = \Phi(\alpha) &= \Phi(\alpha_h) + \underbrace{\Phi(d\gamma)}_{0} \\ \Phi(\alpha) &= \Phi(\alpha_h) \\ &= \int\limits_{S^2} cd\theta d\phi \\ &= c\int\limits_{S^2} d\theta d\phi \\ &= c4\pi \end{aligned}$$

If in fact $\alpha \in ker(\Phi)$ then $0 = c4\pi \Rightarrow c = 0$.
And if $c = 0$, then $\alpha_h = 0$, so $\alpha = d\gamma \Rightarrow ker(\Phi) \subseteq B^2(S^2)$.

If two sets are a subset of each other, just like for $ker(\Phi)$ and $B^2(S^2)$, then $ker(\Phi) = B^2(S^2)$. Now that this has been proven we can make use of the theorem 3.1.

$$H^2(S^2) = \frac{\Omega^2(S^2)}{B^2(S^2)} = \frac{\Omega^2(S^2)}{ker(\Phi)} \cong Im(\Phi) = \mathbb{R}^1.$$

We can for sure conclude that there is one solution for Maxwell's equations for differential 2-forms over S^2, just as for the differential 0-forms, $H^0(S^2)$ and $H^2(S^2)$ has the same cohomology.

5.2 Other topologies

We know that differential equations could have infinitely many solutions. As the introduction alludes to, it would be much easier to count how many solutions there exist, than having to find an explicit solution. And as we have seen on the 2-sphere, the connection between analysis, algebra and topology makes this possible. We are now going to look at some other mathematical tools, in order to benefit from this connection for more complex geometries, like the torus or n-sphere.

We let X be a manifold with some determined properties, a Riemannian manifold with n-dimension [1]. A famous mathematician called Hodge deduced the "Hodge duality" for Riemannian manifolds: $H^p(X) \cong H^{n-p}(X)$. Then Poincaré built further on this with the following "Poincaré duality": $H^{n-p}(X) \cong H_p(X)$. With both duality's together we get:

$$H^p(X) \cong H^{n-p}(X) \cong H_p(X) \cong \mathbb{R}^{b_p} \tag{7}$$

$H_p(X) = p'th$ homology group. This is the group of non-trivial connected and closed sub-p-manifolds modulo deformations. p is the dimension of the submanifold. And b is the betti number in p-dimension, counting the number of sub-manifolds.

By using equation 7, we are going to see how we can find the number of solutions to Maxwell's equations on more complex geometries, starting out with the torus.

Torus: Let us consider the torus, and use equation 7, which states that the homology group is isomorphic to the cohomology group. We have already walked through in the previous section the logic around how many possible solutions there is for the differential equations, and will use the same analogy for the torus.

Let us start out by looking at the 0-forms, which in homology corresponds to a lot of points on the torus. All of these points may be collected together to just one point, hence the $0'th$ homology group which is given by all of these points modulo deformation is $H_0(T^2) \cong \mathbb{R}^1$. Explicitly, the homology group is isomorphic to cohomology group, $H_0(S^2) \cong H^0(T^2) \cong \mathbb{R}^1$.
The differential 1-forms on the torus is represented in figure 11, respectively by the red and purple arrows, corresponding to the red and purple arrows in cohomology. So just by looking at the visual representation of differential 1-forms on the torus we have already taken advantage of the connection between analysis and topology. By adding our knowledge about cohomology and equation 7, we get $H_1(T^2) \cong H^1(T^2) \cong \mathbb{R}^2$, and can conclude that Maxwell's equations for differential 1-forms have 2 solutions on the torus!
What applies to the differential 2-forms will be in principle the same procedure, so there should be no surprise that $H_2(T^2) \cong \mathbb{R}^1$ and one solution. Due to the isomorphism between homology and cohomology, notice how $H^0(T^2)$ and $H^2(T^2)$ has the same cohomology.

***n*-sphere:** We have already established the cohomology groups for 2-sphere in section 5.1, and with equation 7 it is straight forward to count the number of solutions.

- $H_0(S^2) \cong \mathbb{R}^1$, there is only one point modulo deformation $\rightarrow$ one solution
- $H_1(S^2) \cong 0$, every connected and closed 1-loop is trivial $\rightarrow$ no solution
- $H_2(S^2) \cong \mathbb{R}^1$, the hole surface $\rightarrow$ one solution

And this obtained result can in fact be generalized into the n-sphere.

$$H_p(S^n) \cong \begin{cases} \mathbb{R}^1, & p = 0 \\ 0, & 0 < p < n \\ \mathbb{R}^1, & p = n \end{cases}$$

At this level, we are adequately satisfied with the observations and results obtained to settle with this generalization. Anyway, this is a powerful generalization which makes is possible to say how many solutions there are, in addition, we can say with certainty that there is always a spot on the n-sphere where it is windless. Namely due to the duality between the homology group and the cohomology group, where the latter characterise the solutions space to the differential equations.

We have spend a lot of time looking at the wind-field on the 2-sphere and used the connection between analysis, algebra and topology to say something about the wind-field on the n-sphere. Let us go back to the torus and see how the wind-field, which would be the 1-forms, would act on it.

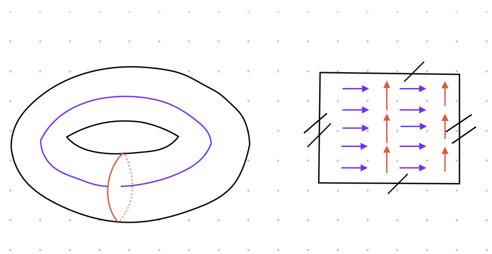


Figure 11: Visual representation on how the wind-field acts on a unfolded torus.

As seen in figure 11 the wind-field can act in a vertical and horizontal direction on the surface. We take it to be a closed, constant field. The purple vectors represents the wind-field that acts along the purple circle, and points in the same direction, correspondingly for the red vectors. From just analysis and topology we can already tell that there is no point on the torus where it is windless. Furthermore, the elements in the homology group is given by the circles modulo deformation, and as we know from equation 7, the number of elements in homology group is equivalent to the cohomology group, which gives us the number of solutions to the differential equations. We have already seen $H_1(T^2) \cong \mathbb{R}^2$, so $H^1(T^2) \cong \mathbb{R}^2$. Just at we excepted from the observation and the homology group, we get two solutions, the constant wind fields represented by the purple and red arrows

5.3 Generalization of Maxwell's

Until now, we have claimed that $H^p(M)$ count the number of solutions to Maxwell's equations for p-forms. Let ut make this connection more explicit.

James Maxwell was a major contributor in physics, and his equation provided us with a mathematical model to describe and understand electromagnetic fields. Those equations are in general restricted to differential 2-forms on M_4, which is the Minkowski space. Field theory is a part of classical physics, and although classical physics has provided us with a lot of information about our world, there is still much left to learn and theories to prove.
Up until now, we have looked at differential forms, and been able to generalize them to apply to manifolds, and made use of their connection with algebra to find the solution space of Maxwell's

equations. By generalizing Maxwell's equation to higher dimensions, mathematicians and physicists believes we could get one tiny step closer to explaining the world.

Maxwell's equations

To illustrate some notation, let us consider Maxwell's equations:
$A_\mu = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 \end{bmatrix}$ is a electromagnetic four-potential, where $t = x_0$ represents time, and x_1, x_2, x_3 are the directions in $\mathbb{R}^3$. $x^\nu \in \{x^0, x^1, x^2, x^3\}$ is an spacetime point with four elements.
We take the derivative of A_μ with respect to x^ν;

$$\frac{\partial A_\mu}{\partial x^\nu} = \partial_\mu A_\nu \qquad \rightarrow \qquad \partial_\nu = \frac{\partial}{\partial x^\nu}$$

The differential of the electromagnetic potential is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $F_{\mu\nu}$ is the antisymmetric matrix

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_2 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

Here $\vec{E} = (E_1, E_2, E_3)$ and $\vec{B} = (B_1, B_2, B_3)$ are the electric and magnetic fields. Then we will have

$$dF \sim \partial_{[\sigma} F_{\mu\nu]} = \frac{1}{3} \left(\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} \right) = 0, \tag{8}$$

And equation 8 gives two of Maxwell's equations, given that we are in vacuum, and the constants $\rho = J = 0$, and $\epsilon_0 = \mu_0 = 1$. And the same will be true for

$$\eta^{\sigma\mu} \partial_\sigma F_{\mu\nu} = 0,$$

using Einstein summation convention, where η is the Minkowski Metric.

We have the electromagnetic potential, $F \in Z^2(X)$. F is a closed 2-form on a n-dimensional curved manifold.
$dF = 0 \Rightarrow \bar{F} \in H^2(X)$, where $\bar{F}$ is the coset in the cohomology group containing elements equivalent to F, such as $dF = 0$. Additionally, to solve Maxwell's we want an element so that $\eta^{\sigma\mu} \partial_\sigma F_{\mu\nu} = 0$ on M_4, which is not curved, but flat.

Generalising Maxwell

Let us assume that X is a Riemannian, then it exist a isomorphism such as:

$$\star : \quad \Omega^p(X) \longrightarrow \Omega^{n-p}(X) \tag{9}$$

where equation 9 is the Hodge duality. We are now going to set the electromagnetic field to be a arbitrary p-from; $F \in \Omega^p(X)$. Then

$$\left. \begin{aligned} dF &= 0 \\ d \star F &= 0 \end{aligned} \right\} \text{Maxwell's equations on a p-forms for n-dimensions}$$

The solutions to Maxwell's equations is called harmonic p-forms: $\mathscr{H}^p(X) = \{F \in \Omega^p(X) \mid dF = 0,\ d\star F = 0\}$. While the harmonic forms describes the solutions to Maxwell's equations, is does not say anything about how many there are.

Theorem 5.1. *Hodge theorem:*
There is a isomorphism between the harmonic p-forms and the p-th cohomology group

$$\mathscr{H}^p(X) \cong H^p(X).$$

By Hodge theorem can we count solutions of Maxwell/harmonic forms using cohomology and topology without knowing what they look like. Let us look at the 2-sphere, and use Hodge theorem to find out how many solutions there is to Maxwell's equations. As we already know, Maxwell's equations are for differential 2-forms, so we get:

$$\mathscr{H}^2(S^2) \cong H^2(S^2) \cong \mathbb{R}^1.$$

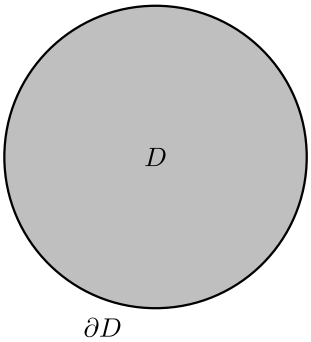
Hence, there is only one solutions to Maxwell's equations on the 2-sphere. For the 3-sphere Maxwell's equations has $\mathscr{H}^2(S^3) \cong H^2(S^3) \cong 0$, so no solutions.

5.4 Boundary value problem and topology

Differential equations that we are familiar with from calculus are not always easy to solve. It is not always given that there exist a solution, and if there does exist a solution it is not given that this is a unique solution. As you may expect at this point, there for sure exists a connection to topology which gives us the answer if there does exists a solution, and in this case, also how many. Let us consider following example:

Example:

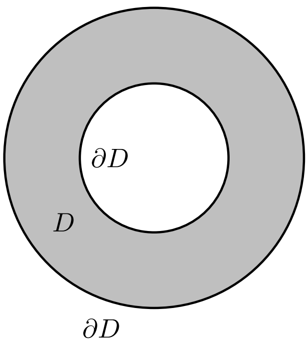
i) Find the harmonic 1-forms on a disk $D \in \mathbb{R}^2$ with the given boundary values:



$v \in \Omega^1(D)$
$dv = 0$
$d \star v = 0$
v takes the same value on all of the ∂D

We can collect all of the points on ∂D and gather them together so they form a ball $\mathscr{H}^1(D) \cong \mathscr{H}^1(S^2) \cong H^1(S^2) \cong 0$. So there is no solution to this boundary value problem.

ii) Find the harmonic 1-forms on a disk $D \in \mathbb{R}^2$ with given boundary values:



$v \in \Omega^1(D)$
$dv = 0$
$d \star v = 0$
v takes the same value on pairwise points on the inner boundary and outer boundary.

Similar to the previous example, we collect pairs of points on ∂D v takes the same value and gather them together so they form a doughnut. $\mathscr{H}^1(D) \cong \mathscr{H}^1(T^2) \cong \mathbb{R}^2$. This boundary value problem has two solutions.

Without even knowing how the solutions looks like, we can still find out how many solutions there are! We see how analysis, boundary value problems and topology are related.

6 Summary

This thesis started out by introducing the need of mathematics in higher dimension in order to have an language to describe physical theories. To do so we have encountered different mathematical branches, and learned how they are all connected together.

$$\begin{matrix} \text{Analysis /} & & \text{Algebra /} & & \text{Topology /} \\ \text{Solutions to diff. eq.} & \longleftrightarrow & \text{Cohomology} & \longleftrightarrow & \text{Geometry} \end{matrix}$$

On the way we have learned a lot of analysis, which gave us the remarkble tool to make it possible to do mathematics in higher dimension. We saw how differential forms is used to describe our differential equations in higher dimensions. From algebra learned some concepts from group theory which gave us the opportunity to define geometric invariants for manifolds known as cohomology groups. And before we connected all the dots we became well acquainted with topology.

We have seen how a manifold is built up by local patches, and can be locally mapped into $\mathbb{R}^n$. Through an example, we saw how the sphere could be mapped into the plane, and how the local coordinate maps are differentiable, and used this to define differential forms on a manifold. We learned that the geometry affects whether a differential closed form also is an exact form, and used our knowledge about factor groups to characterize which closed forms differ from exact forms. This is what we called cohomology groups, which gave us the the number of solutions to Maxwell's differential equations on the manifold. From Hodge duality and Poincarè duality we saw how we could count the number of solutions in more complex geometries. And with these dualities we were able to find the number of solutions to Maxwell on the torus and n-sphere. For more general differential equations, the geometry is given by its boundary conditions, and we saw how the Hodge theorem could be used to count solutions to boundary value problems.

The goal for this thesis was to see and use the connection between analysis, algebra and topology. From here, be able to use this connection in physics, and see how Maxwell's equations could be generalized in higher dimensions, which we successfully have done. The successful generalization of Maxwell's equations in higher dimensions opens up new possibilities for further research and innovation in this field.

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