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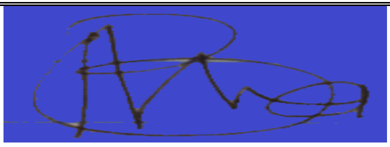
Author: Blessings Alice Muhuwa	

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Investigating opportunities for teacher learning in Lesson Study when promoting geometric reasoning in a Malawi secondary school classroom

Dedication

I dedicate this work to my lovely husband and son (Ackim and Ackim). You sacrificed your happiness for me to stay away from you for my studies. You have been my source of hard work and for all the encouragement I say, thank you. I do not take this for granted. God almighty rewards you more.

To my parents and siblings, I greatly appreciate your endless support, prayers, and encouragement. Dad and mum, thank you for trusting me, I am what I am because you always stood by my side.

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Dad and Mom, you are the kind of rare parents every child deserves to have. I remember sometimes back; you could choose not to buy yourselves luxury things because you needed your daughter to go to school and live a better life like other girls. You never looked down upon me on anything. You always told me “You can make it daughter”. You have been my strength and reason to prove to the world that even a girl child has a bright future if given all the requirements and the opportunity to demonstrate her ability. Thank you so much, I love you. Live long and remain blessed my parents. To my brothers, Chris and Connex, and my sister Eluby I appreciate every support and love you rendered to me. Keep the unity dears and remain blessed.

Abstract

Teacher learning in lesson study has been the focus of recent research by many authors. The current study considered teacher learning as a change in teachers' noticing of their teaching-both on their students' learning and classroom routines. Literature reveals that noticing students' thinking in a class as a whole or a particular student and the teacher's pedagogy is one way of achieving teacher learning. In this study, van Es' (2011) framework for learning to notice students' thinking was used to analyse a group of five teachers in a Malawi secondary school on the teaching of Pythagoras theorem. Though only one cycle was followed, the study aimed at finding the opportunities lesson study offered the teachers to learn as they were trying to promote geometric reasoning in their students. Three opportunities were identified as having the potential to support the teachers' learning. These were the teachers' collaboration, a mistake committed in the lesson, and the presence of a knowledgeable other.

Keywords: teacher learning, lesson study, Pythagoras theorem, noticing, and teacher noticing.

Abbreviations

LS Lesson study

MANEB Malawi National Examination Board

MoEST Ministry of Education Science and Technology

NSD National Centre for Research Data

PGR Promoting Geometric Reasoning

TPT Teacher Professional Development

PD Professional Development

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Chapter 1: Introduction

The current research investigates the opportunities for teacher learning in lesson study when promoting geometric reasoning in a Malawi secondary school classroom in the teaching of Pythagoras' theorem. This chapter gives an overview of the study in terms of background, problem statement, purpose, research questions and significance.

1.1 Background

Japan International Cooperative Agency (JICA) in 2008 launched a three-week Seminar for Mathematics Lesson Evaluation for African educators to learn about the Japanese model of lesson study (hereon LS). The seminar aimed at checking how to deepen and formulate viewpoints necessary for mathematics lesson evaluation and how that contributes to lesson improvement in those countries and Malawi was one of the beneficiaries (Fujii, 2013). It was from this encounter that teacher educators (TE) from Malawi teacher training colleges (TTC) were introduced to LS.

Huang et al. (2018) recognised that TEs in Malawi especially those new to LS had problems in coming up with research questions, predictions, and observations. For the research question, the teachers did not understand the usefulness of having it before planning, while in predictions they were troubled in predicting their students' answers to the given tasks. In observation, the challenge was not so much compared to the two formerly mentioned. In the repeated cycle, improvements were noted in all the areas (research lesson, predictions and observations) (Huang et al., 2018). In agreement with Huang et al. (2018), Fauskanger et al.'s (2019) study discovered that the TEs struggled in focusing on their learning in line with student teachers' learning and that they also had problems predicting student teachers' responses. The findings of these two studies are in harmony with the earlier study by Fujii (2013). Despite the challenges noticed, participants still benefited in other ways. Just like Lewis et al. (2019) found out that LS has four different types of outcomes on teachers which are; knowledge, motivation, self-efficacy, and capacity to enact knowledge of content and teaching in a classroom. The four outcomes observed by Lewis and colleagues stood as the inspiration for the current study that LS impacts teachers in different angles therefore, it was necessary to investigate the opportunities for teacher learning in LS when promoting geometric reasoning in a Malawi secondary classroom context.

The LS was next extended to primary schools in a project called "strengthening numeracy in the early years of primary school through a focus on the professional development of teachers" by the University of Malawi (UNIMA) in collaboration with the University of Stavanger (UiS) (Fauskanger et al., 2021). In their study, Fauskanger and colleagues discovered that the teachers who took part in the LS cycle recognised that they lacked concentration on; how to involve students in mathematics sessions, how to create space for the students' participation, and how to emphasise the value of letting students explore mathematics for themselves (Fauskanger et al, 2021). The realisation of the mentioned weaknesses shows that LS created an opportunity for the teachers to learn which acts as a good sign of progress. Subsequently, the team from UNIMA proceeded with the project to secondary school with the help of a South African project called Wits Maths Connect Secondary Project (WMCS) and the focus was put on geometry (Adler et al., 2023; Mwadzaangati et al., 2022). They named the project Promoting Geometric Reasoning (PGR).

PGR is an introductory project of LS for Malawi secondary school mathematics teachers (Adler et al., 2023; Mwadzaangati et al., 2022). The project's main goal is to provide professional development (PD) experiences to enhance the teaching and learning of geometry (Adler et al., 2023) in secondary schools through LS. PGR is working with two secondary schools in Malawi. This study is a follow-up of the PGR project in one of the two schools' second phase of the LS involving Pythagoras theorem. Adler et al. (2023) and Mwadzaangati et al. (2022) did studies on the first LS phase on the exterior angle of triangles and lines and angles where attention was centred on language use and exemplification respectively (see the next section).

1.2 Problem statement

In Malawi, mathematics is taught in all classes and compulsory for all students at all levels. For instance, the curriculum matrix for both junior and senior secondary school indicates mathematics in category A, a set of core subjects which are mandatory in all schools (Ministry of Education Science and Technology (MoEST), 2015). In Malawi secondary schools, geometry covers up to 33% of the total topics (16 out of 49 topics) of mathematics in both junior and senior secondary curricula making it the second-largest covered area of study after algebra (MoEST, 2013a, 2013b). One of the rationales for teaching mathematics in the Malawi curriculum is that; "mathematics is a vehicle for the development and improvement of a person's intellectual competence in logical reasoning, spatial visualisation, analysis and abstract thought" (MoEST, 2013a, p. xi). Logical reasoning and spatial visualisation are products of learning geometry, this emphasises the significance of successful learning of geometric concepts.

Geometry is one of the most important branches of mathematics that has a direct application to real-life situations and in other areas of study like engineering, art, and geography (Singh & Kumar, 2022). However, the literature shows that it is one of the hard-to-learn and hard-to-understand hence hard-to-teach topics (Serin, 2018; Tachie, 2020). In a study of challenges in teaching geometry in South Africa by Tachie (2020), lack of subject matter content was observed as a major setback which resulted in two problems for teachers 1) chalk-and-talk teaching where teachers were talking and writing a lot without giving explanations of the content for students to understand and 2) the negative attitude of the teachers which was noted as a result of a lack of confidence (see Chapter 2). Similar problems are faced in Malawi, and they affect students' performance in geometry. For instance, the 2019 Malawi National Examination Board (MANEB) chief examiners' report for mathematics indicated that students had difficulties with questions from the core element of space and shape which is the core element where geometry fall (MANEB, 2019). Two consecutive MANEB examiners' reports (MANEB, 2019, 2020) indicated that students were applying the Pythagoras theorem on angles and some were applying it on a non-right angled triangle.

The present study acknowledges that several pieces of research on LS have so far been done with most of the current ones focusing on teacher learning in LS (e.g. Adler et al., 2023; Karlsen, 2022; Mwadzaangati et al., 2022; Uffen et al., 2022). Some studies focus also on teacher noticing of which many focused on pre-service teachers and few on in-service teachers (Santagata et al., 2021). The socio-cultural and cultural-historical approaches have been some of the most used in the studies on teacher learning (Hervas & Medina, 2021; Karlsen, 2022; Lee & Tan, 2020; Uffen et al., 2022).

The few studies the current researcher managed to find for Malawi were on Challenges faced in LS implementation by Fujii (2013) and Huang et al. (2018) who studied the TEs work, challenging primary school teachers' views and understanding LS in teacher educators by Fauskanger et al., (2019, 2021), Adler et al. (2023) and Mwadzaangati et al. (2022). The last two studies were developed on the ongoing PGR project. Mwadzaangati et al. (2022) study investigated how Malawi teachers learn about exemplification in teaching geometry. They put their emphasis on variation in geometry examples with a consideration that in the struggle of promoting learners' geometric reasoning, examples play a greater role.

From their observation, they argued that the PD workshop which the teachers attended instituted learning likewise the presence of an expert educator or a specialised lecturer in teacher reflection commonly known as a knowledgeable other (KO) (Quaresma et al., 2018). Reflection on what was successful or not in the classroom also instituted learning in the teachers (Mwadzaangati et al., 2022). How these contributed to teacher learning was not addressed and the focus was only on exemplification. While in the study by Adler et al. (2023), they described how LS might contribute to teachers' learning of language and the role of theoretically networked teaching frameworks in teacher learning. From their observation, they argued that the teachers changed their way of using words compared to their first lesson plan such that it became more meaningful and detailed in the second lesson plan. They further argued that the teachers also changed the way they were correlating the supporting diagrams to their language. This study emphasised teacher language learning.

In brief, the above-indicated students' challenges by the two MANEB (2019, 2020) reports reveal that the students' problems relate to how they learnt the concepts. To my knowledge, this may mean the students misunderstood the concept and the misconceptions were not observed and addressed by their teachers in the classroom. With consideration of the observations and emphasis made by the two earlier studies for secondary school, Adler et al. (2023) and Mwadzaangati et al. (2022), the current study was designed to investigate teacher learning in LS in the aspect of how the teachers predict, observe, and promote their student's mathematical thinking.

1.3 Purpose of study

The current study aims at investigating the opportunities for teacher learning in LS when promoting geometric reasoning in Malawi secondary classroom in the teaching of Pythagoras theorem, a grade 10 (form 2) topic in Malawi's lower secondary school mathematics curriculum. The study assumed teachers' learning in LS could promote their ability to deliver the concepts and follow up on the students' misconceptions and correct them at an early stage. Data from the ongoing PGR project in the form of lesson plans, field notes, and transcribed video-recordings of lesson planning, classroom teaching and lesson reflection discussions were gathered as tools to be used in the analysis.

1.4 Research questions

The following main research question guided the present study: How does LS offer the opportunity for teacher learning when promoting geometric reasoning in teaching Pythagoras theorem in a Malawi secondary school classroom? Three specific research questions were posed in answering the main question, these are:

1. What do teachers notice when planning in LS in teaching Pythagoras' theorem?
2. How do teachers' predictions and observations in LS promote students' thinking and learning of Pythagoras' theorem?
3. What knowledge do teachers gain from classroom incidents in teaching Pythagoras theorem?

1.5 Significance of the study

The findings of the present study may give a clue to Malawi MoEST through the Department of Teacher Education and Development (DTED) when planning for teacher professional development (TPD). They can adopt LS in the education system by creating academic development programmes that incorporate instruction observation, collaboration, research, and reflection (Hervas & Medina, 2021). In addition, the findings might add knowledge to the earlier studies on teacher learning in LS in Malawi by Adler et al. (2023) and Mwadzaangati et al. (2022). From the current study, the knowledge is focused on what teachers notice when planning in LS, how they make predictions and observations to promote students' learning and the knowledge teachers attain in LS. While as a pedagogical implication, the findings may be useful for teachers to boost understanding and improve the teaching of geometry hence improving mathematics education. Just as Jones (2000) highlighted that for teachers to properly teach geometry at secondary school they need to attain a full understanding of the mathematics to teach.

1.6 Scope of the Study

The core focus of the study was to discover the possibilities of teacher learning in LS when LS was introduced in a Malawian secondary school while they thrive to promote students' reasoning in geometry specifically in the teaching of Pythagoras theorem. This was done by exploring the following things: the teachers' noticing in the two planning and the implementation of the planned research lessons. The teachers' predictions and observations in the planning and presentation of examples to their students were also assessed. No focus was put on individual teachers. The study followed only one cycle of LS (see Chapter 2) which may not be enough to generalise learning as it is a gradual process just like development (Ono & Ferreira, 2010; Van Driel & Berry, 2012). In addition, changing one's traditions in teaching resists as Stigler and Hiebert (1999) articulated that teaching is considered as a cultural activity and is learnt over a long time.

1.7 Chapter Summary`

This chapter has shed light on the background of LS and PGR in Malawi, what motivated the study and the purpose of initiating it, the research and its guiding questions, and the importance of the study. In brief, Malawi adopted LS as a PD beginning with teacher educators then primary school teachers and now to secondary school through the PGR project intending to promote geometry teaching in secondary schools. Two other studies on teacher learning through exemplification and language use are available on the PGR project and the current study wished to investigate the opportunities for learning of the involved teachers by observing how they noticed their predicted and observed work for their students from planning to reflection in their discussion sessions. In terms of importance, the study will add to the existing knowledge on teacher learning in LS, it may give clue to the Malawi MoEST in planning for TPD in the future, and can also support in improving mathematics education in Malawi by increasing teachers' knowledge in the teaching of geometry, Pythagoras theorem, which is one of the longest areas of study in mathematics and has a lot of real-life applications.

The study is segmented into six chapters. The first chapter is the introduction which is followed by the review of the literature and theory used in the study (Chapter 2), methods and methodology followed in the study in Chapter 3. Chapter 4 presents the findings of the analysed data, Chapter 5 covers the discussion of the results, and the last chapter presents the conclusion, pedagogical implication for further research, limitation and recommendations of the study.

Chapter 2: Literature Review and Theory

This chapter presents a brief overview of previous studies related to lesson study (LS) and geometry. The chapter is segmented into four sections of which the first section (2.1) provides a background of LS, its cycle, and a review of studies of LS beyond Japan. Next is a presentation of a brief description of geometry and Pythagoras' theorem in 2.2. A brief review of studies on teacher learning in LS is presented in 2.3 while the last part of the chapter (2.4) focuses on the analytical framework used in the study.

2.1. Lesson Study as an Approach for Professional Development

2.1.1 Background of lesson study

LS is currently among the most common and fast-growing approaches to TPD used (Dudley, 2015; Lewis et al. 2019; Warwick et al., 2016). There are a lot of studies carried out on the introduction of LS, its effectiveness and the challenges faced in its implementation in the adopted countries (Bjuland & Mosvold, 2015; Fujii, 2013; Murata, 2011; Stigler & Hiebert, 1999). This section reviews some of such studies as a foundation for the current study and shows the necessity of continuing with the study.

LS was first discovered in Japan somewhere around the 1880s (Pjanic, 2014) and was adopted outside Japan in the late 1990s (Murata, 2011; Seleznyov, 2018; Stigler & Hiebert, 1999; Takahashi & McDougal, 2016). The Japanese LS was made popular by the study of Stigler and Hiebert (1999), “The Teaching Gap”, where they observed video recordings of mathematics lessons from three countries, Germany, Japan and the United States of America. They discovered that in Japanese lessons the students were actively engaged in mathematics with inquired-based problem-solving activities. In Germany there was less involvement of the students rather the teacher was the one giving explanations and justifications needed. While for the USA, there were lots of interactions but not easy to see the mathematics (Stigler & Hiebert, 1999).

In that study, the teaching gap was found to be in terms of teaching methods, consideration of the teaching itself not the teacher, and the cultural nature of teaching (Stigler and Hiebert (1999). These three things brought the uniqueness of Japan from the other two nations' mathematics lessons. This provoked researchers to further investigate Japanese mathematics lessons. Since its discovery in Japan, LS has been an issue of international interest and in 2002 it came out as one of the foci in the ninth conference of the International Congress on

Mathematics Education (ICME) (Murata, 2011). Later it was adopted in many countries around the world as a new form of TPD.

This form of TPD has been considered unique and effective just as Murata (2011, p.2) argued that lesson study *"*incorporates many characteristics of effective professional development programs identified in prior research: it is site-based, practice-oriented, focused on student learning, collaboration-based, and research-oriented*"*. Agreeing with Murata are Dudley (2013), Lewis et al. (2019) and Qi et al. (2023) who indicated the evidence of teacher learning in LS (see Chapter 2.3).

The universal idea of LS is the coming together of teachers physically, with an agreed question about their students' learning, organising a research lesson that will make their students learn visibly and then they examine and discuss observations made within the lesson (Murata, 2011; Dudley, 2014). This is done to collect data about students' learning to improve instruction (Lewis et al., 2013). LS in Japan has been practised in three categories, depending on shape and size and thus at a small, medium and large scale (Chen & Zhang, 2019; Murata, 2012; Lewis et al, 2013). Small-scale LS happens as school-based, medium-scale happens as district-based while large-scale happens as national activity (Chen & Zhang, 2019; Murata, 2012; Lewis et al, 2013). Large-scale lesson study is vital when a new educational approach, new content, or a sequence of content instruction is introduced. The teachers together try to understand what it means by discussing, asking questions about it and constructing a shared understanding and this can be done through a live public research lesson (Dudley, 2014; Lewis et al, 2013; Murata et al., 2012).

Lewis (2009) noticed three types of knowledge teachers develop in a LS namely, knowledge of the subject matter, development of interpersonal relationships among teachers, and development of teacher personal qualities and disposition. Lewis et al. (2019) also tested the efficiency of LS for 20 years using four theoretical perspectives: knowledge integration environment, self-determination theory, self-efficacy theory, and pedagogies of practice. They discovered the following five pieces of evidence of LS impacts: "impact on students' learning; impact on teachers' knowledge for instance knowledge of tasks and pedagogical content; on teachers' belief; impact on routines and norms of professional learning for instance regarding students' interaction with content not content independently; and on instructional tools and routines" (Lewis et al. 2019, p.17). Lewis et al.'s study also proved that LS works both on the teachers' and students' learning. These findings show that LS is a promising tool for improving the quality of teaching and increasing teacher development hence it is necessary to study how the teachers learn in it.

2.1.2 Lesson Study Cycle

Literature shows different steps in which the LS cycle is conducted but the goal remains the same as indicated above. Stigler and Hiebert (1999) managed to identify eight main steps in the LS process which are: defining the problem; planning the lesson; teaching the lesson; evaluating the lesson and reflecting on its effect; revising the lesson; teaching the revised lesson; evaluating and reflecting again; and sharing the results. These steps make up the LS cycle and can either be fitted in one cycle or can be trimmed. In another review of the literature, Seleznyov (2018) identified sourcing outside experts (KO) as another component of LS. The KO can be someone with more knowledge in the field and who specialised in teachers' reflection (Murata, 2011; Quaresma et al., 2018).

Lewis (2009) presented a four-stage cycle of LS consisting of studying, planning, doing research lessons and reflecting (see also Lewis, 2019; Lewis & Perry, 2017; Murata, 2011; Murata et al., 2012 and Figure 1). These stages involve formulating student learning and long-term development goal, planning together a research lesson which puts the goal/s into action, one member volunteering to teach the lesson while others observe and get facts about students learning and development, reflecting on and discussing the data collected during the lesson to improve the lesson, the unit, or the instruction**.**

Further, the focus on the stages is shown by Lewis et al. (2019). They asserted that the study phase should aim at building team members' knowledge about the topic and helping them develop an understanding of how best to apply the knowledge in practice. According to Murata (2011), goal development for a start can be general and be narrowed down and focused within the process making it a specific research goal. In planning, she said the teachers choose or design a teaching approach which they wish to use to enable students to learn visibly focusing on the set lesson goal to test the effectiveness of the approach and not necessarily making a perfect lesson (Murata, 2011). Fujii (2013) and Stigler and Hiebert (1999) suggested that the choice of a topic should be based on how difficult or easy it is for students or it should be something hard for teachers to teach. The difference in planning this lesson from other lessons is that teachers try to predict students' possible responses and their thinking and reasoning; they also study curriculum materials which may assist in developing content knowledge (Murata, 2011).

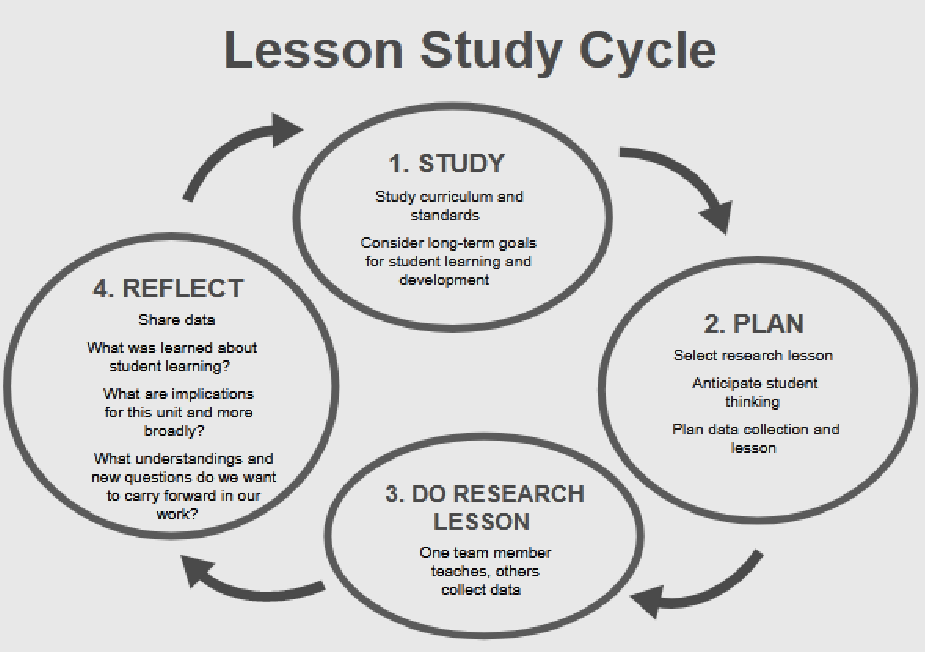


Figure 1: Lesson study cycle

Note: Adapted from Lewis (2009, p.97)

The do research lesson is the stage where one team member teaches the lesson while the rest observe how students are thinking and notice differences in their approaches (Lewis, 2009; Murata, 2011). Stigler and Hiebert (1999) included rehearsing and role-playing the lesson with fellow teachers before the actual day of teaching. The primary objectives of this stage are to test the team's ideas about how to teach the subject matter, cultivate the habit and ability of attentive student observation, and improve the calibre and effect of educators' talk by rooting them in a joint classroom experience (Lewis et al., 2019).

Finally, in the reflection stage, teachers meet at the end and discuss what they have noticed in the lesson, the lessons learnt about students learning, the implications of the unit and their understanding, and decide a new question they want to carry forward (Lewis, 2009; Lewis et al., 2019; Murata, 2011). From Lewis et al (2019), the main objective of the reflection phase is to express what each team member and KO learnt from the LS cycle so that others within the team and outside of it can benefit from this knowledge. This helps the participants to incorporate what they learnt into their thinking and practice, and for educators to reaffirm their commitment to improving their knowledge and practice as well as that of their colleagues (Lewis et al., 2019).

Currently, the LS cycle remains in four stages but there are some modifications for example, the inclusion of repetition of the research lesson (Dudley, 2013, 2015; Banda et al., 2021) and the use of a guiding theorem (Adler & Alshwaikh, 2019; Larssen & Drew, 2015; Schoenfeld et al., 2019). In Dudley's (2013, 2014 & 2015) modification to the Japanese LS, the teachers choose special students within the class, say three students, based on their progress in class (good, average or below average) on top of following all the normal procedures of the Japanese LS. The teachers' attention is put on these case students and at the end of the lesson, the cases are interviewed to obtain evidence of the student's learning.

Due to the differences in the adaptation of LS in different countries which can be a result of what Stigler and Hiebert (1999) said about teaching being a cultural activity, Lewis et al. (2019) presented goals, challenges, and possible ways of overcoming them in each of the LS stages. It can be stated that LS works better depending on set goals and how to move toward them (Lewis et al., 2019). For example, the goal of Japanese LS is to improve teaching and strengthen the teacher's professional community with a focus on problem-solving and student thinking (Lewis et al., 2019; Adler & Alshwaikh, 2019). What you want to achieve in LS should be clear to maintain focus.

An emphasis on the implementation of LS was indicated by Ebaeguin (2015) who reported two approaches: fidelity and a culturally embedded approach. He said the fidelity approach is when the Japanese LS is adopted directly as it is practised in Japan. The cultural-embedded approach is the one that changes the features of Japanese LS which cannot be transferred exactly due to cultural differences, without necessarily changing the whole LS process. Ebaeguin discovered that adapting and critically implementing LS makes it successful and gives teachers opportunities to grow professionally. Most current LS practices are based on the cultural-embedded approach with the addition of a component of guiding the LS with a theory (e.g., Adler et al., 2023; Adler & Alshwaikh, 2019b; Mwadzaangati et al., 2022).

Considering Ebaeguin's (2015) and Lewis et al.'s (2019) studies, knowing what to do in each LS stage and knowing what to adapt depending on your nation's education culture promotes the success of LS outside Japan. Many studies have indicated that LS leads to professional changes in teachers. For instance, Lewis et al. (2009, p. 286) claimed that LS "creates changes in teachers' knowledge and beliefs, professional community, and use of teaching-learning resources,". In addition, LS discussions along with mathematical content and student reasoning, have been shown by Murata and colleagues to aid teachers in developing new mathematical skills for instruction (Murata et al. 2012). Lewis and Perry (2017) also showed that teachers in LS can obtain subject-matter expertise by utilizing specialized mathematics resource packages.

The present study was developed on a medium scale as it involved two different secondary schools within the same district (of which only one is the focus here, see Chapter 3). Even though each school had its topic of concentration, the teachers had a common goal of promoting geometric reasoning, so they were having some common meetings and shared their findings. Adapting from South African LS, the current study followed a repetition cycle, so the lesson was repeated in another class after a second preparation (see Chapter 3.3). The current study aimed at investigating opportunities for teacher learning in LS when promoting geometric reasoning in a Malawi secondary classroom through the teaching of Pythagoras theorem.

2.1.3. Lesson Study outside Japan

After the LS discovery in Japan, other countries have been adopting and implementing it in their own countries. This section highlights some studies on LS in other countries beyond Africa and within Africa in terms of how they differ from each other, their achievements, and what has been done so far in Malawi. The countries are the United States, China, Norway, South Africa, Zambia, and Malawi.

2.1.3.1 Lesson Study beyond Africa

In Asia, three models of LS have been practised: the Japanese LS, the Mainland China LS, and the Hong Kong learning study (Chen & Zhang, 2019). The Japanese LS aims at constructing school culture, focusing on student thinking development and teacher learning. Mainland China's LS aims at expanding students’ learning by improving strategies of teaching, while the Hong Kong learning study aims at scientific investigation and student learning assessment (Chen & Zhang, 2019). The current study follows the Japanese LS, where the development of subject matter and teaching strategies is part of the LS process.

In the US, LS has been applied in different forms; some schools implement it directly as stipulated, while others modify it to fit their system constraints (Lewis et al., 2006). In their study, Lewis and Perry (2013) showed that LS has positively impacted students' and teachers' mathematical knowledge. On the other hand, Schoenfeld et al. (2019) asserted that LS in the US has not been so successful in its implementation. Schoenfeld et al. (2019) indicated that among American teachers, there is less time for combined collaboration among themselves.

This observation contributed to Schoenfeld et al.'s (2019) development of a guide in the form of a framework named Teaching for Robust Understanding (TRU) with LS to help improve collaboration (see 2.4). Nowadays, LS guided by a theory is being practised in many countries including South Africa and Malawi (Adler et al., 2023; Mwadzaangati et al., 2022). Malawi is where the current study is positioned.

In Norway, LS has commonly been practised in teacher education (e.g., Bjuland & Mosvold, 2015; Fauskanger & Bjuland, 2021; Munthe et al., 2016). From Munthe et al.'s (2016) and Larssen's (2019) studies, they realised that introducing improved teaching habits to student teachers contributes to lifelong professional learning and exposes them to the individual realisation of classroom realities and an understanding of how their actions affect the student's learning outcomes. Bjuland and Mosvold (2015) discovered the challenges LS can have in teacher education if not carefully considered before implementation. In their study, they identified four indicators that signal whether the LS will bring positive results or not. These are student teachers' lack of pedagogical content knowledge (PCK); focus on mathematics without a research question; lack of focus on observation; and difficulties in making structured observations (Bjuland & Mosvold, 2015).

The four indicators give hints that when preparing to conduct an LS as a form of PD, especially with student teachers, they at least need to have knowledge of the subject matter and pedagogy of the content as Shulman (1986) highlighted. They should also develop a question they need to investigate with their students throughout the cycle, they need to centre their observations on both the students and their learning as LS focuses on both (Lewis et al., 2019). In LS, student teachers should avoid basing the lesson's attention on their plan rather, the actions of the students during the lessons need to be considered as much as possible. The current study focused on qualified secondary school teachers, but the stated observation appears to be necessary when conducting LS at any level.

2.1.3.2 Lesson Study in Africa

In South Africa, LS has been used as a form of TPD. For instance, the Wits Maths Connect Secondary Project (WMCS), a project in South Africa adapted and shaped LS activities to fit in the SA context (Alshwaikh & Adler, 2017a). In their project, LS is being used as a medium of TPD with an inclusion of a theoretical lens, the lens called Mathematics for Teaching Framework (MTF) (Alshwaikh & Adler, 2017a). These researchers suggested two things from the findings; 1) students' errors should be a component to attend to in the planning of LS and 2) the role of participating observers should be known before the commencement of the teaching. In a related study by Adler and Alshwaikh (2019a), participating teachers expressed their satisfaction with LS. The teachers confessed they learnt about choosing and using examples. Following up on how changes in the examples happened, Adler and Alshwaikh (2019a) identified two ways; when the teachers noticed that the examples were not enough and when they realised that different representations were needed. More evidence of teachers benefiting from LS in South Africa was demonstrated by Helmbold et al. (2021). They discovered that content knowledge, pedagogy content knowledge and general professional development in LS execution were positively influenced by the teachers (Helmbold et al., 2021).

Uganda, Zambia and Malawi are among the African countries that adopted lesson study (Fujii, 2013). After JICA introduced LS to some African countries, the project team members conducted a two weeks follow-up study in the countries like Uganda and Malawi where they interviewed the JICA participants and observed research lessons followed by the post-lesson discussion (Fujii, 2013). In their observation, the team members noticed some misconceptions in the implementation of the LS. Some of their findings were: considering LS as a workshop, not as a teacher-led or bottom-up activity where the teacher initiates the activities; following a lesson plan exactly as planned without accommodating classroom situations; putting the teacher as the focus in the post discussion and not the teaching, and re-teaching a research lesson after evaluation which lead to adding steps to the four basic steps in the Japanese LS. These observations show that apart from having a live observation of Japanese LS and learning from it, countries adopted the development activity but implemented it differently. These modifications in some instances are for the better of the successful implementation of the PD to meet their education system for instance in the USA as indicated by Duez (2018); Schoenfeld et al. (2019).

In Zambia, LS was introduced in 2005 as a tool for science lessons and teacher skills and knowledge development (Banda et al., 2021). Zambia's LS follows a repetition cycle and has eight steps as below; 1) defining the problem or challenge; 2) collaboratively planning a lesson; 3) Implementing demo-lesson; 4) discussing the lesson and reflecting on its effect; 5) revising the lesson; 6) conduct the revised lesson; 7) discuss the lesson and reflect; and 8) compile and share reflections (Ministry of Education, 2010, p. 12). From the time of its introduction in 2005, LS was incorporated in all twenty-nine secondary schools by 2009 as part of the school curriculum (Banda et al., 2021). Despite the claim by Fujii (2013) that the inclusion of repetition of the cycle is a misconception, Banda et al.'s findings indicate that the implementation produced successful results in Zambia as the examination pass rate improved in schools where LS was implemented compared to where it was not. Two things highlighted as contributing factors to the success of LS implementation were the presence of well-trained facilitators and sufficient environments. The high pupil-teacher ratio and heavy teacher workload negatively affected the implementation (Banda et al., 2015).

In Malawi as earlier indicated in Chapter 1, LS was first introduced to TEs (Fujii, 2013), then to primary schools (Fauskanger et al., 2021), and next to secondary school teachers (Mwadzaangati et al, 2022). Despite challenges faced in its implementation in Malawi, LS has shown to be a better resource in teacher development. For instance, in Fauskanger et al.'s (2019) study, primary school teachers discovered what they needed to do in assisting their students to learn as shown in Chapter 1. Malawi like Zambia and South Africa adopted a repeated cycle of LS. Mwadzaangati et al.'s (2022) study used Mathematics Teaching Framework (MTF) to assess how these teachers (the same teachers studied in the current study) use variation against the invariant of examples in teaching geometry. These researchers found that there were variations in two forms: the complexity of examples and the orientation of the diagrams. The teachers realised the importance of giving examples from simple to complex and changing the orientation of diagrams to minimise memorisation.

2.2. Geometry and Pythagoras theorem

2.2.1 Geometry

Geometry is an area of study of mathematics that deals with shapes and space (Jones et al., 2012; Serin, 2018) whereas Duval (1998) defined reasoning as any process that enables someone to draw new information from given information. Thus, reasoning involves coming up with a fresh sequence of facts from what was given. This is done to show understanding, explain and convince others (Hershkowitz et al., 1998). On the other hand, geometric reasoning is the act of "inventing and using formal conceptual systems to investigate shape and space" (Battista, 2007 p.843). For students to develop mathematical reasoning they must be able to make, refine and test their conjectures (Gunhan, 2014). This is in agreement with what Hershkowitz et al. (1998) called 'Mathematising'. Mathematising is "a human activity by which elements of context are transformed into geometrical objects" (p.5). This means the ability to present visually the learnt mathematical concepts and then explain and represent them. Gunhan (2014) noted that having better reasoning skills implies good problem-solving in students. This relatively implies that emphasis on promoting geometric reasoning in students positively impacts their problem-solving skills. Therefore, it is the teachers' role to track the students' thought processes when promoting geometric reasoning.

The teaching of geometry has been hard for a long time for instance; Hershkowitz et al. (1998) observed that geometry has been taught in the context of deductive reasoning. From their scrutiny, it was presided over by two parts; 1) deductive reasoning as part of human culture to be learned and 2) a vehicle for verifying geometrical statements and showing their universality (Hershkowitz et al., 1998). In these aspects, the product which is written proof was considered the most essential part other than the process of proving. This shows that teaching tended to neglect both the visual geometric context and the students, hence encouraging memorisation.

Jones et al.'s (2012) paper concentrated on three issues they considered vital in geometry: the mathematical definitions, representations, and form of teachers' instruction. They highlighted that definitions are important in the deduction of different properties and the determination of reasoning and proving processes in students. But they said care should be taken in defining to avoid memorisation. Instead of giving students definitions, they should be actively involved in coming up with such definitions as supposed by de Villiers (1998). According to Jones et al. (2012), representation is another important thing when "introducing theorems, explaining proofs and posing problems" (p.2390). They added that to avoid mismatches, emphasis and care should be taken when providing external representation of diagrams. The promotion of geometric reasoning was also observed to be supported by teachers' varying ways of teaching instruction techniques and strategies. The strategies should direct the attention of students to facts of the content being taught (Jones et al., 2012).

Introduction of the geometric concepts in the lower classes plays a great role in students' understanding of geometry. As Gülkılık et al. (2015) highlighted, memorisation of mathematical concepts, as stated in the paragraph above, does not mean one has an understanding of the content, rather understanding is a process that develops from informal to formal with a basis of previous knowledge. To improve students' geometric thinking skills, different teaching methods must be practised (Serin, 2018). Therefore, teachers need strong pedagogical approaches on top of content knowledge to assist students to have better and more reasonable thinking. Courtney and Armstrong (2021) elaborated that, in teaching geometry, the teacher should help the students see and appreciate some concrete connection between the content and their previous knowledge, experience and the real world so that they can link with and make sense of it. With such connections, the students can easily learn and understand hence develop their reasoning skills.

In a study by Tachie (2020), teachers' lack of content and pedagogical knowledge were identified as challenges in teaching and learning geometry. This was observed as many teachers deployed to teach mathematics were partly and some were not trained. This made them lack confidence and hence had a negative attitude toward the teaching of Euclidean geometry, a mathematics topic. Many of the teachers were observed using chalk-and-talk teaching, teaching by following the textbook and explaining the concepts for students to understand (Tachie, 2020).

2.2.2 Pythagoras theorem

Promoting Geometric Reasoning in Pythagoras

Pythagoras theorem was named after a Greek philosophic-religious leader sect, Pythagoras, in around 500 BC (van der Waerden, 1978). Pythagoras' theorem is under Euclidean geometry which is a form of geometry that requires students' ability to connect new to existing knowledge (Kotzé, 2007). It is considered one of the most important theorems in geometry as it has more mathematical relationships and applications (Wittmann, 2021) as indicated in Chapter 1. In a right-angled triangle where the values of two sides are known, Pythagoras' theorem is applied to find the value of the third side (Wittmann, 2021).

Literature shows that defining and proving the theorem is done in many ways. For instance, Chambers (1999) indicated four ways of defining the theory and Loomis (1968) indicated up to 370 ways. From Chambers, the theorem can be defined by 1) just stating it in words with drawing ("the square on the hypotenuse is equal to the sum of the squares on the other two sides", p.23), 2) thinking in terms of areas and using a diagram of a right-angled triangle with squares on the sides (C=A+B) Figure 2, 3) using labels on the sides like a, b and c where c is the hypotenuse ($a^2+b^2=c^2$) Figure 3, 4) using labels on the vertices of the right-angled triangle like A, B and C ($AB^2+BC^2=AC^2$) Figure 4 (Chambers, 1999, p.23).

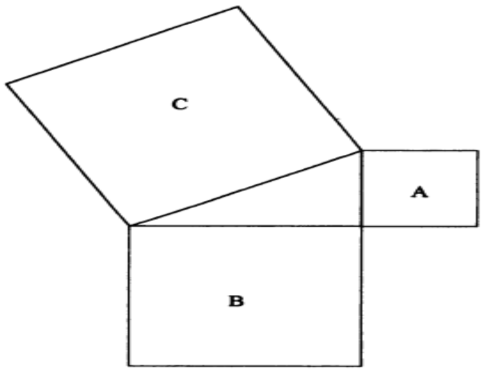


Figure 2: Right-angled triangle a

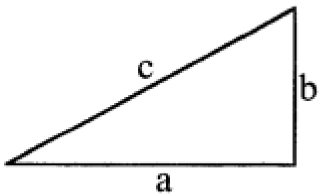


Figure 3: Right-angled triangle b

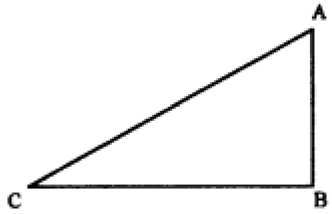


Figure 4: Right-angled triangle c

When teaching any geometry topic, Pythagoras theorem for example, teachers must make sure that the goals of the topic are attained therefore, teachers' awareness in teaching the ancient history of mathematics and the use of scaffolding can help (Wulandari et al., 2021). The students need to appreciate how the theory came in unlike just stating it to them (Chambers, 1999). Similarly, when designing a teaching unit on the Pythagoras theorem, Wittmann (2021, p. 130) proposed that introductory teaching of the topic should be clear with the following boundaries:

1. Students should be faced with a problem that is typical for the use of the Pythagorean Theorem and rich enough to derive and explain (prove) the theorem.
2. The conceptual underpinning of the unit should be as firmly rooted in students' basic knowledge as possible.
3. The setting should be as concrete as possible to account for different levels in the mastery of basic concepts, to stimulate students' ideas and to facilitate checking.

Wittmann also presented a teaching plan on one of the many ways in which a Pythagoras theorem introduction lesson can be planned. The plan starts with presenting the guiding problem, then redefining the problem, specialising the problem, generalising the solution, and finally discussing the formula where the history and importance of the theory are explained.

The highlighted observations show how challenging teaching the Pythagorean Theorem is. Teachers need to understand the concept and know the better definition/s and way/s of proving the theorem for students to understand and promote their reasoning. It is from this understanding that the current study was developed to investigate the opportunities for teacher learning through LS in Malawi in the teaching of Pythagoras theorem a junior secondary (grade 10) mathematics topic.

2.3. Teacher learning in lesson study

Most researchers currently on LS are paying attention to teachers learning in the process (i.e., Alshwaikh & Adler, 2017b; Karlsen, 2022; Huang & Huang, 2023; Qi et al., 2023). Xu and Pedder's (2015) review of the literature recognised that most of the studies analysed what teachers say about their experiences with LS as descriptive evidence for statements about teacher learning instead of focusing on teachers' discussions as an adaptor of learning and development. This is a weakness in the potential of discovering LS as a tool for teacher learning and PD. Another review of literature by Larssen et al. (2018) observed that learning or the details of the learning theory framework were not commonly defined and examined by researchers which resulted in a lack of coherency between results and theory. It is therefore necessary to understand the definition of teacher learning just like Larssen et al. (2018) articulated that observation of learning should be established based on how it is understood and defined.

Bakkenes et al. (2010, p. 536) defined teacher learning as “an active process in which teachers engage in activities that lead to a change in knowledge and beliefs (cognition) and/or teaching practice (behaviour)”. While Tyskerud and Mosvold (2018, p.53) defined teacher learning in LS as a “change in teachers’ discourse on teaching – either in their discourse on student learning or in their routines in the classroom”. From the two definitions, teacher learning can be demonstrated either in the way the teachers communicate with students or in their teaching practices, knowledge, and belief.

It is not yet clear how teachers learn in LS (Lee & Choy, 2017) but, several studies revealed what teachers learn through LS for example, a study by Qi et al. (2023) demonstrated that teachers understood the difference between their authenticity and students’ authenticity. The teachers realised that their assessment criteria in terms of language and expression stood above the knowledge of their students (Qi et al., 2023). Students’ preference was seen to be in working collaboratively with peers.

In another study by Alshwaikh and Adler (2017b), they recognised that learning in LS came about because of errors or mistakes that took place in the classroom caused either by the students or by the teacher. In one research lesson they had, some mathematical errors occurred with the students which the teacher of the lesson corrected. In another lesson, it was the teacher's mistake which the teacher of the lesson did not notice until during the reflection of the lesson (Alshwaikh & Adler, 2017b). They recommended that during planning teachers should foresee learners' errors and misconceptions that may arise and pose a question on whether observers should interfere in a research lesson. Karlsen (2022) on another hand indicated that teacher learning can be represented by a change in teaching approach like changing from the traditional model of receiving expert knowledge. She further claimed that putting together classroom observed evidence and interpreting them offers the opportunity for professional learning.

An earlier study by Cajkler et al. (2014) also disclosed that discussion sessions in LS provided crucial discursive possibilities through planning and in-depth reflection on the standard of teaching and learning. Similarly, consideration of the problems that students have with various areas of their learning of mathematics also helped to influence professional learning in LS Cajkler et al. (2014). Hervas and Medina (2021) highlighted several aspects that contribute to teacher learning in LS. They include sincerity and respect for other teachers' views, making clear underlying understanding, focusing on the group, not individual students, sharing personal pedagogy and professional knowledge, and the presence of a KO. The positive effect of the presence of the KO was also identified by Uffen et al. (2022) although the teachers preferred using a facilitator in sourcing information for the LS team over KO.

Another study by Vermunt et al.'s (2019) assessed the "Impact of lesson study professional development on the quality of teacher learning" following Bakkenes et al.'s (2010) work. They measured the three orientations of learning; application, meaning and problematic-oriented learning (Bakkenes & Vermunt, 2010; Vermunt et al., 2019). Application-oriented is where teachers try to improve teaching by making use of what they have learnt. In meaning-oriented, the teachers attempt to understand the cause and reasons for new practices and ideas, while problematic-oriented is where teachers find challenges in what worked in their teaching and how that happened (Bakkenes et al., 2010). Vermunt et al.'s (2019) realised that meaning and application-oriented learning increased in schools where LS has been practised while the problematic-oriented decreased in such schools, unlike the control school this shows that LS PD influences the quality of teacher learning through meaning and application-orientations.

Dudley (2015) affirmed that LS supports teachers' learning through dialogues, sharing gained insights and the experience of classroom work for both teacher and student. In learning through talks or dialogue, he indicated that throughout the planning, teaching, and reflection the teachers assume they are in the role thus they take responsibility. Within the talks, raising and testing the hypothesis of the lesson opens the teachers' minds hence developing chances of changing their beliefs and practices (Dudley, 2015).

There seems to be a relationship between teacher learning and teacher noticing. For instance, a literature review by Dindyal et al. (2021) considered three contexts of noticing namely: "noticing in the mathematical contents and along content trajectories, noticing in the practices contexts, and noticing in the context of teacher knowledge, skills, and dispositions" (p. 8). The third noticing context relates to teacher learning as both (teacher noticing and learning) deal with teachers' knowledge, skills, and disposition. Both teachers learning and noticing need to be applied in a context like mathematical context which is Dindyal et al.'s (2021) noticing context number one stated above.

On the same note, including an element of teacher learning when promoting teacher noticing suppresses the noticing from being influenced by their personal belief as Schoenfeld (2011) supposed that teacher noticing is intimately connected to the teacher's orientations and resources. He also claimed noticing is consequential, what one sees he or she does, hence leads to transformed practices. Cajkler et al. (2014) also suggested that teachers must be proficient observers to notice and comprehend instructional important elements in difficult classroom settings. These claims show that teacher learning can also be followed and assessed through their noticing of classroom situations. Thus, the current study was designed to investigate the opportunities of teacher learning in LS through observing the teachers noticing of their students' mathematical thinking in the teaching of geometry. A van Es (2011) framework for learning to notice student thinking was applied in the data analysis of the current study (see 2.4.1).

In terms of definition, the present study adapts Tyskerud and Mosvold's (2018) definition of teacher learning that it is the change in teachers' noticing of their teaching-both on their students' learning and classroom routines.

2.4. Theoretical Framework

A framework in research is considered a structure that provides "guidance for the researcher as study questions are fine-tuned, methods for measuring variables are selected and analyses are planned" (Liehr & Smith, 1999, p.13). It acts as a mirror for checking whether findings agree or disagree with the hypothesis or research question/s in explaining the findings (Imenda, 2014). The current study adapted the learning to notice students' thinking framework (hereon noticing framework) by van Es (2011). This section presents some possible frameworks for assessing teacher learning as used by other researchers. The other part of the section is to present the details of the analytical framework used in the analysis of the current study.

Xu and Pedder (2015) called for research to theorise teacher learning in LS of which several studies so far have been done in response, for example (Alshwaikh & Adler, 2017b; Cajkler et al., 2014; Karlsen, 2022; Huang & Huang, 2023; Qi et al., 2023). For example, Qi et al. (2023) used Interconnected Model for Teacher Professional Growth (IMTPG) combined with Bannister's framework. IMTPG believes in four domains in which changes in teachers occur, 1) the personal domain which they adapted as a group domain (e.g., knowledge, beliefs, and altitude), 2) the external domain (i.e., influence from beyond the teacher), 3) the domain of practice which was also modified to group activity experiences and 4) domain of consequences (significant outcome).

Alshwaikh and Adler (2017) and Mwadzaangati et al. (2022) used the Mathematics Teaching Framework (MTF) a mathematics framework developed by Adler and colleagues in the WMCS project in South Africa in 2010. MTF is an adaptation of Mathematical Discourse in Instruction (MDI), and it consists of all components of MDI with adaption to help teachers in planning and as a tool for observation of reflections on the quality of teaching mathematics in LS (Alshwaikh & Alder, 2017; Alder & Alshwaikh, 2019). Alshwaikh and Adler (2017) used MTF in their project as a research tool and a tool for teaching while Mwadzaangati et al. (2022) used it as a guiding framework in the LS.

Teaching for robust understanding with lesson study (TRU-LS) is another feasible framework for exploring teacher learning. It is a synthesis of the TRU framework and LS, the developers viewed that the two can overcome the difficulties of transferring LS from Japan to the United States environment (Schoenfeld et al., 2019). The framework gives teachers the chance to thoroughly investigate students' thinking in their classrooms, as well as to set up and put into practice LS research cycles (Schoenfeld et al., 2019). Teachers can select, study, and improve their research topic and theory of action for "kyouzaikenkyuu" (LS) using the TRU framework. TRU-LS work in such a way that all the stages of the LS cycle are guided by the

TRU conversation and observation guides with a focus on how students perceive the lesson (Schoenfeld et al., 2019).

The interest of the current study is to observe the opportunities for teacher learning in LS through noticing their work from planning to reflection based on how they collectively predict, observe, and promote students' mathematical thinking in the teaching of Pythagoras theorem. The researcher preferred the use of the noticing framework over TRU-LS and the MTF framework to achieve the goal of the study. TRU-LS would have worked better if it was to be applied as a guide for the LS process, but the current study had no option to implement it at the beginning. MTF on the other hand, could also have been a possible framework but for the sake of assessing a different angle of the teachers' learning in the PGR project, the researcher decided to use another analytical framework different from Mwadzaangati et al.'s (2022) study. The subsection below presents the discussion of the noticing framework adopted in this study.

2.4.1. A framework for learning to notice students' thinking

Jacobs et al. (2010, p.172) defined professional noticing as a set of three interrelated skills: "attending to children's strategies, interpreting children's understandings, and deciding how to respond based on children's understandings". So, in education, as defined by Jacobs et al, noticing interest needs to be paid to what students do and their understanding then decisions be made based on the two observations. Van Es (2011) described noticing as being an interpretive type of talk in which teachers attend to and make sense of classroom events and boundaries to make proper decisions on them. In van Es' description, noticing goes together with talking, interpretation and decision-making. Van Es further classified teachers' noticing in three areas which are "what stands out to teachers when they observe teaching, the strategies they use to analyse what they observe and the level of detail at which teachers discuss their observations" (p.137). She related the three areas to two main categories; *what* teachers notice and *how* they notice (van Es, 2011). The w*hat* and *how* of noticing produces four levels of noticing framework: baseline (level 1), mixed (level 2), focused (level 3) and extended (level 4) (van Es, 2011) see Table 1.

Table 1 A framework of teacher noticing students' thinking

	What Teachers Notice	How Teachers Notice
Level 1 (Baseline)	Attend to whole class environment, behavior, and learning and to teacher pedagogy.	Form general impressions of what occurred. Provide descriptive and evaluative comments. Provide little or no evidence to support analysis.
Level 2 (Mixed)	Primarily attend to teacher pedagogy. Begin to attend to particular students' mathematical thinking and behaviors.	Form general impressions and highlight noteworthy events. Provide primarily evaluative with some interpretive comments. Begin to refer to specific events and interactions as evidence.
Level 3 (Focused)	Attend to particular students' mathematical thinking.	Highlight noteworthy events. Provide interpretive comments. Refer to specific events and interactions as evidence. Elaborate on events and interactions.
Level 4 (Extended)	Attend to the relationship between particular students' mathematical thinking and between teaching strategies and student mathematical thinking.	Highlight noteworthy events. Provide interpretive comments. Refer to specific events and interactions as evidence. Elaborate on events and interactions. Make connections between events and principles of teaching and learning. On the basis of interpretations, propose alternative pedagogical solutions.

Note: Adapted from van Es (2011, p.139)

What is noticed includes *who* and the *topic* of discussion. *Who* is concerned with the emphasis of noticing like the whole class, learners in groups, specific learners or teachers themselves and the *topic* of noticing denotes identified matter such as "remarks focused on the pedagogical strategies, behaviour or mathematical thinking, or the classroom climate" (van Es, 2011, p. 138), while *how* noticing consists of the analytical positions and levels of depth of noticing. The analytical position covers the approaches taken by the teachers whether the inquiry is into the teaching and learning as well as evaluation or interpretation of what was observed (van Es, 2011). Evaluating involves judging good or bad or what could have been done differently in the lesson (van Es, 2011). Based on what and how the noticing was, van Es further grouped the levels into low and high levels. Low-level noticing is shown when teachers evaluate, describe or make claims while interpreting, explaining and giving reasons with a focus on students' mathematics representing higher level (Fauskanger & Bjuland, 2021b; Karlsen & Helgevold, 2019; van Es, 2011).

In distinguishing professional noticing from reflection, Criswell and Krall (2017) indicated that professional noticing prioritises recognition and responding to important events immediately while in reflection the focus is on sense-making at the end of the event. Schön (1983) pointed out that the two (professional noticing and reflection) can reinforce each other reciprocally. Professional noticing, on one hand, can support teachers in recognising what to reflect on and establish how to respond, on the other hand, reflection can drive the teachers to critically analyse their interpretation (Schön, 1983).

The latest study by van Es and Sherin (2021) suggested three things in expanding the van Es (2011) noticing framework. The three are 1) inclusion of the aspect of ignoring less impacting classroom situations as another part of attending in addition to choosing instruction observation attributes, 2) investigation of observed phenomena being essential in forming conclusions regarding observed occurrences, and 3) inclusion of shaping as another component in teacher boundaries of noticing.

Noticing has been used differently in different studies. For example, Jacobs et al. (2010) used "noticing-in-the-moment" which is similar to Schön's (1983) "reflection-in-action" where noticing is done within the process of teaching. Santagata (2011) used noticing-after-the-moment thus in the evaluation part of a lesson. Amador et al. (2017) and Choy et al. (2017) proposed that in LS noticing should be considered in all three parts which are; noticing during lesson preparation, lesson implementation and after the lesson (i.e., preparing-to-notice, noticing-in-the-moment, and noticing-after-the-moment). In terms of definition, Bakker et al. (2022, p.3) defined noticing in LS as the "process by which teachers jointly set up reasoning about pupils' subject-related learning (preparing-to-notice) and after (noticing-after-the-moment) the research lesson". In their study, Bakker et al. (2022) considered noticing during the preparation and evaluation of the lesson.

The current study followed Bakker et al.'s (2022), Karlsen and Helgevold's (2019), and Lee and Choy's (2017) studies by analysing preparation and reflection on the lessons. In addition, the study analysed the implementation part of the LS to observe the link between what the teachers planned and how the teacher of the lesson implemented it focusing on the teacher's noticing and supporting of the student's thinking. Unlike other forms of teaching, in LS teachers only collect data from the teaching process which they reflect on after the lesson

(Murata, 2011; 2012). This means there were no teacher-to-teacher interactions established during the teaching phase compared to the planning and reflection phases.

2.5. Chapter Summary

The chapter has presented a review of existing studies on LS as an approach for PD and teacher learning. It has been revealed that outside Japan LS is practised in modified ways for example, there is repetition of the research lesson, use of a guiding theory, and inclusion of case students who are interviewed at the end of the lesson. The chapter has also revealed that teaching geometry, for instance, Pythagoras theorem, has been a challenge for a long time. This in some occurrences was found to be due to the existence of many methods of introducing the concept and the lack of content knowledge in the teachers. It has been observed that teaching using approaches that actively involve students promotes their thinking skills and reduces memorisation. Teacher learning in LS is another thing addressed and it has been noted that teachers change can either be in terms of classroom discourse or classroom routines. Collaboration, classroom mistakes or errors and the presence of KO are some contributors to this learning. To achieve the goal of the current study, a teacher noticing framework was chosen to help in establishing the opportunities teachers have of learning in LS in the teaching of Pythagoras theorem in Malawi secondary school classroom. The next chapter will present the methods and methodology followed by the study and how the chosen framework was used to analyse the empirical data.

Chapter 3: Methodology

The current study is a qualitative case study designed to explore the opportunities for teacher learning in LS. This chapter describes the research design, research methods and instruments, research context, sampling procedure and sample size, data analysis procedure, validation and reliability, and ethical considerations. An important part of this chapter also describes the process of approaching the empirical material, a description of how data was collected and coded. In this process, the analytical framework (see 2.4) was linked to empirical examples from the data material, illustrating how the analytical framework was used to guide the analysis to generate results presented in Chapter 4.

3.1 Research design

Research designs are “plans and procedures for research that span the decisions from broad assumptions to detailed methods of data collection and analysis” (Creswell, 2009, p. 22). Of the three research designs, qualitative, quantitative and mixed methods (Creswell, 2009; Dawson, 2002), the current study adopted a qualitative procedure. Qualitative research is a “means for exploring and understanding the meaning individuals or groups ascribe to a social or human problem” (Creswell, 2009, p. 22). It allows the researcher to investigate the attitude, behaviour and experience of the participants in their setting (Dawson, 2002). Qualitative research is believed to produce data that is culturally specific and contextually rich through its use of open-ended questions and interaction between the researcher and the participant/s (Mack et al., 2005).

To build an in-depth understanding of the opportunities for teacher learning in LS, the current study followed a qualitative case study approach. The case study strategy of inquiry involves the researcher discovering in depth a program, event, activity, process, or individual in its actual occurrence (Cohen et al., 2007; Dawson, 2002).

3.2. Research methods and instruments

Dawson (2002, p.14) defined research methods as “tools used to gather data”. While Yin (2016, p. 138) defined data as “the smallest or lowest entities or recorded elements resulting from some experience, observation, experiment, or other similar situation”. There are several methods of collecting these elements in qualitative research, some of them are; observation, focus group discussions, and interviews (Creswell, 2009; Dawson, 2002; Yin, 2016). Three methods were used to collect data for this study: video observations of planning, teaching, and reflection sessions of two LS cycles (one about Pythagoras theorem and another similar triangle), document analysis in the form of lesson plans, and field notes.

3.2.1 Lesson study observation

Observation in research gives a researcher opportunity to collect first-hand data from a live-occurring situation (Cohen et al., 2007). The roles a researcher plays in observation may differ from being a complete participant, a participant as an observer, an observer as a participant, to a complete observer (Cohen et al., 2007; Yin, 2016). In the current study, the researcher was a participant observer. Participant observation involves researchers dipping themselves into the culture of their participants to make a trusting relationship so that they can be able to gain a deeper understanding of their culture and belief or feeling (Dawson, 2002). As a participant observer, I was observing the teachers planning, teaching, and reflection sessions and participated in the discussions at times.

The videos were recorded in all two iterations of the planning, teaching and reflection sessions of the LS cycle and these videos were the major data collection tool for the study. Video recordings were selected in this study because they are an unassuming way of data collection as they give a direct opportunity to the participants to share their reality, and visual concentration is confined (Creswell, 2014). Some challenges of using video recordings are that the data may be difficult to process and interpret as well as responses may be affected by the presence of the observer (Creswell, 2014). To overcome the two challenges, data analysis was repeated several times until it was understood and taking part as a participant observer also helped much as I had a physical follow-up and field notes to refer to. The second challenge was minimised by establishing a friendly atmosphere with the teachers.

3.2.2 Document analysis

According to Bowen (2009), document analysis involves reviewing or evaluating either printed or electronic documents. This study also used documents in the form of lesson plans which the teachers developed to use in their teaching. The lesson plans were used to trace the examples and flow of the planned work. This method was chosen because it allows the researcher to get the participant's exact language and word use, it is easily accessed at any time when the researcher needs it, and it saves the researchers’ time and costs of transcribing (Creswell, 2014).

3.3 Research context

The present study was developed from a pilot project (PGR project) of introducing LS in Malawi secondary schools where two schools were involved. Participants are serving mathematics teachers in those schools. The PGR project has two LS teams from two different schools. The teams worked on two different geometry topics selected within the group (Pythagoras' theorem and similar triangles). In the teams, I had been observing and participated without interfering with the teachers' discussions. The project started with a workshop for the teachers where they were trained on the promotion of geometric reasoning and were also introduced to LS as a PD. The teachers were then given time within their groups to select a geometry topic they wished to work on. This is when Pythagoras' theorem and similar triangles were selected by the teams from which the current study chose its mathematics focus, Pythagoras theorem. The PGR project LS adapted a theory-guided and repeated cycle from the WITS math connect project in South Africa to suit the Malawi context (Adler et al., 2023) (see Figure 5). The teachers started by studying the curriculum (grade 10 (form 2) books and syllabus). From their study, they identified two books and suggested methods to use in the implementation of the lesson. This was followed by first planning which was done twice; the very first lesson plan (LP1a) was solely done by the teachers which they later sent to the KO for checking. The KO then suggested some modifications to the plan which were discussed afterwards with the teachers, and changes were made where both parties agreed and came up with an improved plan. This is the lesson plan they used during the first lesson implementation (LP1b). For LP2 the KO did not make changes as it was an improvement from what they noticed during the first research lesson reflection.

Participating teacher 4 (P4) volunteered to teach the first lesson (T1) then reflection (R1) followed immediately after T1. The teachers planned the second lesson after the first reflection which was taught by P1, who also volunteered. The second reflection (R2) followed immediately after the second teaching (T2) (see Figure 5). The teachers at the end of R2 had a joint reflection with the other school where they presented their overall reflection of the whole LS cycle (R3). R3 was done in the form of a report on how the teams independently managed to answer their lesson research question/s and lessons learnt. The two research lessons were taught to students of the same grade but in different classes.

The present study consisted of eight participants in the teaching stage which were the researcher (the participant observer addressed as P6), the KO who is an education expert from a University within Malawi, five teachers (one female and four males) and about 100 male and female students (at most 50 per class) of ages 16 to 18. The planning and reflection stages had seven participants excluding the students. The teachers' experiences ranged from three to thirteen years as shown in Table 2: Participating teachers and their mathematics teaching experience. All five teachers in this study once had experience in teaching during the earlier phases of LS, thus, the two who taught in the research lessons of this LS cycle were their second-time experience.

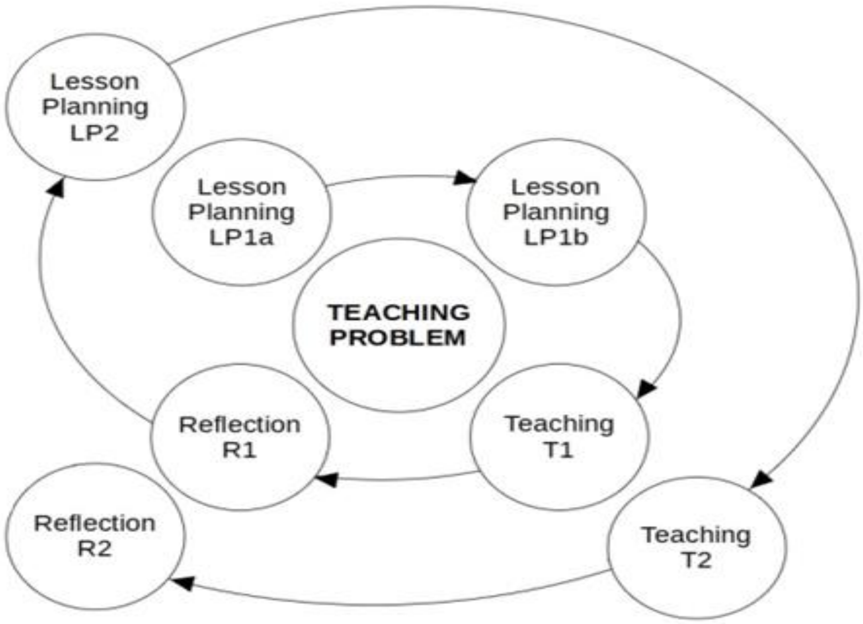


Figure 5: PGR lesson study

Note: Adapted from Mwadzaangati et al. (2022, p.221)

3.4 Sampling procedure and sample size

Mack et al. (2005) claimed that getting valid findings does not necessarily mean data should be collected from all members of a community rather a subset of it can still be a better representation. Research objectives and the characteristics of the population determine the type and size of the sample to use (Mack et al., 2005). With the current study's aim of investigating teacher opportunities for learning in LS, five experienced mathematics secondary school teachers in one of the cities in Malawi were identified (see Table 2). The sample was selected purposefully as the school is one of the two beneficiary schools of the

PGR project. In purposive sampling, the cases are handpicked depending on the researcher's judgement of the needed characteristics (Cohen et al., 2007).

I had access to data for both schools and it could have been interesting to study both groups as the topics they covered are equally important in geometry. However, considering the time factor analysing data for both schools would not be easily accommodated in this study. Instead, I chose the school that was working on Pythagoras' theorem with an interest in their research question of helping students to see and understand the relationship between the hypotenuse and the other two sides of a right-angled triangle. Reflecting on the 2019 and 2020 MANED examiners' reports on how students were confused about when and how to use the Pythagoras theorem as prior indicated supported the choice. The other reason was that I had more time as a participant observer with the Pythagoras theory school than I did with the similar triangles school.

Table 2: Participating teachers and their mathematics teaching experience

Participant teacher (P)	**Mathematics teaching experience in years**
P1	12
P2	4
P3	3
P4	9
P5	13

3.5 Data analysis procedure

Data analysis for the study was done in all the LS stages, planning 1 and 2, teaching 1 and 2 and reflection 1 and 2. The teachers' general reflection of the whole LS cycle was also considered. Each stage of the LS was analysed to facilitate the attempt to answer the three specific research questions of the current study which helped in answering the main research question - how does LS offer the opportunity for teacher learning when promoting geometric reasoning in teaching Pythagoras theorem in a Malawi secondary school classroom?

In the discussions, the teachers were given the freedom to express ideas in any language they felt comfortable with. So, they could alternate between English and Chichewa (the local language). The data was transcribed without editing any grammatical or colloquial language the researcher only translated the local language to English for easy communication with the audience. The translated words were written in square brackets and italicised. The words in curly brackets were used to interpret referenced work for instance if a teacher said, “I wrote that”, the researcher was clarifying what “that” was in the discussion. In the teaching stage, T is for the teacher, BS is for a boy student, GS is for a girl student, and SV was used when I could only hear a student's voice but could not tell the gender. SS was used when the same student was answering a follow-up question and MS for many students when answering as a group. Commentaries are put in round brackets and italicised while translations are put in square brackets and italicised.

The study used the learning to notice students’ thinking framework by van Es (2011) presented in chapter 2.4 to analyse its data. After transcription, further analysis process of the data followed three steps 1) division of the data into episodes and sequences, 2) coding of the episodes and sequences, and 3) content in-depth analysis.

The transcribed data in all the stages of the LS were segmented into episodes and some further into sequences. An episode represented a change in a topic of discussion while a sequence was considered a local shift within the topic of discussion. The use of episodes and sequences was employed as the unit of analysis. It was considered a better way in the process as utterance-by-utterance coding was found hard to use with the noticing framework.

The episodes and sequences identified were then coded. The coding used the noticing framework where ‘*what’* and ‘*how’* teachers noticed in their discussions and teaching were categorised into the four levels of noticing (L1-baseline, L2-mixed, L3-focused, and L4-extended). In some local shifts no noticing was observed as such the researcher coded those sequences as level zero-L0 (no noticing) (Bjuland & Fauskanger, 2023) see the example given below on low and zero-level noticing. This data was further segmented into two, high and low-level, where low-level combined 1 and 2 and high-level combined 3 and 4. This process helped in narrowing the huge amount of data that was available and in the identification of the teachers’ major focus in the LS.

Note that, the evaluation of how teachers noticed was based on six components adopted from Lee and Choy (2017, p.129), showing whether the statement; 1) was general or specific, 2) was descriptive, evaluative or interpretive, 3) was based on evidence, 4) elaborate on events and interactions, 5) made connections between events and principles of teaching and learning, and 6) proposed alternative pedagogical solution considering what the teachers notice.

Relating to van Es' (2011) noticing levels, general, descriptive, and evaluative noticing statements were considered lower level while statements that were specific, interpretive, based on evidence, elaborate events and interactions, made connections between events and principles, and propose alternative pedagogy solutions signify higher level noticing (van Es, 2011). Attending to particular students' thinking in this study was considered in two ways, as the teacher/s' focus on a student's stated idea and as the teacher/s predictions of their students' thinking on a particular concept.

A qualitative in-depth analysis was finally applied to respond to the research question for this study. In this phase, thematic episodes which were developed following the LS principles were assessed for the predictions and observations teachers made that allowed them to learn from the LS process. Content analysis was done. Here, patterns and relationships between the text in the dialogues and the research aim were identified. The consideration was on the display of a better amount of the teachers' predictions, observations, and promotion of their students' thinking on the tasks and examples. Content analysis according to Cohen et al. (2007) is a "process of summarising and reporting written data – the main contents of data and their messages" (p. 475). Since the LS was in two iterations, each stage in the iteration was considered separately then data was related. For example, T1 was coded independently of T2, later the similarity and differences were analysed to answer the research question. The same thing was done with planning and reflection 1 and 2.

Below are examples of zero and low-level coded noticing from planning 1 and reflection1 respectively.

P1: Winaakhale chair bwa *[someone should be the chair*] because it seems I'm the chair and I'm also writing.

P4: But you are doing a good job.

P1: No! Let us all participate.

P1: No, anthumuli [you *people are*] silent. Winaakhale [*someone should be*] chair or else winaazilemba [*someone should be writing*]

P4: The way you are combining we are very impressed.

P all*: (laughing*)

p4: Okay, let us continue.

In the discussion extract above, P1 wanted one of the participants to help him with the role of chairperson of the discussion as he realised he was performing two duties. None of the other participants showed interest and suggested he continued with the role. In this dialogue the teachers were not noticing anything related to students' mathematical thinking therefore it was coded level zero. In the part below, the teachers were reflecting on the first lesson:

P3: So, we are saying in terms of probing, aah probing was done eh?

P1: Was done.

P3: Participation in activities, the students participated in the activities. And did they come up with the right answers or the expected answers? Okay! And the mathematical language?

P1: Yes, was there.

They noticed that probing and involving students to participate happened in the classroom. This discussion shows that the teachers were attending to the whole class environment and did not provide evidence to support their analysis of the observation hence coded level 1.

Noticing in the planning discussion, the teaching of the research lesson, and the reflection discussion was noted to be different. This brought a difference in the use of the framework in the three sessions (planning, teaching, and reflection). Below is a brief explanation of how the differences were addressed in the analysis process.

When coding the planning sessions, the researcher applied only the 'what' part of the noticing framework as traces of 'how' teachers noticed was not clear. Therefore, in the planning sessions, noticing focused only on 'what teachers planned to notice' adopting Fauskanger and Bjuland (2021a). Further analysis was done on the high-level noticing to identify what teachers planned in their predictions and observations on the tasks and examples to give to their students to promote their mathematical thinking. While coding the teaching and reflection sessions, traces of both 'what' and 'how' teachers noticed were visible and all the coded transcripts were further analysed using the noticing framework. The teaching phase was included to observe the link between how the teacher's plan and its implementation in the promotion of students' mathematical thinking were achieved.

Results of the analysis were presented from the episodes. For the episodes that had sequences within them, each sequence was coded independently. Wherever sequences had combined levels (low and high), the episode was considered a high-level episode because the researcher's attention as already stated was on the high-level noticing of the teachers. So, leaving out any high-level code was felt to be a loss in the analysis. To answer the research question of the current study, the selection of episodes presented in the next chapter (See Chapter 4: Findings) was based on the discussion that displayed a better amount of the teachers' predictions, observations, and promotion of their students' thinking on the tasks and examples.

3.6 Validation and reliability

Joppe (2000, p.1) defines reliability as "the extent to which results are consistent over time and an accurate representation of the total population understudy" while validity is the "determination of whether the research truly means that which it was intended to measure or how truthful the research results are". Though studies show that validity and reliability are more difficult issues to demonstrate in qualitative than quantitative research but they are important in showing the trustworthiness of the results (Cohen et al., 2007; Golafshani, 2015). Validity depends on reliability therefore, it is satisfactory to consider reliability first (Lincoln & Guba, 2023).

According to Mathison (1988), triangulation is considered one way of achieving validity and reliability in qualitative research. Triangulation is the "combination of methodologies in the study of the same phenomenon" (Denzin, 1970, p.291). While Joppe (2000) also articulated that, validity can be shown by asking several questions and seeking answers from other research. The current study used a combination of participant observation and documents to attain reliability. In addition, all stages of the LS were recorded and transcribed without editing the teachers' discussion and further analysed. A coding test of a smaller sample of the data was earlier done so that amendments could be implemented before approaching the whole data. This also helps in achieving reliability as highlighted by Weber (1990).

3.7 Ethical consideration

Consent to work with the teachers was requested from the PGR project coordinator (**Appendix** 1). When permission was granted, the teachers were consulted before the commencement of the research. Consent was also taken from the National centre for Research Data (NSD) (**Appendix** 2). The researcher tried in all ways possible to keep, respect, and protect the dignity, privacy, and interests of all participants as highlighted by Cohen et al. (2007). For instance, instead of names the study used number identification and when taking videos students were at all costs avoided. All participants took part voluntarily and were given the authority to withdraw at any time they chose to do so. Cohen et al. (2007) also talked about the purpose and procedure of research that it should be clearly explained which this study has done so that the audience will be able to follow through.

3.8 Chapter Summary

This chapter presented a description of how the study collected its data and how the framework was applied in analysing the data. Issues of validity and reliability have also been discussed as to how the study handled the ethics of research. Just in passing, data was collected in all the stages of LS; the data was in the form of videos, field notes, and documents. The videos were transcribed and coded using the four levels of noticing in all the stages of the LS and through all two iterations of the LS cycle. Episodes coded level 3 or 4 of the noticing framework were the ones considered for further analysis. Finally, content analysis was applied to interpret the data. The presence of teachers' predictions, observations, and promotion of students' thinking leads to the analysis of the data. The next chapter will present the findings of the analysis.

Chapter 4: Findings

The present chapter provides findings of the analysis of the empirical data from the planning, teaching, and reflection of the lessons. The LS analysed in this study was repeated in another class. For easy following, the results of each of the two iterations are presented separately within the chapter. The first three sections (4.1, 4.2 and 4.3) present findings of the first planning, teaching, and reflection while sections 4.4, 4.5, and 4.6 present findings of the second iteration. Section 4.7 provides findings from the examples used in the two lessons, overall reflection is presented in 4.8, and a summary of the chapter is placed in section 4.9. The researcher could not present the whole episode in each section in this chapter therefore, parts of the discussions in the episodes are the ones presented and each section contains two parts. Note that teacher noticing was used as the analytical framework and that high-level episodes were the only ones considered for further analysis and presentation.

4.1. First planning

Coding for this session generated thirteen (13) episodes of which eight contained sequences of high noticing within them. This implies that eight episodes were coded high and eight low, see

Table 3. The two parts of discussion identified in the first planning are the discussion on introducing the hypotenuse and the discussion on the teacher's plan on what to observe in the teaching.

Table 3: Noticing episodes in the first planning

Noticing level	**Low**		**High**	
	Baseline	mixed	Focused	Extended
Number of episodes	0	5	4	4

4.1.1. Introducing hypotenuse

When introducing hypotenuse, some teachers assumed that the students will be able to mention it. They later agreed to let the students draw right-angled triangles and show the hypotenuse. The idea of letting students draw was also discussed earlier in the introduction part where they needed assurance that the students know the triangle they will be working on both in words and diagrammatical form. In the second demand of drawing the triangles, the teacher's concern was on the variation of orientation. The part below is a dialogue that demonstrates what they discussed.

P3: They should draw their triangle,

P4: Mmmh.

P3: And show us the hypotenuse,

P4: Mmmh. But the problem is they will be limited in terms of orientation.

P A: Yah.

P 4: That will be the problem.

P1: True aah on the orientation, most of them I think will just draw aah,

P2: The normal one.

P6: The usual one.

After a time discussing how they should introduce the term hypotenuse, they also thought of whether the students will be able to mention it or not. Some teachers predicted that students will be able to mention it if asked to name the side opposite the right angle (90^o angle) while others thought they may not be able. The dialogue presented above is a continuation of that discussion. P3 suggested that they should allow the students to draw their right-angled triangles to show the side representing the hypotenuse. They all agreed but P4 predicted that the only problem will be on varying orientations of the right-angled triangles which the other three participants, P1, 2 and 6, agreed. They assumed the students would draw the normal/usual right-angled triangle with a 90^o angle located to the left side or what P1 called a ‘right, right triangle’ where the right angle is at the right-hand side of the triangle.

In the episode, the teachers were planning to attend to students’ specific thinking on a right-angled triangle in terms of shape, they also pointed out noteworthy events. The noticing in this episode was coded as focused (level 3) (see Table 1).

4.1.2. Discussion on what to observe in the lesson

In terms of planning what to observe in their lesson, the teachers wanted to promote mathematical language in their students by demanding and supporting them.

P1: [...] so we will still say that's the Pythagoras theorem. And this is how we state it. That is now mathematical language.

P2: Okay.

P1: Yes, and stating it in mathematical language ndekuti ka [*means we are*] supporting or developing language, Waiona imeneyo [*have you seen that*]? So, what we were doing, in asking them about the relationship, means we were demanding language

P2: Demanding.

P1: So now we are supporting language.

P2: Mmmh.

P 1: Yah. Should we go further in developing?

P2: No, we are only supporting.

P1: We are still on supporting only?

P2: Yah.

After Activity 3 where the students were developing the relationship of the three sides of a right-angled triangle to come up with the Pythagoras theorem, P1 elaborates that they still must tell the students that what they have discovered is the Pythagoras theorem and they should state it with better mathematical language. In so doing P1 claimed that they will be developing mathematical language in the students. P1 and P2 agreed with each other on the point claiming that the idea would promote demanding and supporting mathematical language in the lesson.

This discussion paid attention to the relationship between the student's thinking regarding their findings and teaching strategy. Where the teachers plan to demand from the students and support them in case of difficulties represent the teaching strategy. This episode was coded extended (level 4) (refer Table 1).

4.2. First teaching

In this session, six episodes were identified from which three were coded high and three low, see Table 4. Discussion on the complex triangle a student drew and how the teacher handled the observation and the challenge faced in the development of the relationship among sides of a right-angled triangle are the two parts of discussion identified in the first teaching.

Table 4: Noticing episodes in the first teaching

Noticing level	Low		High	
	Baseline	Mixed	Focused	Extended
Number of episodes	1	2	2	1

4.2.1 Right-angled triangle

During the teaching, the teacher followed their developed plan and asked students to draw right-angled triangles as part of the introduction of the lesson. Two students came and drew triangles as in Figure 6.

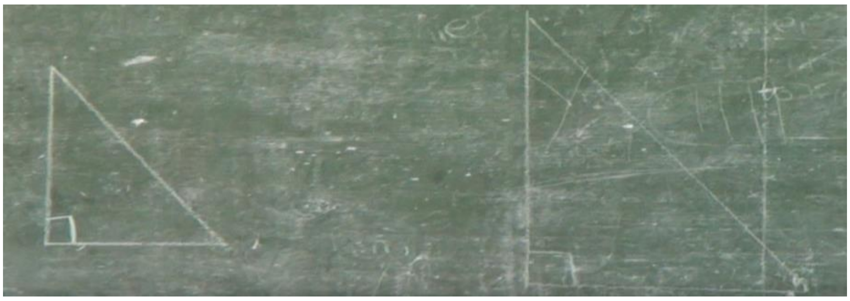


Figure 6: Right-angled triangles students drew

The teacher asked students if the two triangles are different from each other. The students acknowledged that the triangles were similar the only difference was size. Then the teacher demanded different oriented diagrams from the students by saying: “So, do we have another way of drawing a right-angled triangle? Or this is the only way how we draw a right-angled triangle, I just wanted to find out from you” (from empirical data). In response to the teacher's demand, one more diagram was presented (Figure 7) which also raised some controversy within the lesson. Below is the discussion that took place after the diagram was drawn.

Figure 7: The complex and split-up triangle

T: Okay so, we have another triangle, eh, this one (*points at the triangle*). What do you think class? (Silence for about 4 seconds). So, there is a triangle here (points at the triangle drawn by the students), and a right-angled triangle (points at one of the right angles in the triangle). So, what do you think about this triangle? [...] Is it a right-angled triangle?

MS: No.

T: So, why are you saying is not a right-angled triangle? What is the reason? (*Nominates a girl student*).

GS: Because it doesn't have a hypotenuse.

T: It doesn't have the hypotenuse?

SS: Yah.

T: Okay so I will come back to you on the hypotenuse. Someone is saying it's not a right-angled triangle because it doesn't have a hypotenuse. Okay, but any other reason? (*Nominates a boy who raised a hand*).

BS: A right-angled triangle has one slanting edge.

T: It has one slanting edge. So, someone is saying this one is not a right-angled triangle (*points at the big triangle*) because it has more than one slanting edge. Eeh what do you think, anything else?

BS: Okay I think the triangle contains two right-angled triangles.

T: Okay so someone is saying, this one (*points at the big triangle*) contains two right-angled triangles. Ehe maybe you can come in front and show us, the two that you are talking about. The two right-angled triangles that you are talking of or you can just indicate using letters that this one is (*giving chalk to a boy student*).

BS: Aaah, we have a triangle like this one, (*tracing around one of the small triangles*)

T: Mmmh.

SS: And another like this one also *(tracing around the other small triangle)*

T: Okay. So, someone is saying, in this triangle (points at the big triangle), we have got two right-angled triangles. Do you agree with him?

MS: Yes.

As a result of the challenge the teacher made to the students, a diagram to the right side of Figure 7 was drawn. The teacher wanted the students to apply their reasoning skills by asking them to defend whether the triangle was indeed right-angled. Many students (MS) thought it was not until one boy student (BS) suggested that the triangle could be split into two right-angled triangles (diagrams on the left side of Figure 7).

The teacher in the first extract of the episode attended to students' thinking on the similarity and differences between the first two diagrams. In the third diagram, he also attended to students thinking on verification of the presence of a right-angled triangle considering that the symbol of a right angle was shown within the triangle. The teacher tried to highlight noteworthy events and refer to specific events in the lesson. The episode was coded level 3 (refer Table 1).

4.2.2 Developing the Pythagoras theorem

The next observation was on Activity 2 (Figure 10) where the students were supposed to come up with the Pythagoras theorem. There was a mismatch between what the teachers expected in solving Activity 2a (Figure 10) and the results that were found in class. Below is part of the dialogue that took place.

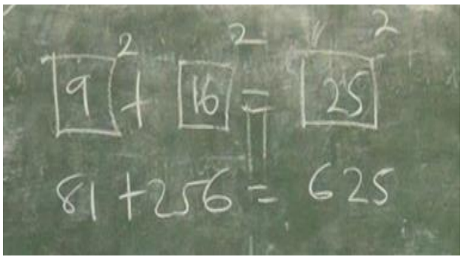


Figure 8: Activity 2 solution a

Figure 9: Activity 2 solution b

T: So, if we add 81 plus 256, is equal to 625?

MS: No.

T: Is what you are telling me, huh? Just add or punch in the calculator and see. What are you getting this side? (*Starts writing on the board*).

BS: 337.

T: Heh?

MS: 337.

T: 3?

MS: 37.

T: So, 337 is equal to 625?

MS: No.

T: So, it means there is something wrong with the figures, right?

MS: Yes.

T: According to our findings, so but we will see. What I wanted to find out from you is, are these the figures that you put in your boxes?

MS: Yes/no (*in unison*) [some said yes others no].

T: Any group that has come up with different figures? (*Nominates a student from those who raised their hands*).

In working on the second activity, the teacher realised that they have missed something in the example, 81 adding to 256 cannot be equal to 625. He had to inquire back from the students about what they found and if it was similar. Some students got the same results (Figure 8) while others found the results as presented in Figure 9. From their planning of Activity 2, the students were asked to find 'areas' of the squares on the sides of the given right-angled triangles. The results were supposed to be filled in the three squared boxes provided. Since the students were instructed to find the area of the squares, they did, and the next step was to insert the digits in the three boxes. But because the boxes had indices of two (2), the students together with the teacher were prompted to square the number (the found area).

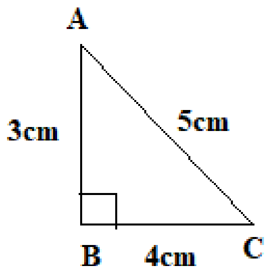
The teacher used learner centred principle of teaching and inquired more from the students. A different result was presented as shown in Figure 9 which was also the expected result by the teacher. This episode shows the relationship between students' geometrical thinking and teaching approach, so it was coded level 4 (refer Table 1).

Activity 2: Establishing the relationship between sides of a right-angled triangle.

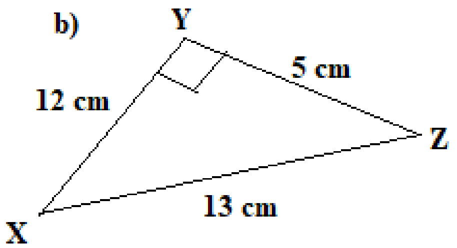
Procedure

- You are provided with two right-angled triangles with known sides.

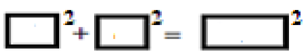
a)



b)



- Draw squares on each side of a triangle
- Find areas of the drawn squares
- Observe the relationship of the areas in each triangle and fill in the boxes below

a) 

b) 

- Explain what you have observed from the filled boxes

____
____
____
____

Figure 10: Activity 2

4.3. First lesson reflection

This reflection session was done in twenty-five (25) minutes and was identified with four episodes; two were coded low and two high-level noticing see Table 5. The two parts of discussion identified in the first reflection are observations by the teacher of the lesson who saw that it was difficult for him to connect two activities they planned and a discussion on the observation on the application of the Pythagoras theorem in solving examples.

Table 5: Noticing episodes in the first reflection

Noticing level	**Low**		**High**	
	Baseline	Mixed	Focused	Extended
Number of episodes	1	1	0	2

4.3.1 Observation by the teacher of the lesson

Two main observations were made in this session. The first one was made by the teacher who taught the lesson on a part where he felt hard to progress. Below is part of the discussion the teachers and the KO had on this observation in the episode.

> **P4:** Okay so on Activity 2 (see Figure 10) on our expected answer eti? [*right?]* aah, that column the last column where it is written the square of the hypotenuse is equal to the sum of squares of the two sides. And I think, then we have Activity 3 {where the students were supposed to generalise the finding from Activity 2 using letters} this point in a right-angled triangle, the area of the square on the hypotenuse side is equal to the sum of the squares on the ‘other two sides. Because I noted that when I was presenting this one {referring to what was on the lesson plan}, I wrote this one {referring to squared boxes he used in the teaching}. So, I noted that maybe if I only stopped on what we planned, then it will confuse the learners because some information is missing. So, I don’t know maybe next time we need to add some information on this one (Activity 2) even to avoid confusing the learners. So, I also think it was difficult to move.
>
> **KO:** So, for me, I thought it’s like building up on that {referring to Activity 2},
>
> **P4:** On theee, okay.

KO: Because this one {referring to Activity 2} they are drawing it from empirical measurements

P4: Mmmh.

KO: From the empirical activity that they have done. So, they are inducing from that to say the square of the hypotenuse is equal to the sum of the squares of the 2 remaining sides,

P4: Aaah, okay, of course, I think it was the learning point, eh?

KO: Yah so.

P4: It was a learning point yah.

KO: Yah! But it is happening after they have done the,

P1: The activity.

KO: The activity of drawing and finding the areas, eh?

P1: Squares yah.

P4: Mmmh.

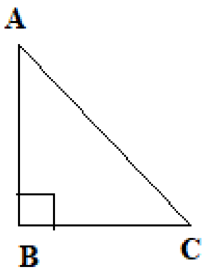
KO: They are concluding finding the areas. So, I think there is a need to include the area because they were finding the areas there.

Activity 1: Identifying the hypotenuse in a right-angled triangle.

Procedure

- You are provided with **four** right-angled-triangles.

a)



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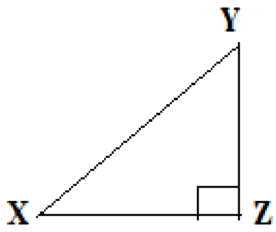

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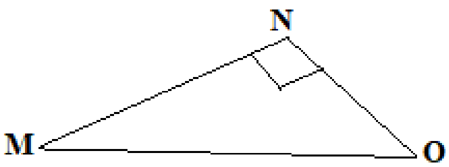

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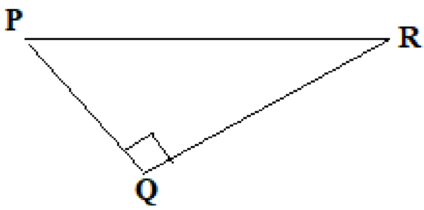


- Identify the longest side in each given triangle

a. ____________________________

b. ____________________________

c. ____________________________

d. ____________________________

Figure 11: Activity 1

In this dialogue, P4, the teacher of the lesson, noticed challenges in coordinating Activity 2 (where students were asked to establish the relationship between sides of a right-angled triangle) see Figure 10, and Activity 3. For him to proceed he noted it would confuse the students. To avoid that he had to add more information on the work within the lesson but still, he suggests they needed to investigate the activity. The KO did not see it as much of a problem as the two activities were following each other. Activity 2 used raw data the students discovered from Activity 1 (Figure 11) whereas Activity 3 was just a generalisation of the results in Activity 2. She then agreed on the need to focus on area not only on squares.

Later in the dialogue P1commented on the use of area by saying:

> "That means when stating the definition of the Pythagoras theorem, we will talk about the areas. As, the sum of the areas of the two squares of the legs, is equal to the area of the square of the hypotenuse. Kenakonde [*then*] in Activity number 3, that's when we will state the exact definition of the Pythagoras theorem".

P1 agreed with the KO that they should emphasise the term area in Activity 2 instead of just talking about squaring so that in Activity 3 they should exactly define the theorem. They planned to introduce the definition in terms of the area of squares of the sides of the right-angled triangle-the sum of the areas of the two squares of the legs is equal to the area of the square of the hypotenuse. In this dialogue, the teachers and the KO provided an elaboration and an interpretation of what happened in the lesson-stagnating of the teacher. They also suggest an alternative pedagogical solution. They attended to the relationship between mathematical concepts and teaching approach hence coded level 4 (Table 1).

4.3.2 Application of the Pythagoras theorem

The second observation was on the application of the theorem in solving mathematical problems. After the theorem was developed, the teachers planned to demonstrate the use of it with the students in solving examples. In the presented part of the discussion, noticed a lack of emphasis on the organisation of the theorem which later the KO assumed was a result of a lack of proportionality of the length of the sides of the sketch of a right-angled triangle and the assigned value the teachers used. See below the figure and the part of the discussion they had.

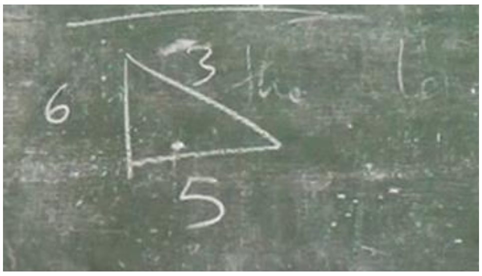


Figure 12: Example e in the lesson

P1: Now on the examples, I think because of time, we were unable maybe to do the other example very well. But this example {Referring to example **e**. where students were asked to show if a triangle with sides 6, 5 and 3 is a right-angled} that you said 6, 5, 3, yah! I liked it. Maybe next time when you are teaching aah tizangoibwerezaso kunena kuti [*we should just repeat it to say*] we are using the Pythagorean triples. Group lija limazanena the arrangement, ndeanazanena kuti [*that group which was stating the arrangement said*] 6 squared plus 5 squared is equal to chani [*what?]* 3 squared. Ndekutiti kuyenera kufotokoza kuti tikugwiritsa ntchito [*so we needed to explain that we were using*] Pythagorean triples to test if the triangle is right-angled, tinayeneranso [*we also needed*] to specify kuti [*tha*t] we have to isolate the longest side and compare its square to the sum of squares of the other numbers.

P all: Mmmh.

P3: That's, the side to be considered to be the longest should be assumed as the hypotenuse.

P1: Yes! The longest side. Zichitekukanika zokhankatimo kuti [*it should fail because*] they are not balancing.

KO: So, maybe it's also the, I think the way we have drawn, if I look aah, I have to look at the longest side,

P1: Yah! (*Laughs*).

KO: By, looking at the diagram itself,

P1: The diagram,

KO: It shows that the longest side is where there is 3 (laughs)

In the discussion of examples used in the teaching, P1 observed that students straggled in presenting their reasoning in example **e** (see Figure 12) where a diagram of a triangle was provided for the students to verify if the given measurements qualify it to be a right-angled triangle. Students failed to; 1) arrange the numbers properly and 2) give a reason for their arrangement. The KO came in with another observation where she assumed that the way the triangle was drawn and labelled - the side that looked long was assigned a lower value, would also be a contributing factor. From the KOs' assumption, the students were just following what they discover as the property of a hypotenuse-always the longest side hence the arrangement $6^2 + 5^2 = 3^2$.

On the same note on diagrams, P1 also suggested that "it shouldn't fully show that it's right-angled". P1 suggested that the diagram should not look exactly like a right-angled triangle instead this should be discovered after calculations when they found that both sides of the equation are equal. P3 also brought another suggestion "This time around, can't we even add two more examples? One, where there is a proof looking at the issues of the triangles, and the other one where you only give three numbers but no diagram" which they agreed to adopt in the next lesson.

The participants in this episode provided elaborative and interpretive comments on specific classroom occasions through P1 and the KO. Participants based their noticing on the relationship between a particular student's mathematical thinking on the arrangement of numbers in using the Pythagoras theorem in the given example and teaching strategy. They noted an important event, interpreted, and elaborated on it and then proposed modification of the teaching pedagogy. The episode was coded level 4 (refer Table 1).

4.4. Second planning

This session was identified with eight episodes, four high and four low (see Table 6). The first discussion presented is on the identification and definition of the hypotenuse and the second is on establishing the relationship of the sides of a right-angled triangle.

Table 6: Noticing episodes in the second planning

Noticing level	**Low**		**High**	
	Baseline	Mixed	Focused	Extended
Number of episodes	1	3	2	2

4.4.1 Identifying and defining hypotenuse

In the second planning discussion on identifying and defining hypotenuse, the teachers identified two issues to address; 1) that the definition should focus on both length and position. 2) To allow students to voice out their reasoning. In their discussions within the episode, they noted that from their previous lesson and some books, the definition of hypotenuse was focused either on it being the longest side or opposite the right angle, but this time they planned to cover both definitions. The dialogue below shows how the teachers planned to give students more opportunities to think on their own.

> **P1:** We should avoid thinking for the students. So, they should tell us what name is given.
>
> **P all:** (*Agrees*).
>
> **P3:** What name is given to the side opposite to a 90-degree angle, in a right-angled triangle yah?
>
> **P1:** [...] okay tachosa word ya hypotenuse [*we have removed the word hypotenuse*]
>
> **P4:** Mmmh.
>
> **P1:** The word should come from the students {referring to the word hypotenuse}. What about those legs? Should we bring them, or should it also come from them?
>
> **P2:** The word should come from the learners.
>
> **P4:** Okay, I don't know kuti ana angalote word imeneyo? [*But can they just dream of that word?]*
>
> **P1:** In the lesson muja sanaitchulepo wina wake [*didn't anyone mention it in the previous lesson?*]

In the dialogue, the teachers planned to allow students to think before the teacher explains to them. P1 suggested that they should avoid putting themselves much in the students' thoughts which may result in limiting the students. They then agree to ask students to provide the name of the side opposite the right angle. They tried to refer to their first lesson and discovered that some students mentioned it. The teachers planned to focus on students thinking by avoiding thinking for them but letting them think and present. The basis of such focus was on the previous lesson where they saw students' ability to mention the sides as legs. The episode was coded level 3 as the teachers were planning to pay attention to students' thinking in the identification of the hypotenuse.

4.4.2. Discussion on how to establish a relationship between the sides of a right-angled triangle

In this episode, the teachers discussed how they needed to help their students develop the relationship between the sides of a right-angled triangle. From the observation made in the first lesson, two amendments were made: provision of triangles with already drawn squares on the sides and removing squares on the boxes given for showing the relationship. Below is part of the discussion on the change of presentation of the second activity.

P4: Mmmh, or after kupeza ma areas ama square, chimenetufuna apange next ndichani? [*After finding the areas of the squares, what do we want them to do next]?*

P3: We want them to observe the relationship between each triangle.

P4: Mmmh. Okay, ndekuti instruction iliapopo izakhala yokuti apange observe relationship yomweyo [*the instruction that is there will be for them to observe the same relationship*].

P all: Yah!

P6: That will be the first one.

P1: And what's that relationship?

P3: That, if they add 2 areas, the sum is equal to the area of the square of the hypotenuse.

P1: Yea uziwa [*you know*] this bullet says, ask learners to observe the relationship of the areas in each triangle and fill in the given boxes. We were not supposed to give squared boxes, because we have already talked about the areas.

P all: (*Agrees*).

P1: And that's what confused the learners when we were teaching. So, these boxes must not have squares.

The students were provided with right-angled triangles from which they requested to find the area of the squares on their sides and then present the relationship observed. From the first planning of the same activity, the teachers provided students with squared boxes where they needed to record their findings. It was noted that the boxes compromised the work. In this plan, students were still asked to find the area and present the results in the given boxes. But this time the boxes had no indices. In the presented extract, P4 wanted to know what the students were supposed to do after finding the areas to which P3 explained that they were to observe the relationships of the area in both triangles. For example, some of the areas of the two sides are equal to the area of one side-the hypotenuse. Their basis of comments and arguments in this episode was the observations made in the first teaching.

The teachers focused on students' particular thinking on the relationship among sides of the triangles. There is a change of strategy regarding the first lesson observation. This episode was coded level 4 as the teachers were attending to the relationship between particular students' mathematical thinking and teaching strategy.

4.5. Second teaching

This session was identified with six episodes; two low and four high (see Table 7). To highlight the findings of the analysis in the second teaching, a discussion on how the teacher helped students to understand that reflex angled is not a type of triangle, and another one on how the teacher was involving students in solving given mathematical examples are presented.

Table 7: Noticing episodes in the second teaching

Noticing level	**Low**		**High**	
	Baseline	Mixed	Focused	Extended
Number of episodes	1	1	4	0

4.5.1 Clearing misconception on reflex angled as a triangle

In this episode, students were asked to mention types of triangles by angles and one student mentioned reflex angled. The teacher did not want to impose on the students that such type of triangle does not exist instead he let the students think and come up with a conclusion. Part of the discussion is presented below.

T: [...] Now, which triangle is reflex-angled? Or can someone define a reflex-angled triangle? Just give it a trial. Can you define a reflex-angled triangle? Yes! (*Nominates a boy student who raised a hand).*

BS: A reflex-angled triangle is, is a triangle which has got, an exterior angle which is more than 180 degrees.

T: Let me write that. A reflex-angled triangle is a triangle which has an exterior angle which is more than 180 degrees (*recites while writing on the board*). That's according to him. You're right about triangles and polygons. The exterior angles of triangles, Aah, what is the sum of the exterior angle of a triangle? We need to clarify this. The sum of exterior angles of a triangle is equal to what? 300 and?

MS: 60 degrees.

T: 360 Degrees. Now for the exterior angles, do we have an exterior angle which is more than 180 degrees? I don't want us to go back to the topic of triangles and polygons. But do we have an exterior angle in a triangle which is greater than 180 degrees?

MS: No!

T: We don't have right?

MS: Yes!

The teacher started by asking the students to state what they know about a reflex angle and one boy student, BS, defined it as above. He then asked for a recall of students' knowledge on the sum of exterior angles of a triangle which adds up to 360 degrees. Finally, he inquired if there is a possibility of having an exterior angle of a triangle being more than 180 degrees which students declined. So, they together conclude that there cannot be an exterior angle with more than 180 degrees therefore there cannot be a reflex-angled triangle. The teacher further emphasised by saying "**T**: All angles in a triangle that we calculate are within 180 and zero exclusive. We do not include 180 because the sum of interior angles in a triangle adds up to what?" and many students responded "**MS**: 180".

In this episode, the teacher paid attention to particular students' thinking on the existence of a reflex-angled triangle by helping them make their discovery. He highlighted noteworthy classroom incidence and provided an elaboration on the incident. The episode was coded level 3 (Table 1).

4.5.2 Students' involvement in solving examples

The next dialogue is from the solving of examples. The part of the discussion below shows how the teacher was following up on particular students thinking in the application of the Pythagoras theorem. Some indicators that demonstrate how the teacher wholly focused on students' mathematical understanding are also highlighted (refer to section 4.7 for more examples).

T. [...] let's look at the first question. Can you identify the hypotenuse in this triangle? Because in a right-angled triangle, the hypotenuse is the longest side so, can you identify the hypotenuse in the triangle given here with sides 6, 8 and p. what is our hypotenuse? Yes! (*Nominates a girl student*)

GS: P is the hypotenuse.

T: Is she correct?

MS: Yes!

T: Aah, what about aaah, who can give me an equation, an equation on the relationship on how to find **P**? To the same student, can you give me the equation, please?

SS: 8 square plus 6 square is equal to **P** square.

T: Reason for that? **(***He asks the questions while writing the solution on the board***).**

SS: Aaah, the sum of the squares of two legs of a triangle is equal to the hypotenuse side.

T: So, what did we say aah the correct name for that?

SS: Pythagoras theorem.

T: Pythagoras theorem, can you give her a hand

In approaching the first example, the teacher started by assessing students' ability to identify hypotenuses and then came up with the Pythagorean Theorem equation. In the above dialogue, we see the teacher inquiring more from the same student. The student was asked to identify the hypotenuse then she had to give a reason for her choice then the teacher asked her to present the formula to be used in solving the problem.

Within the episode but in a different example where they were finding the value of one of the legs, a student presented his response as "hundred will be subtracted from six hundred seventy-six." The teacher intervened on how the student mentioned the number 100 as 'hundred'. He corrected the student that the number is supposed to be mentioned as 'one hundred'.

The teacher in this episode focused on particular students' mathematical thinking by inquiring more on the same student to state and give explains on the example. He also corrected the student who mentioned a number in the wrong mathematical language. Elaboration and interpretation of specific instances in the lesson are demonstrated therefore the episode was coded level 3 (Table 1).

4.6. Second lesson reflection

The second reflection session lasted for twenty (20) minutes and was identified with four episodes; one low and three high (see Table 8). A part of a discussion on what the other teachers observed in the lesson and an observation by the KO on a type of triangle a student mentioned are presented.

Table 8: Noticing episodes in the second reflection

Noticing level	**Low**		**High**	
	Baseline	Mixed	Focused	Extended
Number of episodes	0	1	2	1

4.6.1 Individual teachers' observation

During the second reflection, the chairperson of the session asked the teachers individually to comment on the lesson. The first to comment was the one who taught the lesson followed by the other observing teachers. Below is a dialogue of what P3, P2 and P4 discussed.

P3: Yah! Aah, yah the lesson went well. I could see an emphasis on aah pressing students for justification. There was a sort of consistency on that one to say when a student answers, the teacher made sure that he or she is supposed to justify why he/she is saying so. Things like identification of hypotenuse, aah, basically there are two ways, one is that of aah being an opposite side to the right angle and the other one is that of being the longest side. So, there are situations especially when we are dealing with the example aah where they are not given all the sides so someone could say aah, that one is the longest side, the hypotenuse because the is the longest side, so you could say how is this the longest side because you have got 3, 4 and **y** (*unknown side*). So, later, I mean the teacher could give a chance to the very same student to think of another way to say it. So, I could see that, that aah following the students, was there trying to follow them in terms of even language they could use and the also he made sure that if there is no any mathematical language that has been used, I mean he was identifying that problem the same time at the very same time. Trying to sharpen them to use aah, the mathematical language appropriately.

P2: Alright. I also note that demanding, the teacher was able to demand more from the learners. When the learner has failed to give the correct answer, he tried to ask the same learner so that he has to think more about the answer. When the other one failed, he asked others to help so that she also has to come up to say whatever the other learners said. That I liked most. And also, there is a sense of patience with the learners. A certain learner Anamuzuma kutiiiih *[they murmured at him*] but the teacher came up so that he would have the courage. She had to polish up the statement that she came up with and then that one was correct. I also had missing information on the last part when I look at the conclusion. On a statement saying the Pythagoras theorem that the sum of the squares of the 2 legs is equal to the square of the hypotenuse. There was just a missing mention of a right-angled triangle.

P4: [...] I will focus on the conclusion because I have seen that mmh, the teacher was able to probe more when students are giving half answers and that was good because aah during the conclusion that's where maybe some students are getting what they are supposed to take aah outside. So, that was the last chance for the teacher to come in for those who were having some minor problems. So that was good. The teacher was able to come in and correct some minor mistakes that some of the students had when he was concluding the lesson.

Teachers in this episode expressed their observations on the lesson. P3 observed two things: 1) the teacher’s emphasis on justification from the student. He gave an example of identification of hypotenuse that could be either considering the side opposite the right angle or the longest side for example in the triangle with sides 3, 4 and **y**. 2) The teacher’s encouragement of proper use of mathematical language. The teacher was supporting the students to be using proper mathematical language. P2 observed three things: 1) demanding the students to explain more. She gave an example of asking a student to re-think when s/he gave an incorrect response. 2) The teacher was patient with the students for example she said he could ask the same student to rethink. 3) She observed missing emphasis in stating the Pythagoras theorem that it is ‘only in a right-angled triangle’ when the teacher was concluding the lesson. Finally, P4 commented on the teacher’s probing techniques when the students gave half-baked answers. These observations showed the teachers' emphasis was on a particular student’s mathematical thinking. They gave elaborative and interpretive comments on classroom-specific instances hence coded level 3 (Table 1).

4.6.2 More suggestions on clearing misconception on reflex angled as a triangle

The next discussion is on the KOs first observation of what a student mentioned as a type of triangle, reflex angled. The teacher helped in making it clear to the student that it is not a type of triangle. The KO asked the teachers this “imagine if we are to teach again and come across a child who will also give us a reflex angled triangle as a type of triangle. How would we assist?” See below what the other teachers suggested.

> **P2:** Maybe we may use the logic that when we look at a right-angled triangle that means we are looking at a 90 degrees angle in that triangle. And then an acute-angled triangle means there’s an acute angle inside the triangle so we may try to draw a reflex angle on the board and then ask the learner to draw a triangle where the reflex angle should be inside. So, if they fail to draw that triangle, they will know that this is not correct.
>
> **P all:** *(Laughs*).
>
> **P4:** Yah! Because I was also thinking the same, why not give a chance to the student maybe to come in front and draw that type of triangle, aah because maybe it might be an issue to do with mistaken identity. Maybe it’s failing to identify which one is an

exterior and which one is an interior. So, maybe by drawing the triangle some of the issues might have come out.

P2 suggested that they draw a reflex angle and ask the student to come up with a triangle enclosing the reflex angle. P3 agreed with her and continued to say if the student could not be able to do so it means such type of triangle does not exist. The point for KOs was not opposing the explanation made by the lesson teacher, P1, but rather to explore several ways of promoting students' geometric reasoning as she said, “We are not saying that the way you approached it is not the correct way, but we are just saying assuming that we meet the same situation again”.

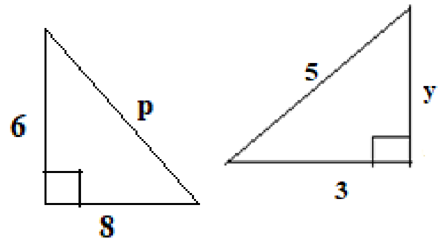
In this episode, the teachers and the KO attended to a particular student and her thinking on the types of triangles, and they suggested alternative pedagogical solutions to the specific observed instance. This episode shows the relationship between particular students’ mathematical thinking, mathematical concepts, and teaching approach hence coded level 4 (refer Table 1).

4.7. Results from analysis of examples used in the lesson plans

An analysis of the lesson plans used in the two teachings showed that the teachers made some changes. Referencing Table 9, four major changes were noticed. 1) Change of side to find its value. In example d in lesson plan 1b the two sides to calculate their values were hypotenuse so they changed one side, **q**, to be a leg. 2) Swap in numbers assigned to the sides of the triangle in **e**. The side assigned number 3 was the longest but was assigned the smallest value. 3) Addition of a question on the proof of the Pythagorean triples. 4) Change in questioning in **f**. The teachers first wanted the students to calculate the perimeter which they changed in the same example in the second lesson plan claiming that the question was out of the concept under assessment. They also included a limit in the approximation of the result.

Table 9: Examples in lesson plan 1b and lesson plan 2

Lesson plan 1B examples	Lesson plan 2 examples
In a, b, c and d the question was to find the length of the unknown side. a. b.	In a, b, c and d the question was to find the length of the unknown side.



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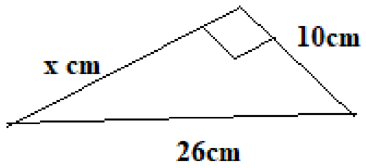
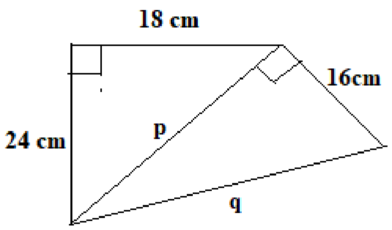
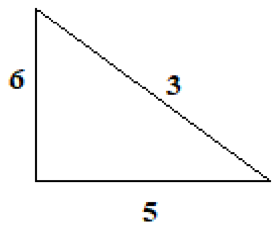


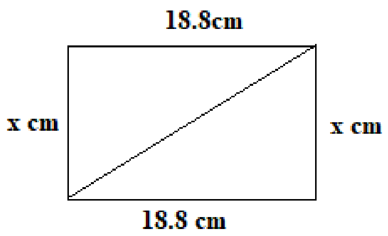
d.



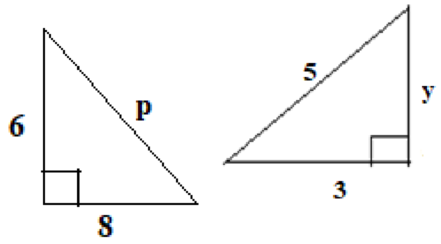
e. Show whether the given triangle is right-angled or not.



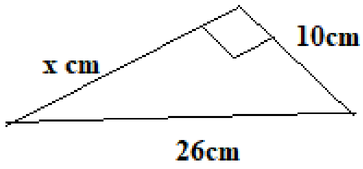
f. The length of a diagonal of a rectangle is 23.7 cm and the length of one side is 18.8cm. Find its perimeter.



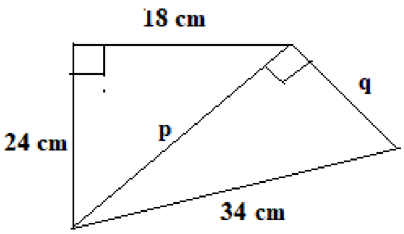
a. b.



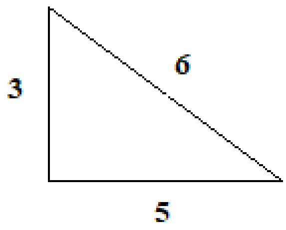
c.



d.



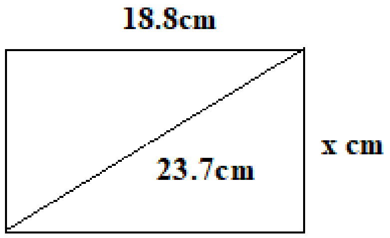
e. Show whether the given triangle is right-angled or not.



f. Which of the following measurements of a triangle gives a right-angled triangle?

i. 27m, 50m, 35m
ii. 14m, 50m, 48m

g. A rectangle is 18.8cm long and its diagonal is 23.7cm. Calculate the width of the

	rectangle. Give the answer to 4 significant figures. 

4.8. General reflection

The teachers were investigating how they would help learners to establish and understand the relationship between the hypotenuse and the other two sides of a right-angled triangle. In the general reflection session where both teams were present (the Pythagoras theorem and similar triangles team), P3 presented the agreements of the discussion they had within their LS team. Four things were highlighted, which are: 1) Choosing examples that will help students to understand - from simple to complex and giving students word problems. 2) Being patient with the students allows them to reason and express their ideas. 3) Need for much emphasis on mathematical language, they suggested demanding more and varying activities can be helpful. 4) Careful presentation of diagrams, they suggested from simple to complex and realistic representation.

The teachers' general reflection after the LS cycle showed that they came to realisation that the co-planning discussion, the teaching, and the reflection sessions they had in the two lessons brought some changes wealthy to support them in improving their professionalism. The area of changes and improvements are illustrated in their way of thinking about students and the way they could present mathematical concepts to the students in the four ways stated above.

4.9 Chapter Summary

The chapter has presented findings of the analysis of the data which was collected for the current study. The LS followed a repetition cycle, so it was repeated in another class. Results for each stage have been presented in a separate section. In a nutshell, episodes which were coded level 3 (focused) and level 4 (extended), qualified to show what and how teachers noticed in the planning, teaching, and reflection sessions. But for the goal of the current study which was to investigate the opportunities for teacher learning in LS, teachers' predictions, observations, and promotion of students' thinking in the suggested or given tasks and examples were prioritised and presented in the chapter.

The following twelve themes, two from each of the repeated planning, teaching and reflection sessions have been presented: introducing hypotenuse and discussion on what to observe in the lesson for planning one; right-angle triangle and developing the Pythagoras theorem in teaching one; observation by a teacher of the lesson and application of the Pythagoras theorem for reflection one; identifying and defining the hypotenuse and establishing the relationship of the sides of a right-angled triangle for planning two; clearing misconception on reflex angled as triangle and students involvement in solving examples for teaching two; and individual teachers lesson observation and strategy suggestions on clearing misconception on a reflex-angle as a triangle for reflection two. Results of analysis of the example used in the two lesson plans were presented as well as the general reflection of the cycle. The next chapter will present a discussion of the presented results.

Chapter 5: Discussion

The present study was developed to investigate the opportunity for teacher learning in LS when promoting geometric reasoning in Malawi secondary classroom in the teaching of Pythagoras theorem. The study aimed to investigate ways in which the opportunity is offered to the teachers involved in LS in one secondary school. To establish the ways, three supporting research questions were developed. These are: what do teachers notice when planning in LS in teaching Pythagoras' theorem? How do teachers' predictions and observations in lesson study promote students' thinking and learning of Pythagoras' theorem? What knowledge do teachers gain from classroom incidents in teaching Pythagoras' theorem?

The current study considered teacher learning as a change in teachers' awareness of their teaching, including its effects on students' learning and daily classroom routines. The findings of the present study reveal the possible opportunities teachers had to learn to change in those two angles from the LS process under study. It showed that all the stages of the LS offered the teachers at least a new thing to learn from. The study agrees with a discourse suggestion made by Lee and Choy (2017) that attention and interpretation of mathematically important teaching and learning features are associated with high levels of noticing. The focus of the analysis was based only on high-level noticing (focus-level 3 and extended-level 4). How the possibilities were observed in the LS is discussed within the chapter with each of the three research questions discussed separately in sections 5.1 to 5.3. Further discussion of the findings is presented in section 5.4 and the chapter summary in 5.5.

5.1 What do teachers notice when planning lesson study in teaching Pythagoras' theorem?

To answer this question, teachers' predictions and plans for observations in the selection of examples and tasks to give their students were considered. From the findings of the two planning sessions, the teachers were observed planning to attend to their students' mathematical thinking and also its relationship with teaching strategy (noticing levels 3 and 4). In the first planning session, eight episodes of high-level noticing were identified from a total of thirteen episodes while in the second planning, four of six were identified. The findings agree with Fauskanger and Bjuland (2021) who found that in co-planning teachers are concerned with students' thinking alone or students thinking in relationship with pedagogy (high-level) more than focussing on their pedagogy (low-level). However, the findings defer from the two authors' current article (Bjuland & Fauskanger, 2023) which discovered that there was less high-level noticing in their study but they acknowledged that the lower-level noticing acted as important starting points of the high-level noticing.

We can wonder why the difference. The current study suggests that the differences may be in terms of the focus of analysis. In Fauskanger and Bjuland (2021), they focused on what the teachers noticed only while in their latest article, they focused on both what and how teachers notice (Bjuland & Fauskanger, 2023). The current study focused its analysis only on what teachers noticed in the planning session like Fauskanger and Bjuland (2021) thus where the current study assumes the similarity comes from. But further focus on the unit of analysis might help to minimise the difference so that an enhanced conclusion can be drawn.

Another observation reveals that there was coordination among the teachers in their planning. This was demonstrated in the way they were allowing each other to share ideas on an approach then together they were brainstorming with a focus on particular students' mathematical thinking (van Es, 2011) and come up with one agreed approach to be applied in the lesson. In their discussion, the teachers were observed using the phrases like '*what should they..., what do we want them to...*' These phrases may show that the teachers were planning to expect more from the students than they could give them. The teachers debating and coming to one agreement supports Karlsen and Helgevold's (2019) claim that teachers in LS jointly make sense of the observations gathered from the research lesson. Sincerity and respect for each other's raised ideas were observed agreeing with Herva and Medina (2021) which may be the reason for the teachers' progress in their planning. This also supports Lewis et al.'s (2013) claim that in LS, different ideas are debated, contested, and ultimately accepted to produce co-construction of the teachers' knowledge, which is then passed to and fixed in the knowledge of the individuals..

In terms of the teachers' basis of predictions, in the first planning, the teachers based their predictions on personal knowledge which was unlike the second planning where they based the predictions on the success and failure of the previous research lesson. The teachers act in the second planning may agree with the suggestion pointed out by Bjuland and Fauskanger (2023) of testing activity in a class of one of the participants before the co-planning session. This they assumed would inspire the teachers attending to students' mathematical thinking (high-level noticing) which the current study assumes supported the teachers in second planning as it is noted that there was a high percentage of high-level noticing in second planning compared to first planning. This also may mean the teachers demonstrated learning through application orientation (Bakkenes & Vermunt, 2010; Vermunt et al., 2019) as they were improving their research lesson using the experiences from the previous lesson.

Results of the analysis of the examples from the lesson plans also show that the teachers were focussing their attention on the student's mathematical thinking by presenting the examples from simple to complex. For instance, the first example (see Table 9) demanded the students calculate the value of the hypotenuse-the theorem here is applied directly ($c^2= a^2+b^2$), then one leg-where the theorem needs rearrangement then introduce square root, either $a^2= c^2- b^2$ or $b^2= c^2- a^2$, until the last one where they were asked to prove the Pythagoras triple-where they needed a careful arrangement of the given set of numbers to fill in the formula. From the examples, it was also observed that the teachers were changing the orientation of the shapes from one example to another (see examples **a** to **d** in Table 9. This finding agrees with Mwadzaangati et al. (2022) who discovered that the teachers were varying the use of examples from simple to complex. The variation in this study was observed to be based on the experience from the classroom or the reflection of the research lesson when they noticed deficiency. Using diagrams and varying orientations also agrees with Jones et al. (2012) who claimed that drawings act as a better mathematical representation in the teaching of geometry.

Further analysis of the examples in comparison of lesson plans one and two, it was observed that the teachers made some adjustments. For instance, in example **d** from plan one, the two unknown sides were both hypotenuses, so they changed in plan two to make the other side a leg. There was also inclusion of approximation in example **g** which was example **f** in plan one (see Table 9). One more example was added in plan two where they wanted the students to prove if the given three numbers were Pythagoras triples. These changes may mean the teachers utilised what Lewis (2009) articulated that planning discussions offer teachers opportunities to check the quality of teaching and learning for specific planning and focused reflection of their work.

Another observation from the findings of the planning is on defining what to observe in the LS cycle. The teachers within their first planning session discussed that they wanted to promote the use of mathematical language through demanding and supporting techniques. This may contribute to the teachers' focused attention on their students' mathematical thinking (van Es, 2011) just as Bjuland and Mosvold (2015) study claimed that student teachers need to have clearly defined observations to focus on in LS to make it profitable. This could apply to serving teachers too as revealed in the current study.

5.2 How do teachers' predictions and observations in lesson study promote students' thinking and learning of Pythagoras' theorem?

The findings from the teaching sessions reveal that the teachers were giving students opportunities to express their ideas and justify their responses. It is also observed that the teachers referred students' raised questions and ideas to the whole class to look at them, brainstorm, and conclude. The teachers were observed following through on the students' work to establish a common understanding. In one of the lessons when the teacher was solving an example together with the students, he noticed there was a problem in the solution of the example. He then invited other students to present their findings from their work. The teacher's actions demonstrate that he focused on particular students' thinking and the relationship between teaching strategy and students' mathematical thinking (van Es, 2011).

In the discussion in 4.2.1, the teacher observed a noteworthy event in the lesson where the diagram a student drew had a symbol for the right angle but wondered if the students could identify it as really being a right-angled triangle. He asked the students to observe and think and then present their geometric thinking. A similar incident was observed in the second research lesson (see 4.5.1) where the teacher asked other students to explain whether a reflex-angled triangle mentioned by one of the students exists. Just as Wittmann (2021) proposed that students should be challenged with typical and rich enough problems that require them to derive and explain to make the underpinning of the unit firmly rooted.

It was also noted that the teachers made connections between the classroom events and principles of teaching and learning (van Es, 2011). The shifting of the discussion from general class discussion to asking individuals to present their discovery in the findings in the discussion presented in section 4.2.2 is an example of such a connection. However, the findings show that it was rare to see the teacher of the research lesson attending to the relationship between particular students' mathematical thinking and between teaching strategies and students' mathematical thinking (level 4). The researcher managed to identify one episode of the three high-level episodes coded in the first teaching and zero of the four in the second teaching. This according to Lee and Choy (2017) means the teacher of the research lesson was not relating his observations and interpretations focused on instructional decisions which may harm students' mathematical thinking within the lesson. Further attention needs to be put on this part.

5.3 What knowledge do teachers gain from classroom incidents in teaching Pythagoras' theorem?

Findings from the reflections of the lessons conquer van Es and Sherin's (2021) proposition that teachers use their understanding of mathematics, learners, learning, and teaching mathematics, as well as their prior experiences as math teachers and once students, to interpret student thought as a manifestation of a greater mathematical idea. The findings revealed that the teachers in their discussion were attending to particular students' mathematics thinking and the relationship between particular students' thinking with teaching strategy (van Es, 2011). This was observed when they were highlighting significant incidents that took place in the classroom and interpreting each other's comments while elaborating on the events (van Es, 2011). They were also observed making connections between the events and principles of teaching and proposing alternative pedagogy (van Es, 2011) to replace the one which they noticed had some weaknesses in the previous research lesson.

For example, in the first part of the discussion presented in 4.3.1, the teacher of the first research lesson noticed the mistake that took place in his lesson which he assumed was due to the teaching approach they used. He suggested they improve the teaching approach which they discussed and agreed to improve the approach to attend to how the students were thinking concerning how the approach was presented. The teachers' adjustments of instruction were observed to be based on the evidence of the students' struggle displayed in the lesson.

Another interesting finding was the contributions the KO provided in the reflection sessions. The KO was seen in several instances starting a discussion for the teachers to reflect on. In one instance the KO was observed playing a role in helping the teachers to advance their attention on their students' particular mathematical thinking (Mwadzaangati et al., 2022; van Es, 2011). She asked the teachers to think of another possible pedagogy in helping to clear misconceptions in students' thinking. This gave the teachers time to think of the relationship between their students' thinking and pedagogy.

Two of the four discussions presented in this study were initiated by the KO echoing Adler and Alshwaikh's (2019b) argument that the human resource provided by the KO in LS is of great value. The KO was observed supporting the teachers to refer further to specific events by asking them to think of more ways of supporting their students' reasoning on clearing misconceptions. She was not quick in suggesting alternative pedagogy but encouraged the teachers to consider the flow of the activities first and then based on interpretation make modifications to the approach (see 4.3.1 and 4.6.2). The KO's role observed here agrees with the claims made by Bjuland and Helgevold (2018), Hervas and Medina (2021), and Uffen et al. (2022) that KO supports teachers' learning in LS by increasing their attention and inter-thinking.

Finally, the results of the LS's general reflection show that the teachers' planning, teaching, and reflection sessions for the two research lessons helped them to see how they could improve their observation of their students' thinking and how they could communicate mathematical concepts to the students. For example, they mentioned "being patient with the students" as one lesson learnt. This may be one way of allowing the teachers to follow through with a particular student's mathematical thinking (van Es, 2011). This is in agreement with Dudley (2015) that teachers' minds get open from the talks and the raising and testing of the hypothesis of the lesson hence, developing chances of changing their beliefs and practices.

5.4 Further discussion

The observation shows that a larger percentage of the findings, about 58%, of the episodes was coded high level. As indicated above, this is in agreement with Fauskanger and Bjuland (2021) but disagrees with Bjuland and Fauskanger (2023). Further scrutiny on the existence of more high-level episodes than low ones may be due to the influence of two more things; the teachers having a clear goal of what to notice in their students (Bjuland & Mosvold, 2015) and the LS being guided by a theory. The LS in the PGR project which the current study is following was guided by MTF theory with a focus on exemplification and explanatory talk adapting the South Africa LS by Alshwaikh and Adler (2017a and b). These suggestions are in agreement with what Star et al. (2011) argued that teachers observe a broad range of occurrences and facts, both relevant and irrelevant to the purposes of LS if they do not have an apparent guiding focus. So having a guiding theory and structured observations could mean the teachers had a target.

These teachers as presented in 4.1.2, wanted to observe the promotion of mathematical language in their students. It may be possible that in their quest of promoting the language they were paying attention to particular students' mathematical thinking which is what the current study was trying to observe in relationship with the opportunities teachers had to learn from the LS process.

The findings reveal that the teachers were concerned with students' mathematical thinking and the relationship between students' mathematical thinking and teaching strategy (van Es, 2011) in all the stages of the LS. In the first planning stage, the teachers were conscious of what to give to the students and better ways of presenting while in the second planning, they were concerned with rectifying the research lessons' weaknesses. During the teaching phase, they were observant of the implementation of the research lesson and its success and failure. Whereas in the first reflection session, the teachers were more inquisitive in their noticing than they were in the second reflection. This may be as a result that the teachers were contented with the improvements made that far as also observed by Cajkler et al. (2015, p.21)

> "While a quarter of the discussion about the second research lesson was focused on the responses of learners during the lesson, their engagement in the tasks and their progress, the second evaluation meeting acted more as a celebration of success, which boosted teacher morale but offered only limited evidence of detailed critical evaluation".

But further research focusing on the major cause is needed to make a general conclusion.

Interestingly, the teachers were making links in all the happenings in the LS cycle and showed to have a universal desire to improve and change their next research lesson just like Karlsen's (2019) study asserted.

5.5 Chapter Summary

The chapter has presented discussions of the findings from Chapter 4. The findings reveal that the LS offered the teachers an opportunity to learn in all the stages. Planning offered them the opportunity to brainstorm and condense their discussion to a single possible approach. Predictions were at first from their personal experience then from the observation noticed in the first research lesson. Finally, in reflection, the KO played a major role in supporting the teachers to reflect more on their work and attending more to their students' mathematical thinking.

Chapter 6: Conclusion, Implication, Limitations, and Recommendation

The study has discovered that LS offers teachers the opportunity to learn. The present chapter presents a summary of the study, the pedagogical implication of the study on further research, limitations faced in the process, and what the study recommends.

6.1 Conclusion

The present study aimed at investigating how LS offer the opportunity for teacher learning when promoting geometric reasoning in teaching Pythagoras theorem in a Malawi secondary school classroom. Three specific research questions were developed addressing the stages of the LS cycle. A single cycle of one LS group consisting of five teachers was followed. The group was researching how they would assist learners to understand the relationship between the hypotenuse and the other two sides of a right-angled triangle. Video recordings from the stages of the LS cycle in the two iterations were used as the main data which were coded and further analysed using the teacher learning to notice framework by van Es (2011). The teachers noticing progressed from level 0 (no noticing), level 1 (baseline), level 2 (mixed), level 3 (focused), to level 4 (extended). The levels were further categorised as low (1 and 2) and high (3 and 4). The current study paid attention to the high-level coded episodes to observe the teachers' noticing in the two planning sessions, the implementation of the planned research lessons, and the reflection sessions. The study also observed the teachers' predictions and observations when given examples and activities to their students.

This study suggests that the LS stood as a possible platform for teacher learning agreeing with some studies on teacher learning in LS (e.g., Alshwaikh & Adler, 2017; Hervas & Medina, 2021; Mwadzaangati et al., 2022; Uffen et al., 2022). The findings of the study indicated three ways through which traces of teacher learning opportunities in the study arose. These are the teachers' collaboration, classroom mistakes, and the presence of the KO.

The teachers' collaboration was displayed in their teamwork in the LS process by accommodating each other's views and combining them to come up with what they considered a possible way of paying attention to students' mathematical thinking. The existence of a mistake in one of the research lessons was noticed and discussed by the teachers in their first reflection and it was also a point of reference for change of approach in the second planning. The presence of the KO also supported the teachers by asking the teachers probing questions which required them to think over while considering their students.

6.2 Pedagogical and methodological implication of the study on further research

The teaching of geometry has been observed to be difficult in literature (Serin, 2018; Tachie, 2020) even though geometry covers a larger percentage of the total topics in mathematics, especially in Malawi. Jones (2002) claimed that a knowledgeable teacher can teach the topic better for students to understand. From these findings of the current study, it may be possible for other teachers to gain the knowledge needed in their teaching of Pythagoras theorem and learn from the experiences of the teachers in reducing the challenge presented by Serin (2018) and Tachie (2020) in chapter 2.

On another hand, teachers' noticing their work provides the opportunity to check back on the work and understand the students' actions just like Jacobs et al. (2010) and van Es (2011) said that noticing works together with talking, interpretation and decision making. It may therefore mean that, if teachers can notice their students' learning, then they can be able to make the right decisions that support students' mathematical thinking hence promoting geometric reasoning in their students.

The methodological implication of this study is that the use of the teacher noticing framework in observing the teachers' focus on their students' geometric reasoning helped the researcher to concentrate on what teachers' pay attention to when planning, teaching, and reflecting on research lessons in LS. Lee and Choy (2017) said teachers learn in LS but what brings the learning is not clear so, observing what teachers notice displayed to be a better way of observing the opportunities teachers have to learn. The application of the noticing framework in the analysis was one way of inspecting and interpreting the teachers' work.

6.3 Limitation

The study used only one LS cycle which may not be enough to display clear evidence and long-time teacher learning. Since teaching is regarded as a cultural activity and takes time to develop (Stigler & Hiebert, 1999) and that teacher learning has been considered as a change in teachers noticing in their teaching, then more than one cycle can help in the establishment of what the teachers learn in LS. In addition, the promotion of geometry reasoning cannot just be concluded in the teaching of one geometry topic like Pythagoras theorem covered in this study rather a combination of two or more LS groups working on different geometric topics can be helpful to strengthen the findings.

The findings in this study were generated from the teachers' collaborative discussions from the planning, teaching, and reflection work, this means the traces of learning observed are for the team, not individual teachers. For further studies, it would be interesting to follow the teachers' conversation utterance by utterance and use interviews as a research tool to capture individual teachers' views and experiences from the PD. Finally, the current study used episodes as units of analysis which the study later realised were bigger units. The researcher suggests the use of smaller units for example using sequences in another study.

6.4 Recommendation

The researcher can claim that LS in this study has shown to be one form of TPD which has promising effects on teacher learning. The researcher may recommend Malawi MoEST consider introducing LS on a wider range so that many secondary school mathematics teachers can have the experience of learning from each other. This may help in the promotion of the teaching of geometry and promoting students' interest in mathematics. The findings of this study have recognised collaboration as one-way teachers learn to improve their professionalism. So, the study recommends that teachers within the school or at the zone level can be working in groups to support each other on how they can teach or help their students in the teaching of geometry.

To the teacher educators and mathematics experts and professors, the study has also recognised the importance of their closeness to the teachers; therefore, the researcher recommends that spending some time with the teachers in the schools is another way of supporting them professionally. They can be organising short seminars for them to meet the teachers and discuss how to overcome some geometry teaching challenges they meet in their teaching process.

Finally, for the PGR project, the findings of this study display some potential strength of the achievements of the project in impacting Malawi secondary school mathematics teachers with teaching knowledge learnt from within the teachers. The study recommends the extension of the project to other secondary schools and in other mathematics topics to reduce the challenges in students' mathematics performance especially geometry as the study noted from the MANEB reports referred to in Chapter 1.

References

Adler, J., & Alshwaikh, J. (2019a). A case of Lesson Study in South Africa. In R. Huang, A. Takahashi, & J. P. da Ponte (Eds.), *Theory and Practice of Lesson Study in Mathematics* (pp. 317–342). Springer International Publishing. https://doi.org/10.1007/978-3-030-04031-4_16

Adler, J., & Alshwaikh, J. (2019b). Working with example sets: A productive focus in Lesson Study. *Preceding of the Eleventh Congress of the European Society for Research in Mathematics Education*, 3199–3198.

Adler, J., Mwadzaangati, L., & Takker, S. (2023). From defining as assertion to defining as explaining meaning: Teachers' learning through theory-informed Lesson Study. *International Journal for Lesson and Learning Studies*, *12*(1), 38–51. https://doi.org/10.1108/IJLLS-02-2022-0029

Alshwaikh, J., & Adler, J. (2017a). Researchers and teachers as learners in Lesson Study. In M. K. Mhlolo, S. N. Matoti, & B. Fredericks (Eds.), *SAARMSTE Book of Long Papers* (pp. 2–14). Free State, South Africa: Central University of Technology.

Alshwaikh, J., & Adler, J. (2017b). Researchers and teachers as learners in Lesson Study. In M. K. Mhlolo, S. N. Mtoti, & B. Fredricks (Eds.), *Book of Long Papers, 25th Annual meeting of Southern Africa Association for research in mathematics, science and technology education (SAARMSTE)* (pp. 2–14).

Amador, J. M., Carter, I., Hudson, R. A., & Galindo, E. (2017). Following a teacher's mathematical and scientific noticing across career progression from field experiences to classroom teaching. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks* (pp. 161–181). Springer International Publishing. https://doi.org/10.1007/978-3-319-46753-5_10

Bakkenes, I., & Vermunt, J. D. (2010). Teacher learning in the context of educational innovation: Learning activities and learning outcomes of experienced teachers. *Learning and Instruction*, *20*(6), 253–248.

Bakker, C., de Glopper, K., & de Vries, S. (2022). Noticing as reasoning in Lesson Study teams in initial teacher education. *Teaching and Teacher Education*, *113*. https://doi.org/10.1016/j.tate.2022.103656

Banda, B., Mudenda, V., Tindi, E., & Nakai, K. (2021). Lesson Study Practice of Science Teachers in Zambia: Its Effects, Enhancing and Hindering Factors. *Conference: New Perspectives in Science Education 3rd International Conference*. Italy.

Battista, M. T. (2007). The development of geometric and spatial thinking. In *F Lester (ed). Second Handbook of Research on Mathematics Teaching and Learning*. Reston, VA/USA: National Council of Teachers of Mathematics.

Bjuland, R., & Fauskanger, J. (2023). What and how mathematics teachers notice while co-planning instruction. *Accepted for Presentation at Congress of the European Society for Research in Mathematics Education* (*CERME) 13*. Budapest.

Bjuland, R., & Helgevold, N. (2018). Dialogic processes that enable student teachers' learning about pupil learning in mentoring conversations in a Lesson Study field practice. *Teaching and Teacher Education*, *70*, 246–254. https://doi.org/10.1016/j.tate.2017.11.026

Bjuland, R., & Mosvold, R. (2015). Lesson study in teacher education: Learning from a challenging case. *Teaching and Teacher Education*, *52*, 83–90. https://doi.org/10.1016/j.tate.2015.09.005

Bowen, G. A. (2009). Document analysis as a qualitative research method. *Qualitative Research Journal*, *9*(2), 27–40. https://doi.org/10.3316/QRJ0902027

Cajkler, W., Wood, P., Norton, J., & Pedder, D. (2014). Lesson study as a vehicle for collaborative teacher learning in a secondary school. *Professional Development in Education*, *40*(4), 511–529. https://doi.org/10.1080/19415257.2013.866975

Cajkler, W., Wood, P., Norton, J., Pedder, D., & Xu, H. (2015). Teacher perspectives about lesson study in secondary school departments: A collaborative vehicle for professional learning and practice development. *Research Papers in Education*, *30*(2), 192–213. https://doi.org/10.1080/02671522.2014.887139

Chambers, P. (1999). Teaching Pythagoras' theorem. *Mathematics in School*, *28*(4), 22–24.

Chen, X., & Zhang, Y. (2019). Typical practices of lesson study in East Asia. *European Journal of Education*, *54*(2), 189–201. https://doi.org/10.1111/ejed.12334

Choy, B. H., Thomas, M. O. J., & Yoon, C. (2017). The FOCUS Framework: Characterising Productive Noticing During Lesson Planning, Delivery and Review. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks* (pp. 445–466). Springer International Publishing. https://doi.org/10.1007/978-3-319-46753-5_26

Cohen, L., Manion, L., & Morrison, K. (2007). *Research methods in education* (6th ed). Routledge.

Courtney, S. A., & Armstrong, B. (2021). Promoting geometric reasoning through Artistic constructions. *European Journal of STEM Education*, *6*(1), 09. https://doi.org/10.20897/ejsteme/11332

Creswell, J. W. (2009). *Research design: Qualitative, quantitative, and mixed methods approaches* (3rd ed). Sage Publications.

Creswell, J. W. (2014). *Research design: Qualitative, quantitative, and mixed methods approaches* (4th ed). Sage Publications.

Criswell, B., & Krall, R. M. (2017). Teacher noticing in various grade bands and contexts: Commentary. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks* (pp. 21–30). Springer International Publishing. https://doi.org/10.1007/978-3-319-46753-5_2

Dawson, C. (2002). *Practical Research Methods: A User-friendly Guide to Mastering Research*. How To Books. http://dlib.ptit.edu.vn/handle/HVCNBCVT/2369

de Villiers, M. (1998). To teach definitions in geometry or teach to define. *In A. Olivier & K. Newstead (Eds), Proceedings of the Twenty-Second International Conference for the Psychology of Mathematics Education*, *2*, 248–255.

Denzin, N. K. (1970). *The research act: A theoretical introduction to sociological methods*. Aldine.

Dindyal, J., Schack, E. O., Choy, B. H., & Sherin, M. G. (2021). Exploring the terrains of mathematics teacher noticing. *ZDM – Mathematics Education*, *53*(1), 1–16. https://doi.org/10.1007/s11858-021-01249-y

Dudley, P. (2013). Teacher learning in Lesson Study: What interaction-level discourse analysis revealed about how teachers utilised imagination, tacit knowledge of teaching and fresh evidence of pupils learning, to develop practice knowledge and so enhance their pupils’ learning. *Teaching and Teacher Education*, *34*, 107–121. https://doi.org/10.1016/j.tate.2013.04.006

Dudley, P. (2014). *Lesson Study: A handbook*. www.lessonstudy.com.uk

Dudley, P. (Ed.). (2015). *Lesson study: Professional learning for our time*. Routledge.

Dues, E. (2018). Global application of the Japanese 'Lesson Study' teacher education and training model. *International Dialogues on Education*. 5(1). 65-73. ISSN 2198-5944

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana & V. Villani (Eds.), *Perspectives on the Teaching of Geometry for The 21st Century* (pp. 37–52).

Fauskanger, J., & Bjuland, R. (2021a). Learning professional noticing by co-planning mathematics instruction. *SMDF, Swedish Society for Research in Mathematics Education*, 65–72.

Fauskanger, J., & Bjuland, R. (2021b). Learning to notice learners' mathematical thinking while coenacting instruction. *SAARMSTE*, 57–68.

Fauskanger, J., Helgevold, N., Kazima, M., & Jakobsen, A. (2021). Challenging Malawian primary teachers' views on mathematics teaching and learning through lesson study. *International Journal for Lesson & Learning Studies*, 2046–8253.

Fauskanger, J., Jakobsen, A., & Kazima, M. (2019). Malawi Mathematics Teacher Educators' Understanding of Lesson Study. *International Journal for Lesson and Learning Studies*, *8*(1), 48–59.

Fujii, T. (2013). Implementing Japanese Lesson Study in Foreign Countries: Misconceptions Revealed. *Mathematics Teacher Education and Development*, 17.

Golafshani, N. (2015). Understanding Reliability and Validity in Qualitative Research. *The Qualitative Report*. https://doi.org/10.46743/2160-3715/2003.1870

Gülkılık, H., Uğurlu, H. H., & Yürük, N. (2015). Examining Students' Mathematical Understanding of Geometric Transformations Using the Pirie-Kieren Model. *Educational Sciences: Theory & Practice*. https://doi.org/10.12738/estp.2015.6.0056

Gunhan, B. C. (2014). A case study on the investigation of reasoning skills in geometry. *South African Journal of Education*, *34*(2), 19.

Helmbold, E., Venketsamy, R., & van Heerden, J. (2021). Implementing Lesson Study as a professional development approach for early grade teachers: A South African case study. *Perspectives in Education*, *39*(3). https://doi.org/10.18820/2519593X/pie.v39.i3.14

Hershkowitz, R., Duval, R., Bussi, M. G. B., Boero, P., Lehrer, R., Romberg, T., Berthelot, R., Salin, M. H., & Jones, K. (1998). Reasoning in geometry. In C. Mammana & V. Villani (Eds.), *Perspectives on the Teaching of Geometry for the 21st Century* (Vol. 5, pp. 29–83). Springer Netherlands. https://doi.org/10.1007/978-94-011-5226-6_3

Hervas, G., & Medina, J. (2021). Learning and developing during lesson study through professional conversations. *International Journal for Academic Development*, *26*(3). https://doi.org/10.1080/1360144X.2021.1931872

Huang, R., Takahashi, A., Clivaz, S., Kazima, M., & Inprasintha, M. (2018). Lesson study in mathematics: Current status and future directions. *PROC. INT. OF MATH*, *1*, 1125–1156.

Huang, X., & Huang, R. (2023). Characterizing teachers' collective learning through lesson study as document development: A case study in China. *International Journal for Lesson & Learning Studies*, *12*(1), 92–105. https://doi.org/10.1108/IJLLS-03-2022-0035

Imenda, S. (2014). Is there a conceptual difference between theoretical and conceptual frameworks? *Journal of Social Sciences*, *38*(2), 185–195. https://doi.org/10.1080/09718923.2014.11893249

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). *Professional noticing of children's mathematical thinking", Journal for Research in Mathematics Education*. *41*(2), 169–202.

Jones. (2000). Teacher knowledge and professional development in Geometry. *Proceedings of the British Society for Research into Learning Mathematics*, *20*(3), 109–114.

Jones, K., Fujita, T., & Kunimune, S. (2012). Promoting productive reasoning in the teaching of geometry in lower secondary school: Towards a future research agenda. *Pre-Proceedings of the 12th International Congress on Mathematical Education (ICME-12*, 2387–2396.

Joppe, M. (2000). The research process. *Retrieved February*. https://www.academia.edu/930161/The_research_process

Karlsen, A. M. F., (2022). Investigating teacher learning in Lesson Study: The important link between reported observations and change of plans. *Professional Development in Education*, *48*(1), 53–69. https://doi.org/10.1080/19415257.2019.1685564

Karlsen, A. M. F., & Helgevold, N. (2019). Lesson Study: Analytic stance and depth of noticing in post-lesson discussions. *International Journal for Lesson and Learning Studies*, *8*(4), 290–304. https://doi.org/10.1108/IJLLS-04-2019-0034

Kotzé, G. (2007). Investigating shape and space in mathematics: A case study. *South African Journal of Education*, *27*(1), 19–35.

Larssen, D., & Drew, I. (2015). The influence of a Lesson Study cycle on a 2nd grade EFL picture book-based teaching practice lesson in Norway. *Nordic Journal of Modern Language Methodology*, *3*(2). https://doi.org/10.46364/njmlm.v3i2.143

Larssen, D. L. S., Cajkler, W., Mosvold, R., Bjuland, R., Helgevold, N., Fauskanger, J., Wood, P., Baldry, F., Jakobsen, A., Bugge, H. E., Næsheim-Bjørkvik, G., & Norton, J. (2018). A literature review of lesson study in initial teacher education: Perspectives about learning and observation. *International Journal for Lesson and Learning Studies*, *7*(1), 8–22. https://doi.org/10.1108/IJLLS-06-2017-0030

Lee, L. H. J., & Tan, S. C. (2020). Teacher learning in Lesson Study: Affordances, disturbances, contradictions, and implications. *Teaching and Teacher Education*, *89*, 102986. https://doi.org/10.1016/j.tate.2019.102986

Lee, M. Y., & Choy, B. H. (2017). Mathematical teacher noticing: The key to learning from Lesson Study. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks* (pp. 121–140). Springer International Publishing. https://doi.org/10.1007/978-3-319-46753-5_8

Lewis, C. (2009). What is the nature of knowledge development in lesson study? *Educational Action Research*, *17*(1), 95–110. https://doi.org/10.1080/09650790802667477

Lewis, C., Friedkin, S., Emerson, K., Henn, L., & Goldsmith, L. (2019). How does Lesson Study work? Toward a theory of Lesson Study process and impact. In *Theory and Practice of Lesson Study in Mathematics: An International Perspective* (pp. 14–35). 978-3-030-04031-4.

Lewis, C., & Perry, R. (2017). Lesson Study to scale up research-based knowledge: A randomized, controlled trial of fractions learning. *Journal for Research in Mathematics Education*, *48*(3), 261–299. https://doi.org/10.5951/jresematheduc.48.3.0261

Lewis, J. M., Fischman, D., Riggs, I., & Wasserman, K. (2013). Teacher learning in Lesson Study. *The Mathematics Enthusiast*, *10*(3), 583–620. https://doi.org/10.54870/1551-3440.1281

Liehr, P., & Smith, M. J. (1999). Middle range theory: Spinning research and practice to create knowledge for the new millennium. *Advances in Nursing Science*, *21*(4).

Lincoln, Y. S., & Guba, E. G. (2023, March 2). *Naturalistic Inquiry*. SAGE Publications Ltd. https://uk.sagepub.com/en-gb/eur/naturalistic-inquiry/book842

Loomis, E. S. (1968). *The Pythagorean Proposition*.

Mack, N., Woodsong, C., MacQeen, K. M., & Namey, E. (2005). *Qualitative research methods: A data collectors field guide*.

Malawi Nation Examination Board. (2019). Malawi School Certificate of Education (MSCE) mathematics I and II (M131) chief examiner's report. *Unpublished*, 1–14.

Malawi Nation Examination Board. (2020). Malawi School Certificate of Education (MSCE) mathematics I and II (M131) chief examiner's report. *Unpublished*, 1–13.

Mathison, S. (1988). Why triangulate? *Educational Researcher*, *17*(2), 13–17. https://doi.org/10.3102/0013189X017002013

Ministry of Education. (2010). School-based continuing professional development implementation guidelines. *Ministry of Education and Japan International Cooperation Agency, Zambia*.

Ministry of Education Science and Technology (MoEST). (2013a). *Syllabus for mathematics forms 1 and 2* (Unpublished). Malawi Institute of Education (MIE).

Ministry of Education Science and Technology (MoEST). (2013b). *Syllabus for mathematics forms 3 and 4* (Unpublished). Malawi Institute of Education (MIE).

Ministry of Education Science and Technology (MoEST). (2015). *Secondary School Curriculum and Assessment Framework*. Malawi Institute of Education (MIE).

Murata, A. (2011). Introduction: Conceptual Overview of Lesson Study. In L. C. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson Study Research and Practice in Mathematics Education* (pp. 1–12). Springer Netherlands. https://doi.org/10.1007/978-90-481-9941-9_1

Murata, A., Bofferding, L., Pothen, B. E., Taylor, M. W., & Wischnia, S. (2012). Making connections among student learning, content, and teaching: Teacher talk paths in elementary mathematics Lesson Study. *Journal for Research in Mathematics Education*, *43*(5), 616–650. https://doi.org/10.5951/jresematheduc.43.5.0616

Mwadzaangati, L., Takker, S., & Adler, J. (2022). Teacher learning about exemplification in Geometry through Lesson Study. *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education*, *3*, 219–226.

Ono, Y., & Ferreira, J. (2010). A case study of continuing teacher professional development through lesson study in South Africa. *South African Journal of Education*, *30*(1), 59–74. https://doi.org/10.15700/saje.v30n1a320

Pjanic, K. (2014). The origins and products of Japanese lesson study. *Inovacije u Nastavi*, *27*(3), 83–93. https://doi.org/10.5937/inovacije1403083P

Qi, C., Lai, M. Y., Liu, L., Zuo, S., Liang, H., & Li, R. (2023). Examining teachers' learning through a project-based learning lesson study: A case study in China. *International Journal for Lesson & Learning Studies*, *12*(1), 106–119. https://doi.org/10.1108/IJLLS-09-2022-0127

Quaresma, M., Winsløw, C., Clivaz, S., da Ponte, J. P., Ní Shúilleabháin, A., & Takahashi, A. (Eds.). (2018). *Mathematics Lesson Study Around the World: Theoretical and Methodological Issues*. Springer International Publishing. https://doi.org/10.1007/978-3-319-75696-7

Santagata, R. (2011). From teacher noticing to a framework for analysing and improving classroom lessons. In M. Sherin, V. Jacobos, & R. Philipp (Eds.), *Mathematics Teacher Noticing: Seeing through teachers' eyes* (1st ed., pp. 152–168). Routledge.

Santagata, R., König, J., Scheiner, T., Nguyen, H., Adleff, A.-K., Yang, X., & Kaiser, G. (2021). Mathematics teacher learning to notice: A systematic review of studies of video-based programs. *ZDM – Mathematics Education*, *53*(1), 119–134. https://doi.org/10.1007/s11858-020-01216-z

Schoenfeld, A., Dosalmas, A., Fink, H., Sayavedra, A., Tran, K., Weltman, A., Zarkh, A., & Zuniga-Ruiz, S. (2019). Teaching for Robust Understanding with Lesson Study. In R. Huang, A. Takahashi, & J. P. da Ponte (Eds.), *Theory and Practice of Lesson Study in Mathematics* (pp. 135–159). Springer International Publishing. https://doi.org/10.1007/978-3-030-04031-4_7

Schoenfeld, A. H. (2011). Noticing matters. A lot. Now what? In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics Teacher Noticing* (pp. 323–338). Routledge.

Schön, D. A. (1983). *The reflective practitioner: How professionals think in action*. Basic Books.

Seleznyov, S. (2018). Lesson study: An exploration of its translation beyond Japan. *International Journal for Lesson and Learning Studies*, *7*(3), 217–229. https://doi.org/10.1108/IJLLS-04-2018-0020

Serin, H. (2018). Perspectives on the Teaching of Geometry: Teaching and Learning Methods. *Journal of Education and Training*, *5*(1). https://doi.org/10.5296/jet.v5i1.12115

Singh, N. A., & Kumar, N. (2022). Geometry and its use in day-to-day life. *International Research Journal of Modernization in Engineering Technology and Science*, *4*(5).

Star, J. R., Lynch, K., & Perova, N. (2011). Using video to improve preservice mathematics teachers' abilities to attend to classroom features. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 117–133).

Stigler, J. W., & Hiebert, J. (1999). *The Teaching Gap: Best ideas from the world's teachers for improving education in the classroom*. Free Press.

Tachie, S. A. (2020). The Challenges of South African Teachers in Teaching Euclidean Geometry. *International Journal of Learning, Teaching and Educational Research*, *19*(8), 297–312. https://doi.org/10.26803/ijlter.19.8.16

Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM*, *48*(4), 513–526. https://doi.org/10.1007/s11858-015-0752-x

Tyskerud, A., & Mosvold, R. (2018). Scrutinizing teacher-learner interactions on volume. *Nordic Studies in Mathematics Education*, *23*(2), 49–67.

Uffen, I., de Vries, S., Goei, S. L., van Veen, K., & Verhoef, N. (2022). Understanding teacher learning in lesson study through a cultural–historical activity theory lens. *Teaching and Teacher Education*, *119*, 103831. https://doi.org/10.1016/j.tate.2022.103831

van der Waerden, B. L. (1978). *Die Pythagoreer*. Zürich.

Van Driel, J. H., & Berry, A. (2012). Teacher Professional Development Focusing on Pedagogical Content Knowledge. *Educational Researcher*, *41*(1), 26–28. https://doi.org/10.3102/0013189X11431010

van Es, E. A. (2011). A framework for learning to notice student thinking. In M. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics Teacher Noticing. Seeing through Teachers' Eyes* (pp. 164–181).

Van Es, E. A., & Sherin, M. G. (2021). Expanding on prior conceptualizations of teacher noticing. *ZDM – Mathematics Education*, *53*(1), 17–27. https://doi.org/10.1007/s11858-020-01211-4

Vermunt, J. D., Vrikki, M., van Halem, N., Warwick, P., & Mercer, N. (2019). The impact of Lesson Study professional development on the quality of teacher learning. *Teaching and Teacher Education*, *81*, 61–73. https://doi.org/10.1016/j.tate.2019.02.009

Warwick, P., Vrikki, M., Vermunt, J. D., Mercer, N., & van Halem, N. (2016). Connecting observations of student and teacher learning: An examination of dialogic processes in Lesson Study discussions in mathematics. *ZDM*, *48*(4), 555–569. https://doi.org/10.1007/s11858-015-0750-z

Weber, R. P. (1990). *Basic Content Analysis*. SAGE.

Wittmann, E. C. (2021). Designing Teaching: The Pythagorean Theorem. In E. C. Wittmann, *Connecting Mathematics and Mathematics Education* (pp. 95–160). Springer International Publishing. https://doi.org/10.1007/978-3-030-61570-3_7

Wulandari, S., Syahbana, A., Tanzimah, T., Shang, Y., Weinhandl, R., & Sharma, R. (2021). Analysis of students' thinking level in solving Pythagoras' theorem problems based on Van Hiele's theory. *Malikussaleh Journal of Mathematics Learning (MJML)*, *4*(2), 124. https://doi.org/10.29103/mjml.v4i2.3905

Xu, H., & Pedder, D. (2015). Lesson Study. An international review of research. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (pp. 29–58).

Yin, R. K. (2016). *Qualitative research from start to finish* (Second edition). The Guilford Press.

Appendices

Appendix 1: PGR consent letter

Vice Chancellor: P. O. Box 280, Zomba, MALAWI

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MATHEMATICS AND SCIENCE EDUCATION SECTION

27 June 2022

TO WHOM IT MAY CONCERN

LETTER OF PERMISSION: MS BLESSINGS MUHUWA

This is to certify that Ms Blessings Muhuwa is a Master of Education (Mathematics Education) student at the University of Malawi. Ms Muhuwa is in her second year of the master's programme and is researching within the Promoting Geometric Reasoning (PGR) project based in the Mathematics and Science section. She has permission from the University to use the PGR project data for her master's thesis and to work with the project schools and teachers for further data that she might need.

Please feel free to contact me if there are any queries or questions regarding Ms Muhuwa's data.

MERCY KAZIMA

Head of Mathematics and Science Section

Appendix 2

Consent from the National Centre for Research Data (NSD)

Notification form / An investigation of teacher development through lesson study i... / Assessment

Assessment of processing of personal data

Reference number	Assessment type	Date
820699	Standard	28.11.2022

Project title
An investigation of teacher development through lesson study in geometric reasoning

Data controller (institution responsible for the project)
University of Stavanger / Faculty of Education and Humanities / Department of Primary School Teacher Education, Sports and Special Education

Project leader
Raymond Bjuland

Student
Blessings Muhuwa

Project period
15/09/2022 - 01/08/2023

Categories of personal data
General

Legal basis
Consent (General Data Protection Regulation art. 6 no. 1 a)

The processing of personal data is lawful, as long as it is carried out as stated in the notification form. The legal basis is valid until 01.08.2023.

Notification Form

Comment
ABOUT OUR ASSESSMENT
Data Protection Services has an agreement with the institution where you are carrying out research or studying. As part of this agreement, we provide guidance so that the processing of personal data in your project is lawful and complies with data protection legislation.

We have now assessed the planned processing of personal data in this project. Our assessment is that the processing is lawful, as long as it is carried out as described in the Notification Form with dialogue and attachments.

IMPORTANT INFORMATION
You must store, send and secure the collected data in accordance with your institution's guidelines. This means that you must use online surveys, cloud storage, and video conferencing providers (and the like) that your institution has an agreement with. We provide general advice on this, but it is your institution's own guidelines for information security that apply.

TYPE OF DATA AND DURATION
The project will process general categories of personal data, special categories of personal data about until 01.08.2023.

LEGAL BASIS
The project will gain consent from data subjects and from the parents of the pupils to process their personal data. We find that consent will meet the necessary requirements under art. 4 (11) and 7, in that it will be a freely given, specific, informed and unambiguous statement or action, which will be documented and can be withdrawn.

The legal basis for processing general categories of personal data is therefore consent given by the data subject, cf. the General Data Protection Regulation art. 6.1 a).

PRINCIPLES RELATING TO PROCESSING PERSONAL DATA
We find that the planned processing of personal data will be in accordance with the principles under the General Data Protection Regulation regarding:

- lawfulness, fairness and transparency (art. 5.1 a), in that data subjects and the parents will receive sufficient information about the processing and will give their consent
- purpose limitation (art. 5.1 b), in that personal data will be collected for specified, explicit and legitimate purposes, and will not be processed for new, incompatible purposes
- data minimization (art. 5.1 c), in that only personal data which are adequate, relevant and necessary for the purpose of the project will be processed
- storage limitation (art. 5.1 e), in that personal data will not be stored for longer than is necessary to fulfill the project's purpose

THE RIGHTS OF DATA SUBJECTS
We find that the information provided to data subjects about the processing of their personal data will meet legal requirements for form and content, cf. species. 12.1 and art. 13.

As long as data subjects can be identified in the collected data they will have the following rights: access (art. 15), rectification (art. 16), erasure (art. 17), restriction of processing (art. 18) and data portability (art. 20).

We remind you that if a data subject contacts you about their rights, the data controller has a duty to reply within a month.

FOLLOW YOUR INSTITUTION'S GUIDELINES
Our assessment presupposes that the project will meet the requirements of accuracy (art. 5.1 d), integrity and confidentiality (art. 5.1 f) and security (art. 32) when processing personal data.

To ensure that these requirements are met you must follow your institution's internal guidelines and/or consult with your institution (ie the institution responsible for the project).

NOTIFY CHANGES
If you intend to make changes to the processing of personal data in this project it may be necessary to notify us. This is done by updating the information registered in the Notification Form. On our website we explain which changes must be notified. Wait until you receive an answer from us before you carry out the changes.

FOLLOW-UP OF THE PROJECT
We will follow up the progress of the project at the planned end date in order to determine whether the processing of personal data has been concluded.

Good luck with the project!
Contact person: Henriette S. Munthe-Kaas