FACULTY OF SCIENCE AND TECHNOLOGY

MASTER'S THESIS

Study programme / specialisation: Master's in Mathematics and Physics/Mathematics	The *spring*/ semester, *2023* June 2023
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Thesis title: Mathematical and Numerical Aspects of Scalar Nonlinear Conservation Laws	
Credits (ECTS): 60	
Keywords: Conservation law, Method of Characteristics(MOC), Upwind Scheme, Lax-Friedrichs (LF) scheme, Riemann Problem, Burger Equation, Courant-Friedrichs-Lew(CFL) Condition	Pages: 69 Stavanger, *June 2023*.

Approved by the Dean 30 Sep 21
Faculty of Science and Technology

Mathematical and Numerical Aspects of Scalar Nonlinear Conservation Laws

June 13, 2023

Abstract

This thesis delves deeply into numerical solutions to nonlinear conservation laws. It focuses largely on the Method of Characteristics, its application in solving conservation laws, and the implementation of solutions in MATLAB using the Lax-Friedrichs scheme.

The first chapter offers numerous examples of linear conservation rules and examines its numerical scheme. The equations' stability qualities are carefully examined. In this chapter, the Method of Characteristics is extensively used to solve conservation laws, and the efficiency of the Upwind Scheme is proved.

The generic solution to the nonlinear conservation law, $u_t + f(u)_x = 0$, is investigated in Chapter 2. The Characteristics Method is used to deduce the criteria for $u_t + f(u)_x = 0$ and to examine the Lax-Friedrichs scheme. The Rankine-Hugoniot condition, the development of similarity and shock wave solutions, and the distinctions between convex and concave flux are all covered in this chapter. It also provides a thorough comparison of the Method of Characteristics and the Finite Difference Technique. The chapter concludes with an examination of inadequate conservation legislation remedies.

The third chapter tackles a more complicated Riemann issue, presenting a thorough solution as well as MATLAB-based visualization. The use of prior chapters' knowledge and methodologies to this more complicated issue illustrates the methods' adaptability and robustness.

In conclusion, this thesis makes an important addition to the understanding and application of the Method of Characteristics and the Lax-Friedrichs scheme in the solution of nonlinear conservation laws, as evidenced by practical MATLAB implementation. The comparison of numerical systems, as well as the expansion to complicated Riemann problems, improve the study's value in improving numerical analysis in general.

Contents

Chapter 1

Basics for the Conservation law

In this starting section we are going to give a brief introduction to form the basics for the conservation law.

1.1 Introduction

A PDE that portrays the time evolution of some quantity/quantities that is/are conserved in time is known as a conservation law.[5]
To see the emergence of the conservation laws from a physical phenomenon, we will start by thinking about the most manageable problem of fluid dynamics, in which a gas or fluid is streaming through a one-dimensional pipeline with some known velocity $u(x,t)$, which is supposed to fluctuate just with x, the distance along the pipeline and time t. Usually, we mainly remain concerned about the direction of the flow of the fluid in the problems that arise in fluid dynamics, i.e., the velocity function $u(x,t)$, as a part of the solution. Suppose now this is known, and we wish to model the concentration or density of some chemical present in this liquid (in tiny amounts that don't influence the fluid elements). Suppose that $q(x,t)$ is the density of the chemical present inside the fluid, and we must find this function. As we are dealing here with the fluid's one-dimensional flow, the unit being utilized for q here is mass per unit length or grams per meter.
Then,

$$\int_{x_1}^{x_2} q(x,t)dx \tag{1.1}$$

(1.1) give us the tracer's total mass in the pipe's section from x_1 to x_2 at time t and possesses the units in mass. It is essential that whenever you are coping with problems in which chemical kinetics is being involved, then we will use the units of mass in terms of moles instead of grams, and density in moles per meter or moles per cubic meter, because we are pretty much interested in the molecules present in the chemical and not in the mass of the chemical. We will speak about mass for simplicity, but the conservation laws still apply in these other units. Consider the following section of the pipe $x_1<x<x_2$ and how integral (1.1) changes over time. If we are interested in the study of a substance that is neither created nor destroyed within this section, the total mass within this section can only change due to the flux or flow of particles through the section's endpoints at (x_1 and x_2. For $i = 1, 2$, let F_i(t) be the rate at which the tracer flows past the fixed point x_i (measured in grams per second, say). We use the convention that F_i(t)> 0 represents flow to the right, whereas F_i(t)< 0 represents flow to the left, at a rate of $|F_i(t)|$ grams per second. Because the change in the total mass in section $[x_1, x_2]$ is due to the fluxes at the endpoints. we have

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = F_1 - F_2 \tag{1.2}$$

Note that $+F_1(t)$ and $-F_2(t)$ are the fluxes into the section.
Equation (1.2) represents the conservation law in the form of the integral equation. We need to find out the relation between the $F_j(t)$(flux function) and the $q(x,t)$ to get an equation that is easy to solve for q. The fluid flow that we have mentioned above is a case in which the flux at time t at any point x is the product of $q(x,t)$(density) and $u(x,t)$(velocity).
i.e.

$$\text{flux} \quad \text{at } (x,t) = u(x,t)q(x,t) \tag{1.3}$$

Since, $u(x,t)$ is known so,

$$\text{flux} = f(q,x,t) = u(x,t)q \tag{1.4}$$

Here we have a particular case to discuss in which the velocity $u(x,t)$ is not dependent on x and t. Then, in that case, the u is constant, and we can write u as $\bar{u}$. Then the flux will take the form,

$$\text{flux} = f(q) = \bar{u}q \tag{1.5}$$

The determination of the flux at any point and time can be done directly from the conserved quantity at that point, and the flux does not depend at all on the location of the point in space-time. The equations made from the above-mentioned case are called autonomous.
The general case of the autonomous flux $f(q)$, which has a dependency only on the value of q, the conservation law mentioned in (1.2) can be re-written as,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} q(x,t)dx = f(q(x_1,t)) - f(q(x_2,t)) \tag{1.6}$$

By using calculus, we can write the above equation as,

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = -f(q(x,t)|_{x_1}^{x_2} \tag{1.7}$$

This abbreviation will be helpful when the flux has a complex form. It also suggests the calculations are done out below, which result in the differential equation for q.
Once the flux function $f(q)$ is defined, for the most basic instance, by (1.5), We have an equation for q that we can attempt to solve. This equation ought to be true, for any values of x_1 and x_2, over each interval $[x_1, x_2]$. It is unclear what to do, locating a function $q(x,t)$ that meets this requirement. Instead of attempting to solve this issue in general, we convert it into a manageable partial differential equation, through methods that are common. We must make the assumption that $q(x,t)$ and $f(q)$ is smooth enough for the following manipulations to work. Remember this as we examine non-smooth approaches to these equations.
This equation can be rewritten as follows if we make an assumption that f, as well as q, are smooth

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} q(x,t)dx = -\int_{x_1}^{x_2}\frac{\partial}{\partial x}f(q(x,t))dx \tag{1.8}$$

this equation implies that,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} q(x,t)dx + \int_{x_1}^{x_2}\frac{\partial}{\partial x}f(q(x,t))dx = 0 \tag{1.9}$$

the simplest form of the above can be written by utilizing some integration rules,

$$\int_{x_1}^{x_2}[\frac{\partial}{\partial t}q(x,t)) + \frac{\partial}{\partial x}f(q(x,t))]dx = 0 \tag{1.10}$$

for all the values of x_1 and x_2 this integral will possess a value equal to zero so,

$$\frac{\partial}{\partial t}q(x,t) + \frac{\partial}{\partial x}f(q(x,t)) = 0 \tag{1.11}$$

This equation is named Conservation law and equation(1.11) is the differential form of conservation law. Equation(1.11) can also be written as follows,

$$q_t(x,t) + f(q(x,t))_x = 0. \tag{1.12}$$

1.2 Some examples of nonlinear conservation laws

1.2.1 Traffic flow model

A mathematical depiction of how cars move along a road is called a traffic flow model. The principle of vehicle conservation, which asserts that the overall number of cars on a given length of road must remain constant until there is a change in the number of vehicles entering or exiting the route, is frequently the foundation of these models. This idea is comparable to the conservation principles of physics, which dictate that in a closed system, the total amount of a given quantity—such as mass or energy—must remain constant. In traffic flow models, the conserved quantity is the number of vehicles, and the variables that are typically considered include traffic density, speed, and flow rate.
The mathematical explanation of the traffic flow model will be as follows,

- $\bar{u}(x,t)$ represents the density of cars (number/meter)
- u_M represents the maximal number
- $u(x,t) = \frac{\bar{u}(x,t)}{u_M} \in [0,1]$
- $V(x,t)$ represents the macroscopic velocity

mass balance equation can be written as follows,

$$u_t + (uv)_x = 0 \tag{1.13}$$

If we write the equation in terms of relation between V,V_{max} and u then we can write it as,

$$V(x,t) = V_{max}(1-u) \tag{1.14}$$

This equation describes the relationship between the velocity of vehicles V and the traffic density u. The parameter V_{max} is the maximum velocity that vehicles can travel under free-flow conditions. The Mathematical equation demonstrates that vehicle velocity reduces as traffic density (u) increases. This equation is frequently applied in models of macroscopic traffic flow that depict the normal behavior of lots of cars.
The final equation for the conservation law for this model will be as follows,

$$u_t + f(u)_x = 0 \tag{1.15}$$

Equation(1.13), is a partial differential equation known as the conservation of vehicles equation. It describes how the traffic density u changes over time t and space x. The function $f(u)$ represents the flux of vehicles, which is the product of velocity and density. According to the equation, the difference between the rate at which vehicles are entering and departing a spot in space and time equals the rate at which traffic density at that location changes over time.This equation is frequently used to simulate the dynamics of traffic on a highway.

In traffic flow models, these equations are frequently combined to explain the spatiotemporal evolution of traffic density and velocity.To evaluate traffic patterns, forecast traffic congestion, and create traffic control systems, these models are employed.[3]

The Lighthill-Whitham-Richards (LWR) flux function is a typical representation of the function $f(u)$ in the context of traffic flow,

$$f(u) = uv(u) \tag{1.16}$$

where the traffic velocity, $v(u)$, is a decreasing function of u that increases as u approaches zero, reaching a maximum value, v_{max}.In particular, the Greenshields model is frequently used to model $v(u)$,

$$v(u) = v_{\max}(1 - \frac{u}{\rho_{\max}}) \tag{1.17}$$

where the maximum car density on the route is $\rho_{\max}$.

Since the velocity function $v(u)$ is declining and linear, the LWR flux function is often concave in the region of interest (i.e., the region of density where traffic flow occurs).Though there may be changes in the form of $f(u)$ that reflect certain characteristics of the traffic flow model or the road network under consideration, the precise shape of $f(u)$ can depend on the specific choice of velocity model.

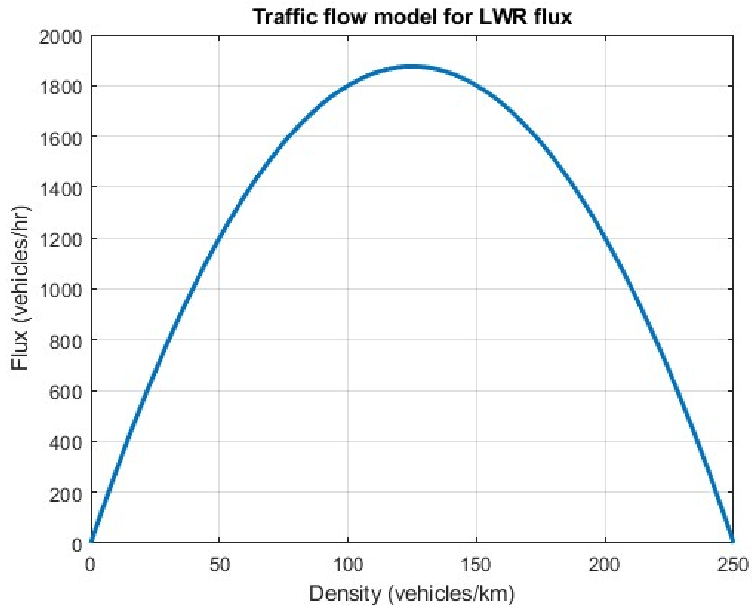

Figure 1.1: The graphical interpretation for the traffic flow model

1.2.2 Displacement of two fluids in a reservoir

The action of one fluid (typically of lower density) driving another fluid (usually denser) out of a porous solid, such as silt or rock, is referred to as fluid displacement. This phenomenon is caused by pressure differences between the fluids, which propel them through the material's interconnecting pore spaces. Additional forces, such as gravity or capillary forces, may aid in the displacement process in some cases. This phenomena may be seen in a variety of settings, including oil reservoirs, aquifers, and geothermal systems, where it plays an important role in resource management, environmental conservation, and energy generation.

- s_w represents the volume fraction of water
- s_o represents the volume fraction of oil
- $\phi \cdot \partial_t s_w + \partial_x (u_w) = 0$
- $\phi \cdot \partial_t s_o + \partial_x (u_o) = 0$

These cases are for mass balance,

$$u_w = -\lambda_w (s_w) (\partial_x P + r_w), \tag{1.18}$$
$$u_o = -\lambda_o (s_o) (\partial_x P + r_o). \tag{1.19}$$

the equation of conservation law for this case will be as follows,

$$s_t + f(s)_x = 0 \tag{1.20}$$

with $s = s_w$ and $1 - s = s_o$

$$f(s) = \frac{s^2}{(s)^2 + (1-s)^2 M} \tag{1.21}$$

where as,

$$M = \frac{\mu_w}{\mu_o} \tag{1.22}$$

"M" is known as the mobility ratio and is equal to the ratio of the dynamic viscosity of the water phase μ_w to the dynamic viscosity of the oil phase μ_o. A fluid's viscosity is a measure of its resistance to flow, so the mobility ratio represents the relative ease of flow between the two fluid phases.
The mobility ratio is an essential quantity in the displacement of two fluids in porous medium because it indicates the relative speed at which the two fluids travel through the porous media. If the mobility ratio is less than one, the water phase is less viscous than the oil phase and may easily displace it. If the mobility ratio is larger than one, the oil phase is less viscous than the water phase and can easily displace it. The mobility ratio may also be utilized to forecast the period when one fluid phase first shows at the production well. The mobility ratio is important in calculating fluid displacements and flow rates in porous media and is utilized in models that explain fluid flow in these systems.
If we want to discuss the nature of $f(s)$ then we can do it as follows,
Taking the second derivative of the function $f(s)$ of equation (1.21) with respect to s,

$$f''(s) = \frac{2(1-M)(2s-1)}{[(s)^2 + (1-s)^2 M]^3}. \tag{1.23}$$

Since $1 - M$ is positive and the denominator of $f''(s)$ is always positive, the sign of $f''(s)$ is determined solely by the numerator $(2s - 1)$. This numerator is zero when $s = 1/2$, and positive when $s > 1/2$, and negative when $s < 1/2$. Therefore, $f(s)$ is concave for $0 < s < 1/2$, convex for $s > 1/2$, and neither convex nor concave at $s = 1/2$. The graphicval representation of $f(s)$ given in equation (1.21) for $M = 1$ will be as follows,

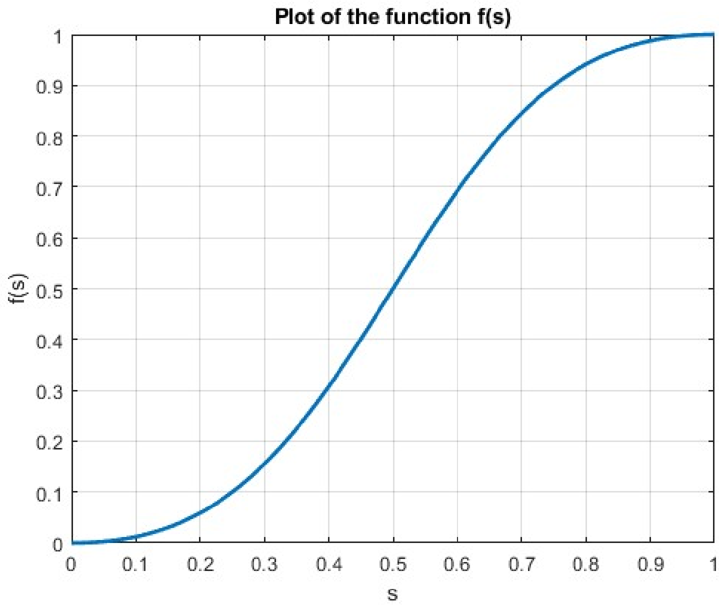

Figure 1.2: The Graph for $f(s)$ in equation (1.21) when $M = 1$

1.3 The Advection Equation

Before digging into the material for studying nonlinear conservation law, we will briefly recall how to use the method of characteristics to solve linear advection equation.
For the flux that we have mentioned in equation(1.5),for that flux the equation(1.11) becomes as follows,

$$q_t + \bar{u}q_x = 0 \tag{1.24}$$

Equation(1.24) is referred to as Advection Equation Since it simulates the advection of a tracer along with the fluid. A chemical that is present in the fluid at very low quantities, such that the magnitude of the concentration essentially has no impact on the fluid dynamics, is referred to as a tracer.

1.3.1 Example

In this section, we are going to discuss some of the problems in which f(flux function) is linear. We are going to solve the linear from of conservation law by the method of characteristics.

1.3.2 Example 1

First of all we have the equation in the form,

$$u_t + au_x = 0 \tag{1.25}$$

and the initial condition:

$$u(x, t = 0) = u_0(x) \tag{1.26}$$

we are going to solve this equation by the method of characteristics, first of all consider that $u(x, t)$ along the path in $x - t$ space which is given by,

$$\frac{dx(t)}{dt} = a, x(t = 0) = x_0 \tag{1.27}$$

Now we will do the integration as follows,

$$\int_{x_0}^{x(t)} dx = a \int_0^t dt \tag{1.28}$$

we will get,

$$x(t) - x_0 = at \tag{1.29}$$

which implies,

$$x(t) = x_0 + at \tag{1.30}$$

Now we will check that how $u(x, t)$ vary along this $x(t)$. Remember that we are going to use the equation (1.27) in the below equation i.e. $\frac{dx(t)}{dt} = a$

$$\frac{du(x(t), t)}{dt} = u_x \frac{dx}{dt} + u_t \frac{dt}{dt} = u_x a + u_t = 0 \tag{1.31}$$

Equation(1.31) implies that

$$u(x(t), t) = constant = u(x(t = 0), t = 0) \tag{1.32}$$

as we know that $x(t) = x_0 + at$ so, $x_0 = x(t) - at$ we have equation (1.32) as

$$u(x(t), t) = u_0(x(t) - at) \tag{1.33}$$

Finally,

$$u(x, t) = u_0(x - at) \tag{1.34}$$

The Explanation of the Matlab code and Figure 1.3 generated after the execution of the code is as follows,

- In the beginning of the code script the number of time steps iS represented by $Ntime = 50 * 4$, the final time by $T = 0.5$, and the number of grid cells by $N = 500$.
- The grid spacing dx and the time step dt are totally dependent on the values of T, $Ntime$, and N.
- The function `fun_initial` is utilized for the initial data.
- Red dotted line was used for the plotting of the initial data.
- The computation of the Numerical solution was done by looping over the time steps using an upwind scheme.
- When the loop got finished, computation of the exact solution was plotted with the green solid line.
- We have used legend in the graph to see the difference between initial data, numerical solution and the exact solution.

There is no predetermined number of grid cells that guarantees a perfect approximation since the quality of the approximation relies on a variety of factors, including the numerical method employed, the time step, and the features of the problem. However, in general, the accuracy of the approximation may be increased by increasing the number of grid cells N and decreasing the time step dt. In fact, it is crucial to strike a balance between accuracy and effectiveness since, as should be mentioned, increasing grid resolution also raises computing costs.
Figure 3 shows the real, shape-unaltered translation of the initial condition to the right.This behavior can be seen as the lines move to the right while maintaining the same form as the original data (red dashed line) in both the exact solution (green solid line) and the numerical approximation (orange line with circle markers).

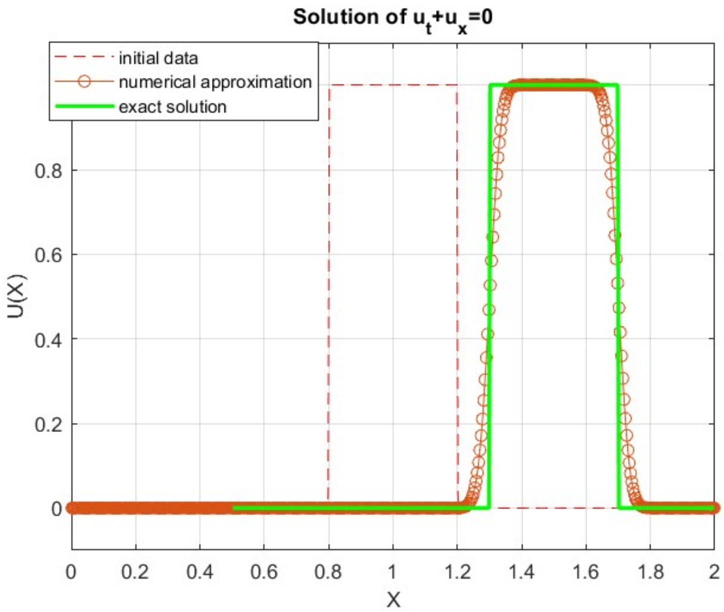

Figure 1.3: Graphical representation for Example 1

1.3.3 Figure in $x - t$ space

The Figure 1.4 describes the trends of the characteristics in $x - t$ space for $u_t + au_x = 0, a = 1$.

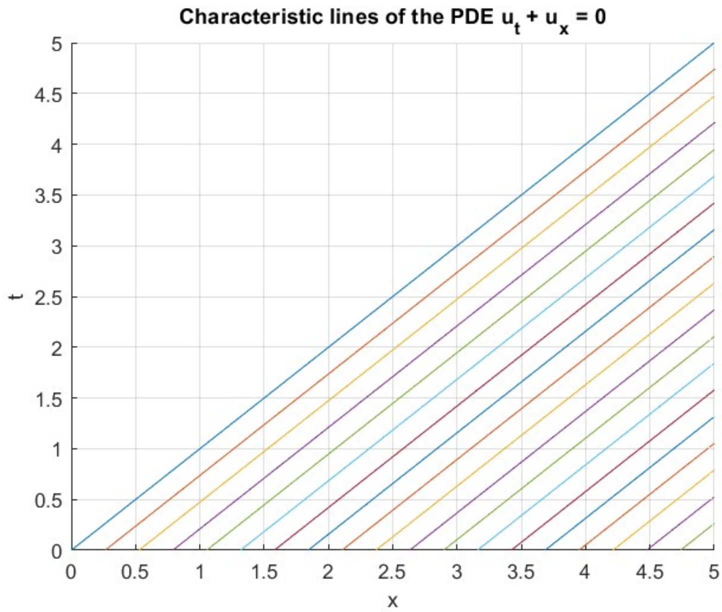

Figure 1.4: Characteristic in $x - t$ for Example 1

1.3.4 Numerical scheme for $u_t + au_x = 0$

The numerical scheme for the equation, Suppose that $a > 0$,

$$u_t + au_x = 0 \tag{1.35}$$

with the initial data

$$u(x, t = 0) = u_0(x) \tag{1.36}$$

is as follows

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n}{\Delta x} = 0, j = 2, 3....., M-1 \tag{1.37}$$

Now, we have to define the upwind principle as follows for the case $a > 0$,

$$u_{j+\frac{1}{2}}^n = u_j^n, u_{j-\frac{1}{2}}^n = u_{j-1}^n \tag{1.38}$$

we have $u_1^{n+1} = u_0(x_1)$ and $u_M^{n+1} = u_0(x_M)$ for the first and last cell respectively. we choose

$$a\frac{\Delta t}{\Delta x} \leq 1 \tag{1.39}$$

for the time step $= \Delta t = t^{n+1} - t^n$.
You can adjust the upwind principle in accordance with the instance where

$a < 0$ to achieve a numerical scheme. You should apply the upwind Scheme given below, since the information will spread in the other direction,

$$u^n_{j+\frac{1}{2}} = u^n_{j+1}, u^n_{j-\frac{1}{2}} = u^n_j \tag{1.40}$$

By substituting this upwind scheme in the equation (1.37), we get,

$$\frac{u^{n+1}_j - u^n_j}{\Delta t} + \frac{u^n_{j+1} - u^n_j}{\Delta x} = 0, j = 2, 3, ..., M - 1 \tag{1.41}$$

The suitability of the above numerical scheme depends upon the case when $a < 0$.[4]

1.3.5 Stability properties for $u_t + au_x = 0$

For the given PDE with a constant a [5],

$$u_t + au_x = 0, x \in (0, 2) \tag{1.42}$$

Boundary conditions are as,

$$u(0, t) = u(2, t) = 0, u(x, t = 0) = u_0(x) \tag{1.43}$$

we going to perform the integration to check the behavior of the true solution

$$\int_0^2 u_t dx + \int_0^2 (au_x) dx = 0. \tag{1.44}$$

Utilization of the boundary conditions is performed as follows,

$$\frac{d}{dt} \int_0^2 u dx = -\int_0^2 (au_x) dx = -au(2, t) + au(0, t) = 0. \tag{1.45}$$

Doing the integration from 0 to t,

$$\int_0^2 u(., t) dx = \int_0^2 u(., 0) dx = \int_0^2 u_0(.) dx. \tag{1.46}$$

In the general form we can write as follows,

$$\int_0^2 |u(., t)| dx \leq \int_0^2 |u_0(.)| dx. \tag{1.47}$$

1.3.6 Stability of Numerical scheme for $u_t + au_x = 0$ when $a > 0$

$$u(0,t) = u(2,t) = 0, u(x,t=0) = u_0(x) \tag{1.48}$$

We have the numerical scheme as follows,

$$u_j^{n+1} = u_j^n - \alpha(u_j^n - u_{j-1}^n), \quad j = 1, \dots, M, \quad \alpha = a\frac{\Delta t}{\Delta x}, u_0 = u_{M+1} = 0. \tag{1.49}$$

In order to analyze the stability we have,

$$u_j^{n+1} = u_j^n(1-\alpha) + \alpha u_{j-1}^n. \tag{1.50}$$

By using the triangular inequality and the absolute value. Remember one thing that $\alpha \geq 0$ and $(1-\alpha) \geq 0$ therefore the changes in the inequality will be as follows.

$$|u_j^{n+1}| = |u_j^n(1-\alpha) + \alpha u_{j-1}^n| \leq |u_j^n(1-\alpha)| + |\alpha u_{j-1}^n| = (1-\alpha)|u_j^n| + \alpha|u_{j-1}^n|. \tag{1.51}$$

As $a > 0$, the condition for stability will adopt the form $\alpha = a\frac{\Delta t}{\Delta x} \leq 1$, i.e., $\Delta t \leq \frac{\Delta x}{a}$. By using this inequality and modifying the index of summation,

$$\sum_{j=1}^{M} |u_j^{n+1}| \leq (1-\alpha)\sum_{j=1}^{M} |u_j^n| + \alpha \sum_{j=1}^{M} |u_{j-1}^n|. \tag{1.52}$$

As we know that $\alpha \geq 0$, therefore $1-\alpha \geq 0$. So, α and $1-\alpha$ will came out of the absolute value.

$$\sum_{j=1}^{M} |u_j^{n+1}| \leq (1-\alpha)\sum_{j=1}^{M} |u_j^n| + \alpha \sum_{j=0}^{M-1} |u_j^n| \leq \sum_{j=1}^{M} |u_j^n|. \tag{1.53}$$

To Conclude,

$$\sum_{j=1}^{M} |u_j^{n+1}| \leq \sum_{j=1}^{M} |u_j^n| \leq \cdots \leq \sum_{j=1}^{M} |u_j^0|, \tag{1.54}$$

To put it another way, Our analysis demonstrates the stability of our numerical scheme. The requirement for the stability is as $\alpha = a\frac{\Delta t}{\Delta x} \leq 1$. When a is positive then the requirement for the stability will take the form $\Delta t \leq \frac{\Delta x}{a}$. Therefore, if the condition $\Delta t \leq \frac{\Delta x}{a}$ holds, the stability of the numerical scheme is assured for the provided PDE $u_t + au_x = 0$, with the required boundary and initial conditions.[5]

1.3.7 Effectiveness of the Upwind Scheme

[5] Upwind schemes are resilient approaches for dealing with partial differential equations (PDEs), which are frequently encountered in transport phenomena and computational fluid dynamics (CFD). They are especially useful for convection-dominated issues. Here are a few reasons why upwind techniques are effective.

- Upwind schemes, as opposed to central-difference schemes, provide more stability over a larger variety of circumstances. They excel at dealing with shocks, contact discontinuities, and steep slopes without causing undesired oscillations or instabilities, making them appropriate for simulations incorporating these phenomena.
- One of the key advantages of upwind schemes is their inherent adherence to conservation rules, such as those of mass, momentum, and energy. These concepts are critical for developing accurate models, especially when dealing with compressible flows or long-term temporal integration.
- When compared to alternative solutions, upwind approaches display more resilience and less reliance on grid quality. Because they can support skewed or non-orthogonal meshes, they are an ideal choice for complicated geometries.
- Upwind schemes, intuitively, align with the natural flow of information. They provide convergence to the right physical solution even in the presence of significant flow gradients by discretizing the convective component according to the local flow direction.
- Upwind schemes are often used in a variety of technological applications because they are easier to build and less computationally intensive than more complex approaches such as high-order schemes and discontinuous Galerkin methods.

However, there are certain disadvantages to upwind designs. They are highly diffusive, which might possibly blur strong gradients and affect solution accuracy. To address this, higher-order upwind schemes or alternative tactics such as TVD (Total Variation Diminishing) and MUSCL (Monotonic Upstream-centered Scheme for Conservation Laws) might be used. These strategies can improve accuracy while maintaining stability.[2],[12]

1.4 Example 2

The equation in this case is as,

$$u_t + xu_x = 0, x \in [0, 2] \tag{1.55}$$

with the initial conditions,

$$u(x; t = 0) = \phi(x) \tag{1.56}$$

To solve the conservation law equation $u_t + xu_x = 0$ by the method of characteristics, we need to find a family of characteristic curves in the $x - t$ plane, along which the solution remains constant.
We can write the characteristic as,

$$\frac{dx}{dt} = x(t), x(t = 0) = x_0 \tag{1.57}$$

Now we will take the integration as follows,

$$\int_{x_0}^{x(t)} \frac{1}{x} dx = \int_0^t dt \tag{1.58}$$

After integration we get,

$$\ln(x) - \ln(x_0) = t \tag{1.59}$$

and,

$$e^{ln(\frac{x}{x_0})} = e^t \tag{1.60}$$

Equation (1.60) implies that,

$$x(t) = x_0 e^t \tag{1.61}$$

Now we will check that how $u(x, t)$ vary along this $x(t)$. Remember that we are going to use the equation (1.57) in the below equation i.e. $\frac{dx(t)}{dt} = x(t)$

$$\frac{du(x(t), t)}{dt} = u_x \frac{dx}{dt} + u_t \frac{dt}{dt} = u_x x + u_t = 0 \tag{1.62}$$

Equation(1.62) implies that

$$u(x(t), t) = constant = u(x(t = 0), t = 0) \tag{1.63}$$

As we know that $x(t) = x_0 e^t$ so, $x_0 = x(t)e^{-t}$ we have equation (1.63) as

$$u(x(t), t) = \phi(x_0) \tag{1.64}$$

Hence,

$$u(x(t), t) = \phi(x(t)e^{-t}) \tag{1.65}$$

Remember, The solution moves along the characteristic arcs at a constant velocity defined by the value of x at each location.

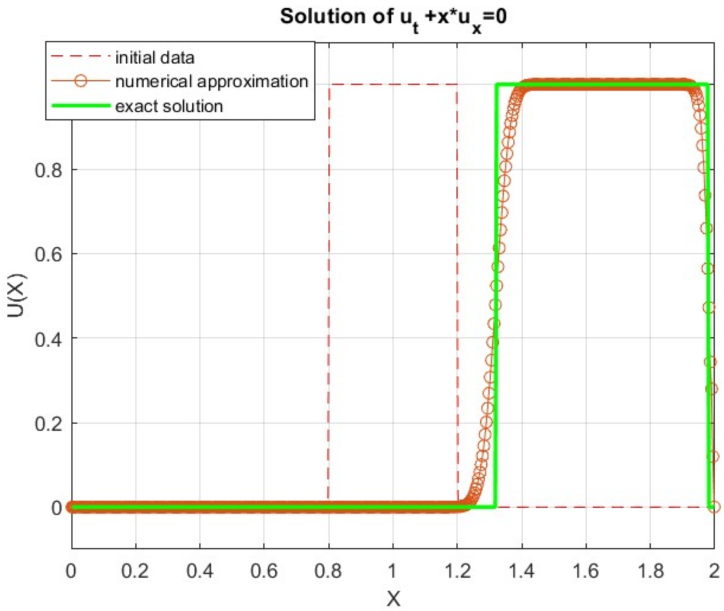

Figure 1.5: Graphical representation for Example 2

1.4.1 Numerical scheme for the $u_t + xu_x = 0$

Given the linear advection equation:

$$u_t + xu_x = 0 \tag{1.66}$$

with the initial condition $u(x, 0) = u_0(x)$, we can use the following numerical scheme to approximate its solution,
Let $x_j = j\Delta x$ and $t_n = n\Delta t$, where Δx and Δt are the grid spacing in space and time respectively. Then, we can define the numerical solution $u_j^n \approx u(x_j, t_n)$. Using a forward difference for time and an upwind scheme for space, we have,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + x_j \frac{u_j^n - u_{j-1}^n}{\Delta x} = 0 \tag{1.67}$$

Note that for stability of the scheme, we need to ensure the CFL condition,

$$x_j \frac{\Delta t}{\Delta x} \leq 1$$

The boundary conditions can be discretized as follows,

$$u_0^n = u_0(x_0), \quad u_M^n = u_0(x_M)$$

where x_0 and x_M are the locations of the left and right boundary points.[5]

1.4.2 Characteristics in $x - t$ for $u_t + xu_x = 0$, $x \in [0; 2]$

The Figure 1.6, explains the characteristics in $x-t$ for $u_t+xu_x = 0$, $x \in [0 : 2]$

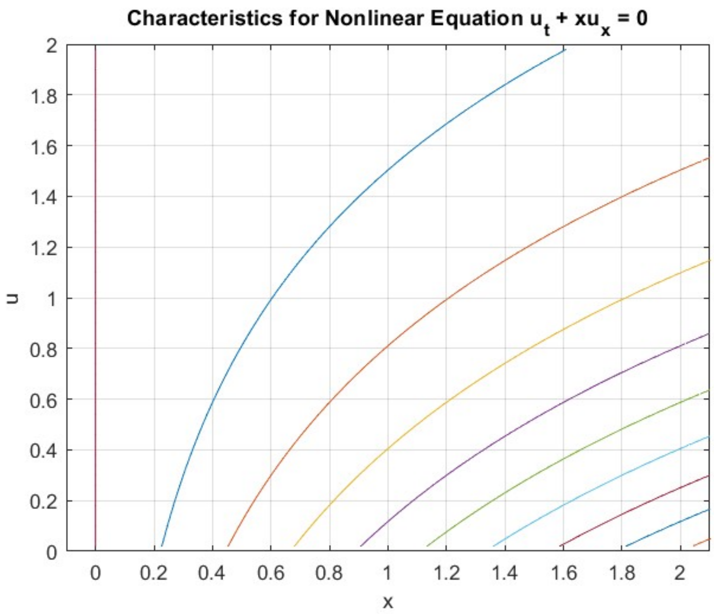

Figure 1.6: Characterstics for Example 2

1.4.3 Stability properties for $u_t + xu_x = 0$

$$u_t + xu_x = 0, x \in (0, 2) \tag{1.68}$$

with the initial condition:

$$u(x, t = 0) = u_0(x) \tag{1.69}$$

In order to check the behavior of the true solution we have to perform the integration in the following way.

$$\int_0^2 u_t dx + \int_0^2 (xu_x) dx = 0. \tag{1.70}$$

Performing integration by parts,

$$\int_0^2 xu_x dx = xu\Big|_0^2 - \int_0^2 udx. \tag{1.71}$$

Utilizing the conditions of boundary $u(0,t) = u(2,t) = 0$

$$\frac{d}{dt}\int_0^2 udx = \int_0^2 udx. \tag{1.72}$$

Let $I(t) = \int_0^2 udx$. Therefore,

$$\frac{d}{dt}I(t) = I(t). \tag{1.73}$$

For, $I(t)$ it is a differential equation of order first. AS, $I(0) = \int_0^2 u_0(x)dx$, therefore,

$$I(t) = \int_0^2 u(x,t)dx = I(0)e^t = \int_0^2 u_0(x)dx \cdot e^t. \tag{1.74}$$

The behavior of the actual solution, $u(x,t)$, may be controlled by this equation at any moment when $t > 0$. The initial state $u_0(x)$ determines the behavior of the solution in an integrated sense (in the L1-norm) and it decays exponentially with time.[5]

1.4.4 Stability of Numerical scheme for $u_t + xu_x = 0$

For the equation that we are using is $u_t + xu_x$, now we will discuss the stability for the more general case then our equation will take the form for the general case as follows,

$$u_t + a(x)u_x = 0 \tag{1.75}$$

with the initial and boundary conditions are as follows,

$$\begin{cases} u(0,t) = u(1,t) = 0 \\ u(x,t=0) = u_0(x) \end{cases} \tag{1.76}$$

Then the numerical scheme for the time frame $[0,T]$ will be as follows,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a_j}{\Delta x}(u_j^n - u_{j-1}^n) = 0, j = 0,1,2,\ldots,M \tag{1.77}$$

The sequence $t^0, t^1, t^n, ..., t^N$ provides the relevant time discretization in such a way that $N\Delta t = T$. Each t^n in this example denotes a discrete time step

in the series, and n stands for the time index. Each time step has a size of Δt and there are N total time steps. The last time step, t^N, is equal to the time interval's upper bound, T.
If we assume that $\alpha = \frac{\Delta t}{\Delta x}$, then

$$u_j^{n+1} = u_j^n - \alpha a_j (u_j^n - u_{j-1}^n) \tag{1.78}$$

Taking the absolute value on both sides,

$$|u_j^{n+1}| = |u_j^n - \alpha a_j (u_j^n - u_{j-1}^n)| \tag{1.79}$$

The aforementioned equation may be expressed as follows,

$$|u_j^{n+1}| = |(1 - \alpha a_j) u_j^n + \alpha a_j u_{j-1}^n)| \tag{1.80}$$

we will make use of the triangular inequality, $\alpha a_j \geq 0$ as well as $(1 - \alpha a_j) \geq 0$ so,

$$|u_j^{n+1}| \leq (1 - \alpha a_j)|u_j^n| + \alpha a_j |u_{j-1}^n)| \tag{1.81}$$

Making use of the summation sign we have,

$$\sum_{j=1}^{M} |u_j^{n+1}| \leq \sum_{j=1}^{M} (1 - \alpha a_j)|u_j^n| + \alpha \sum_{j=1}^{M} a_j |u_{j-1}^n| \tag{1.82}$$

if $j ----- > j + 1$, then

$$\alpha \sum_{j=1}^{M} a_j |u_{j-1}^n| \leq \alpha \sum_{j=1}^{M} a_{j+1} |u_j^n| \tag{1.83}$$

so again writing the equation,

$$\sum_{j=1}^{M} |u_j^{n+1}| \leq \sum_{j=1}^{M} |u_j^n| + \alpha \sum_{j=1}^{M} (a_{j+1} - a_j)|u_j^n| \tag{1.84}$$

Now, as the $a(x)$ is decreasing so, $a_{j+1} \leq a_j$ so,

$$\sum_{j=1}^{M} (a_{j+1} - a_j)|u_j^n| \leq 0 \tag{1.85}$$

It is important to note that the above equation will hold only when $a(x)$ is decreasing or non-increasing, The other for which the above equation does

not hold is the case when the $a(x)$ is increasing.
Hence,

$$\sum_{j=1}^{M} |u_j^{n+1}| \leq \sum_{j=1}^{M} |u_j^n| \tag{1.86}$$

More generally if we consider that, $|a^{'}(x)| \leq K$, Then,

$$\alpha \sum_{j=1}^{M} (a_{j+1} - a_j)|u_j^n| \leq \Delta t K \sum_{j=1}^{M} |u_j^n| \tag{1.87}$$

Now the equation (1.84) will take the form as follows,

$$\sum_{j=1}^{M} |u_j^{n+1}| \leq \sum_{j=1}^{M} |u_j^n| + \Delta t K \sum_{j=1}^{M} |u_j^n| \tag{1.88}$$

Also we can write it as follows,

$$\sum_{j=1}^{M} |u_j^{n+1}| \leq (1 + \Delta t K) \sum_{j=1}^{M} |u_j^n| \tag{1.89}$$

utilizing the inequality given below,

$$\sum_{j=1}^{M} |u_j^1| \leq (1 + \Delta t K) \sum_{j=1}^{M} |u_j^0| \tag{1.90}$$

$$\sum_{j=1}^{M} |u_j^2| \leq (1 + \Delta t K) \sum_{j=1}^{M} |u_j^1| \tag{1.91}$$

$$\sum_{j=1}^{M} |u_j^3| \leq (1 + \Delta t K)^2 \sum_{j=1}^{M} |u_j^0| \cdots \tag{1.92}$$

$$\sum_{j=1}^{M} |u_j^{n+1}| \leq (1 + \Delta t K)^{n+1} \sum_{j=1}^{M} |u_j^0| \tag{1.93}$$

So, equation(1.87) will adopt the form,

$$\sum_{j=1}^{M} |u_j^{n+1}| \leq (1 + \Delta t K)^{n+1} \sum_{j=1}^{M} |u_j^0| \tag{1.94}$$

We know that $(1 + x)^n \leq e^{nx}$ $x \geq 0$ and $n \geq 0$. The application of this inequality for $(1 + \Delta t K)^{n+1}$, will be as follows,

$$(1 + \Delta t K)^{n+1} \leq e^{(n+1)\Delta t K} \tag{1.95}$$

Hence,

$$\sum_{j=1}^{M} |u_j^{n+1}| \leq e^{(n+1)\Delta t K} \sum_{j=1}^{M} |u_j^0| \tag{1.96}$$

Where as $(n+1)\Delta t \leq N\Delta t = T$.Then the above equation (1.96) will conclude as follows,

$$\sum_{j=1}^{M} |u_j^{n+1}| \leq e^{TK} \sum_{j=1}^{M} |u_j^0| \tag{1.97}$$

The equation (1.97), is the discrete approximation of the continuous case of the function $u(x,t)$ that we have discussed in equation (1.74).[5],[4]

1.5 Example 3

Here we are going to solve another type of conservation law with the method of characteristics.The new feature is the inclusion of a source term on the right-hand side that does not include derivatives.

$$u_t + u_x = x, x \in [0;2] \tag{1.98}$$

with the initial condition

$$\phi(x; t = 0) = \phi(x_0) \tag{1.99}$$

To solve the equation $u_t + u_x = x$ using the method of characteristics,

$$\frac{dx(t)}{dt} = 1, x(t = 0) = x_0 \tag{1.100}$$

now we will do the integration as follows,

$$\int_{x_0}^{x(t)} dx = a \int_0^t dt \tag{1.101}$$

we will get,

$$x(t) - x_0 = t \tag{1.102}$$

which implies,

$$x(t) = x_0 + t \tag{1.103}$$

Now we will check that how $u(x,t)$ vary along this $x(t)$. Remember that we are going to use the equation (1.100) in the below equation i.e. $\frac{dx(t)}{dt} = 1$

$$\frac{du(x(t),t)}{dt} = u_x \frac{dx}{dt} + u_t \frac{dt}{dt} = u_x 1 + u_t = x(t) \tag{1.104}$$

Equation(1.103) implies that, as we know that $x(t) = x_0 + t$ so,

$$\frac{du(x(t),t)}{dt} = x_0 + t \tag{1.105}$$

Now, we will apply integration on equation(1.105)

$$\int_{u(x(t=0),t=0)}^{u} du = \int_0^t (x_0 + t)dt \tag{1.106}$$

As we know, $u(x(t = 0), t = 0) = \phi(x_0)$ so, equation (1.106) can be written as,

$$\int_{\phi(x_0)}^{u} du = \int_0^t (x_0 + t)dt \tag{1.107}$$

Now by integration and applying the lower and upper limits of integration we can have the equation(1.107) as follows,

$$u(x(t),t) - \phi(x_0) = x_0 t + \frac{1}{2}t^2 \tag{1.108}$$

As $x(t) = x_0 + t$ so, equation (1.108) will be as,

$$u(x,t) = \phi(x - t) + xt + \frac{1}{2}t^2 \tag{1.109}$$

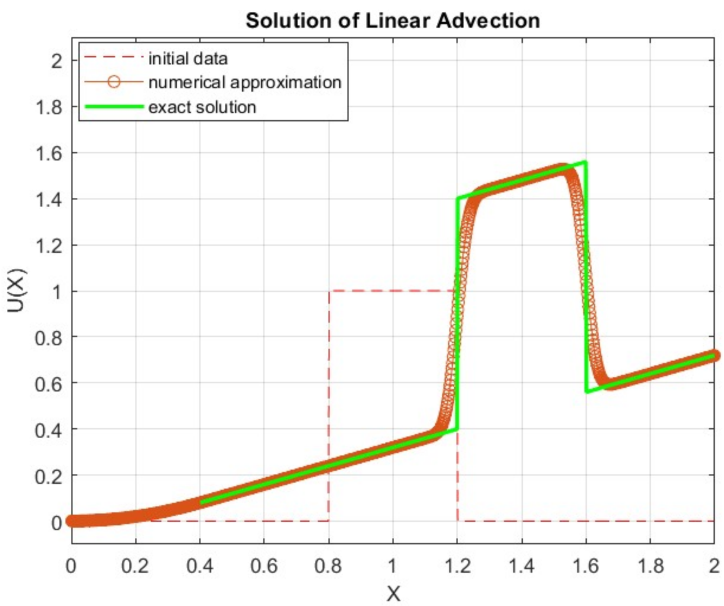

Figure 1.7: Graphical interpretation for the Example 3

1.5.1 Numerical scheme for the $u_t + u_x = x$

The numerical scheme for the equation,

$$u_t + u_x = x \tag{1.110}$$

with the initial data

$$u(x, t = 0) = \phi(x_0) \tag{1.111}$$

is as follows

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n}{\Delta x} = x, j = 1, 2....., M \tag{1.112}$$

Now, we have to define the upwind principal as follows,

$$u_{j+\frac{1}{2}}^n = u_j^n, u_{j-\frac{1}{2}}^n = u_{j-1}^n \tag{1.113}$$

we have $u_1^{n+1} = u_0(x_1)$ and $u_M^{n+1} = u_0(x_M)$ for the first and last cell respectively. we choose

$$\frac{\Delta t}{\Delta x} \leq 1 \tag{1.114}$$

for the time step $= \Delta t = t^{n+1} - t^n$.[5]

1.5.2 Comparison between Figure 3, Figure 5, and Figure 7

Because of the non-constant advection speed and greater spatial resolution, the behavior of the solution in Figure 1.7 is more complicated and possibly more correct. Figures 1.3 and 1.5, on the other hand, exhibit simpler behaviors due to constant advection speed, with Figure 3 being more precise and steady due to the greater temporal precision.

1.5.3 Stability properties for $u_t - u_x = 0$

$$u_t - u_x = 0, x \in (0, 1) \tag{1.115}$$

The boundary conditions are as follows

$$u(0, t) = u(1, t) = 0, u(x, t = 0) = u_0(x) \tag{1.116}$$

We are going to check the attitude of the true solution by performing the integration in the following way,

$$\int_0^1 u_t dx - \int_0^1 u_x dx = 0. \tag{1.117}$$

Utilizing (1.116) we get,

$$\frac{d}{dt}\int_0^1 u dx = \int_0^1 u_x dx = u(1,t) - u(0,t) = 0. \tag{1.118}$$

By doing integration from 0 to t

$$\int_0^1 u(.,t)dx = \int_0^1 u(.,0)dx = \int_0^1 u_0(.)dx. \tag{1.119}$$

Generally,

$$\int_0^1 |u(.,t)|dx \leq \int_0^1 |u_0(.)|dx. \tag{1.120}$$

1.5.4 Stability analysis for the numerical scheme of $u_t - u_x = 0$

The numerical scheme for the given equation is,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{\Delta x}(u_{j+1/2}^n - u_{j-1/2}^n), u_0 = u_{M+1} = 0. \tag{1.121}$$

Using the upwind scheme, we assign $u_{j+1/2}^n = u_{j+1}^n$ and $u_{j-1/2}^n = u_j^n$ because the flow is moving from left to right. Thus, the numerical scheme can be written as,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{1}{\Delta x}(u_{j+1}^n - u_j^n) \tag{1.122}$$

Let $\alpha = \frac{\Delta t}{\Delta x}$, then the above equation will take the form,

$$u_j^{n+1} = u_j^n(1-\alpha) + \alpha u_{j+1}^n \tag{1.123}$$

Usage of the absolute value,

$$|u_j^{n+1}| = |u_j^n(1-\alpha) + \alpha u_{j+1}^n| \tag{1.124}$$

Property of triangular inequality can be applied as follows,

$$|u_j^{n+1}| \leq |u_j^n(1-\alpha)| + |\alpha u_{j+1}^n| \tag{1.125}$$

Sum over all j from 1 to M, As $\alpha \geq 0$ so $|(1-\alpha)| = 1-\alpha$ and $|\alpha| = \alpha$

$$\sum_{j=1}^{M} |u_j^{n+1}| \leq (1-\alpha)\sum_{j=1}^{M} |u_j^n| + \alpha \sum_{j=1}^{M} |u_{j+1}^n| \tag{1.126}$$

we can write as,

$$\sum_{j=1}^{M}|u_j^{n+1}| \leq (1-\alpha)\sum_{j=1}^{M}|u_j^n| + \alpha\left(\sum_{j=1}^{M-1}|u_{j+1}^n| + |u_{M+1}^n|\right) \tag{1.127}$$

Since $u_{M+1}^n = 0$, then the form of the inequality will be as follows,

$$\sum_{j=1}^{M}|u_j^{n+1}| \leq (1-\alpha)\sum_{j=1}^{M}|u_j^n| + \alpha\sum_{j=1}^{M-1}|u_{j+1}^n| \tag{1.128}$$

By Supposition $j+1 \rightarrow j$,

$$\sum_{j=1}^{M}|u_j^{n+1}| \leq (1-\alpha)\sum_{j=1}^{M}|u_j^n| + \alpha\sum_{j=2}^{M}|u_j^n| \tag{1.129}$$

Spliting the seconds sum of the right hand side,

$$\sum_{j=1}^{M}|u_j^{n+1}| \leq (1-\alpha)\sum_{j=1}^{M}|u_j^n| + \alpha\left(\sum_{j=2}^{M}|u_j^n| + |u_1^n| - |u_1^n|\right) \tag{1.130}$$

Now,

$$\sum_{j=1}^{M}|u_j^{n+1}| \leq \sum_{j=1}^{M}|u_j^n| - \alpha|u_1^n| + \alpha|u_1^n| \tag{1.131}$$

Finally,

$$\sum_{j=1}^{M}|u_j^{n+1}| \leq \sum_{j=1}^{M}|u_j^n| \tag{1.132}$$

The stability of the numerical solution depends upon the above inequality (1.132).[5]

Chapter 2

Solution to the nonlinear Conservation law

The journey towards unraveling the complexities of nonlinear conservation laws begins in this chapter. Conservation laws are fundamental principles governing the behavior of various physical systems, and their nonlinear nature often poses significant challenges in obtaining accurate and efficient solutions. In this section we are going to solve nonlinear conservation, especially the burger's equation. Burgers' equation is a basic partial differential equation that appears in many academic disciplines, such as fluid dynamics, traffic flow, and shock wave production. It combines the diffusion and advection processes and offers a straightforward yet flexible model for comprehending the behavior of increasingly complicated systems. Due to the equation's nonlinearity and connections to crucial mathematical ideas like conservation laws, weak solutions, and the Riemann problem, it has received a great deal of attention. The goal of this research is to analyze various approaches for resolving Burgers' equation, look into the characteristics of its solutions, and clarify the fundamental mathematical ideas that control their behavior. The chapter will cover a variety of subjects in order to do this, such as,

1. The solution of Burgers' equation using the method of characteristics and the finite difference method.

2. The ideas of conservation laws and weak solutions, with a focus on the distinction between convex and concave flux.

3. The Riemann problem for scalar conservation rules.

4. The significance of the Rankine-Hugoniot criterion in identifying weak solutions.

5. As a standard for choosing weak solutions that are physically applicable, the entropy condition.

This chapter tries to expand our grasp of the mathematical structures that underpin Burgers' equation and the various solutions by giving a thorough discussion of these issues. Not only will the knowledge gathered from this study add to the corpus of current knowledge in this field, but it will also provide the groundwork for future studies on more intricate systems and issues.

2.1 General solution of $u_t + f(u)_x = 0$

We first need to comprehend the characteristic curves in order to use the method of characteristics to determine the general solution of the partial differential equation (PDE) $u_t + f(u)_x = 0$. With the use of these curves, we may convert the PDE into a set of simpler ordinary differential equations (ODEs). Since the PDE in this instance represents a conservation law, a common method for resolving such equations is the method of characteristics. We have the equation in the general form as,

$$u_t + f(u)_x = 0 \tag{2.1}$$

with the initial condition,

$$u(x, t = 0) = u_0(x) \tag{2.2}$$

We know that the equation (2.1) can also be written as folows,

$$u_t + f^{'}(u)u_x = 0 \tag{2.3}$$

Then by applying the method of characteristic are,

$$\frac{dx}{dt} = f^{'}(u) \tag{2.4}$$

By applying the integration on the above equation (2.4) we get,

$$\int_{x_0}^{x(t)} dx = \int_0^t f^{'}(u)dt \tag{2.5}$$

After solving the above integral we get,

$$x - x_0 = f^{'}(u_0)t \tag{2.6}$$

Which can be written as follows,

$$x = x_0 + f^{'}(u_0)t \tag{2.7}$$

Now we will check that how u(x,t) vary along this x(t). Remember that we are going to use the equation (2.4) in the below equation i.e. $\frac{dx(t)}{dt} = f^{'}(u)$

$$\frac{du(x(t),t)}{dt} = u_x\frac{dx}{dt} + u_t\frac{dt}{dt} = u_x f^{'}(u) + u_t = 0 \tag{2.8}$$

So,

$$u(x(t),t) = constant = u(x(t=0), t=0) \tag{2.9}$$

as we know that $x(t) = x_0 + f^{'}(u_0)t$ so, $x_0 = x(t) - f^{'}(u_0)t$ we have,

$$u(x(t),t) = u_0(x(t) - f^{'}(u)t) \tag{2.10}$$

Finally,

$$u(x,t) = u_0(x - f^{'}(u)t) \tag{2.11}$$

2.1.1 Condition for $u_t + f(u)_x = 0$

We are going to calculate the condition for which equation (2.11) is the solution of $u_t + f(u)_x = 0$.
As we know that $x = x_0 + f^{'}(u_0(x_0,t))t$, therefore,

$$u(x,t) = u_0(x_0) = u_0(x - f'(u(x,t))t) \tag{2.12}$$

We are going to introduce a variable as follows,

$$w = x - f'(u(x,t))t \tag{2.13}$$

Now equation (2.12), will adopt the form as follows,

$$u(x,t) = u_0(w) \tag{2.14}$$

By chain rule for above equation,

$$u_t = \frac{\partial u_0}{\partial w} \cdot \frac{\partial w}{\partial t} \tag{2.15}$$

Let's do the calculations to find out the $\frac{\partial w}{\partial t}$

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}(x - f'(u(x,t))t) \tag{2.16}$$

Then,
$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial t}(f'(u(x,t))t) \tag{2.17}$$
Now,
$$\frac{\partial}{\partial t}(f'(u(x,t))t) = f''(u(x,t)) \cdot \frac{\partial}{\partial t}u(x,t) \cdot t + f'(u(x,t)) \tag{2.18}$$
Hence equation (2.17) will adpot the form,
$$\frac{\partial w}{\partial t} = -f''(u(x,t)) \cdot \frac{\partial}{\partial t}u(x,t) \cdot t - f'(u(x,t)) \tag{2.19}$$
Now by utilizing the equations (2.15) and (2.19) we get, u_t,
$$u_t = u_0'(w) \cdot (-f''(u(x,t)) \cdot \frac{\partial}{\partial t}u(x,t) \cdot t - f'(u(x,t))) \tag{2.20}$$
In order to do the calculations for u_x, we will rewrite the equation
$$u(x,t) = u_0(x - f'(u(x,t))t) \tag{2.21}$$
and defining w as,
$$w = x - f'(u(x,t))t \tag{2.22}$$
In order to do the calculations for $u_x = \frac{\partial}{\partial x}u_0(x - f^{'}(u(x,t)t)$, we can rewrite,
$$u(x,t) = u_0(w) \tag{2.23}$$
By the use of chain rule, we can do the computation for u_x,
$$u_x = \frac{\partial u_0}{\partial v} \cdot \frac{\partial w}{\partial x} \tag{2.24}$$
Firstly, find out the partial derivative w w.r.to x
$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(x - f'(u(x,t))t) \tag{2.25}$$
Again by chain rule,
$$\frac{\partial w}{\partial x} = 1 - \frac{\partial}{\partial x}(f'(u(x,t))t) \tag{2.26}$$
By further simplifications,
$$\frac{\partial}{\partial x}(f'(u(x,t))t) = f''(u(x,t)) \cdot \frac{\partial}{\partial x}u(x,t) \cdot t \tag{2.27}$$

Hence equation (2.26) will adopt the form,

$$\frac{\partial w}{\partial x} = 1 - f''(u(x,t)) \cdot \frac{\partial}{\partial x}u(x,t) \cdot t \tag{2.28}$$

Utilizing the equation (2.23) and (2.27) we have,

$$u_x = u_0'(v) \cdot (1 - f''(u(x,t)) \cdot \frac{\partial}{\partial x}u(x,t) \cdot t) \tag{2.29}$$

Now, as weh have u_x as,

$$u_x = u_0'(w) \cdot (1 - f''(u(x,t)) \cdot \frac{\partial}{\partial x}u(x,t) \cdot t) \tag{2.30}$$

Now, we need to multiply u_x with $f^{'}(u)$:

$$f'(u)u_x = f'(u(x,t)) \cdot u_0'(w) \cdot (1 - f''(u(x,t)) \cdot \frac{\partial}{\partial x}u(x,t) \cdot t) \tag{2.31}$$

This is the expression for $f^{'}(u)u_x$.
As we know that $f^{'}(u(x,t)\frac{\partial}{\partial x}u(x,t) = f(u)_x$. So,

$$f'(u(x,t))u_x = u_0'(w) \cdot (f'(u(x,t)) - f(u)_x \cdot t) \tag{2.32}$$

Now,

$$u_t + f(u)_x = -\, u_0'(w) \cdot \left(f''(u(x,t))\frac{\partial}{\partial t}u(x,t) \cdot t\right) \tag{2.33}$$

$$+\, u_0'(w) \cdot (f'(u(x,t)) - f''(u(x,t))f(u)_x \cdot t) \tag{2.34}$$

$$u_t + f(u)_x = -u_0'(w) \cdot (f''(u(x,t)) \cdot \frac{\partial}{\partial t}u(x,t) \cdot t + f''(u(x,t))f(u)_x \cdot t) \tag{2.35}$$

$$u_t + f(u)_x = -u_0'(w)f''(u(x,t))t[u_t + f(u)_x] \tag{2.36}$$

Finally,

$$(u_t + f(u)_x).[1 + u_0^{'}(w)f^{''}(u)t] = 0 \tag{2.37}$$

From equation (2.37) we can conclude that $u_t+f(u)_x = 0$ if $1+u_0^{'}(w)f^{''}(u)t \neq 0$ where as $w = x - f'(u(x,t))t$.

2.2 Lax-Friedrichs scheme

The Lax-Friedrichs (LF) technique, named after the mathematicians Peter D. Lax and Kurt Otto Friedrichs, is a well-known method for numerically solving hyperbolic partial differential equations (PDEs). This approach combines the notion of finite difference approximation with a novel averaging mechanism, assuring the calculated solution's stability.
To understand the LF method intuitively, let's consider a standard hyperbolic PDE in conservation form,

$$u_t + f(u)_x = 0 \tag{2.38}$$

The LF system consists of two major phases,

- The first stage The Average step computes an average of the nearby grid point values, resulting in a smoothed variation of the solution for each grid point. This may be stated numerically as,

$$u_j^{n+1/2} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) \tag{2.39}$$

- In the second phase, a forward Euler step is used to update the solution based on the relaxed values derived in the first step. These newly calculated relaxed values are used to analyze flux differences. These flux differences are used to calculate the updated values of u at each grid point. This is mathematically expressed as and this step is known as transport step,

$$u_j^{n+1} = u_j^{n+1/2} - \frac{\Delta t}{2\Delta x}(f(u_{j+1}^n) - f(u_{j-1}^n)) \tag{2.40}$$

Collectively, these two steps mentioned in equations (2.39) and (2.40) comprise the Lax-Friedrichs scheme,

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\Delta t}{2\Delta x}(f(u_{j+1}^n) - f(u_{j-1}^n)) \tag{2.41}$$

Despite its simplicity and easy implementation, the LF scheme does have its downsides, such as introducing substantial numerical diffusion. This could potentially over-smoothen the solution, causing the loss of critical details.
Lets talk about some general form of LF scheme by utilizing hyperbolic PDE in one spatial dimension,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \tag{2.42}$$

The Lax-Friedrichs scheme may be written as follows, where $f(u)$ is the flow function,

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\Delta t}{2\Delta x}(f(u_{j+1}^n) - f(u_{j-1}^n)) \tag{2.43}$$

With a correction factor for the time and space steps, this method updates the solution at each grid point based on the average of the solutions at surrounding grid points and the differences of the fluxes at neighboring places. Please be aware that the Lax-Friedrichs technique significantly increases numerical diffusion, which might smooth out the solutions. Other, more advanced numerical techniques could be better suitable for applications where this is a concern. Additionally, for this scheme to be stable, the Courant-Friedrichs-Lewy (CFL) condition must be met. Where a is the maximum wave speed, the formula is $|a|\frac{\Delta t}{\Delta x} \leq 1$.[7],[9],[10]

2.3 Solution to some examples

In this part, we will look at how effective mathematical technique known as the method of characteristics and how it may be used to solve the inviscid Burgers' equation, a key non-linear PDE.
The Burgers' equation, which illustrates various important phenomena including shock waves and solitons, is a fundamental equation in fluid dynamics and nonlinear acoustics. This equation's inviscid form, which ignores the effects of viscosity, is an especially intriguing problem because of its nonlinear term. Many common solution methods cannot be used in a straightforward manner because of this nonlinear term.
Our solution uses a modified variation of the method of characteristics, a potent mathematical trick that converts the initial partial differential equation into a set of ordinary differential equations along characteristic curves. We use a particular characteristic form mentioned in equation (2.11), as opposed to the method of characteristics in its conventional form.
This section will walk you through a thorough application of this modified technique of characteristics to the inviscid Burgers' problem. We give a step-by-step explanation of the procedure, emphasizing the computing specifics and the mathematical justification for each action. As part of this investigation, we will also talk about the circumstances in which solutions may be found and the difficulties in expressing the answer in the original variables.
We want you to have a thorough comprehension of this improved technique of characteristics by the end of this section, as well as its effectiveness in tackling challenging nonlinear partial differential equations like the inviscid

Burger's equation.

2.3.1 Example 2.1

In this section, Using the characteristics technique, a solution was first found analytically. This entailed solving an associated ordinary differential equation system and then utilizing the Initial conditions to arrive at a specific solution. This solution was discovered to be dependent on the initial conditions, providing insight into the solution's development through time.
Following the analytical solution, the Lax-Friedrichs method was used to provide a numerical solution to the Burgers' problem. This finite difference technique estimates the convective component at half-integer time levels using a simple average, allowing this PDE to be solved numerically.
To highlight the influence of the discretization size on the solution's correctness, comparisons were done between solutions with different grid sizes. It was demonstrated that a finer grid produces a more accurate numerical solution, but at the expense of more processing effort.
Characteristic lines were also plotted to show how information spreads along these lines. The characteristic lines aided in comprehending the impact of the starting conditions on the solution's future state.
In summary, the Burgers' equation has been intensively explored, both analytically and numerically, yielding insights into the behaviors and features of this fundamental fluid mechanics equation. First of all, we have the equation as follows,

$$u_t + \frac{1}{2}(u^2)_x = 0 \tag{2.44}$$

with the initial condition,

$$u_o(x) = \begin{cases} 2x, x \in [0, \frac{1}{2}] \\ 2(1-x), x \in (\frac{1}{2}, 1] \end{cases} \tag{2.45}$$

here we have $f(u) = \frac{1}{2}u^2$ then $f^{'}(u) = u$ The characterstics for the above equation will be as follows,

$$x(t) = x_o + f^{'}(u_o(x_o))t = x_o + u_o(x_o)t \tag{2.46}$$

The solution will be as follows,

$$\begin{cases} u(x,t) = u_o(x_o) = u_o(x - u_o(x_o)t) \\ x = x_o + u_o(x_0)t \end{cases} \tag{2.47}$$

Now, the characterstics will become,

$$x = x_o + t \begin{cases} 2x_0, x_0 \in [0, \frac{1}{2}] \\ 2(1 - x_0), x_0 \in (\frac{1}{2}, 1] \end{cases} \tag{2.48}$$

then,

$$x = \begin{cases} x_o(1 + 2t), x_o \in [0, \frac{1}{2}] \\ x_o(1 - 2t) + 2t, x_o \in (\frac{1}{2}, 1] \end{cases} \tag{2.49}$$

to find the expression for the x_o we have from above equation ,

$$x_0 = \begin{cases} \frac{x}{1+2t}, 0 \le x_0 = \frac{x}{1+2t} \le 1/2 \\ \frac{x-2t}{1-2t}, 1/2 < x_0 = \frac{x-2t}{1-2t} \le 1 \end{cases} \tag{2.50}$$

From equation (2.47) and (2.50) we have,

$$u(x,t) = u_o(x_o) = \begin{cases} u_o(\frac{x}{1+2t}), 0 \le x_0 = \frac{x}{1+2t} \le 1/2 \\ u_o(\frac{x-2t}{1-2t}), 1/2 < x_0 = \frac{x-2t}{1-2t} \le 1 \end{cases} \tag{2.51}$$

by utilizing the definition of $u_o(x)$ we have the final solution as follows,

$$u(x,t) = \begin{cases} \frac{2x}{1+2t}, 0 \le \frac{x}{1+2t} \le 1/2 \\ \frac{2(1-x)}{1-2t}, 1/2 < \frac{x-2t}{1-2t} \le 1 \end{cases} \tag{2.52}$$

The final solution with the calculations with the limits of the solution will take the form,

$$u(x,t) = \begin{cases} \frac{2x}{1+2t}, 0 \le x \le 1/2 + t \\ \frac{2(1-x)}{1-2t}, 1/2 + t < x \le 1 \end{cases} \tag{2.53}$$

The graphical representation of the Burger's equation for three different time, $T_1 = 0.125$,$T_2 = 0.25$,$T_3 = 0.375$ will be as follows,

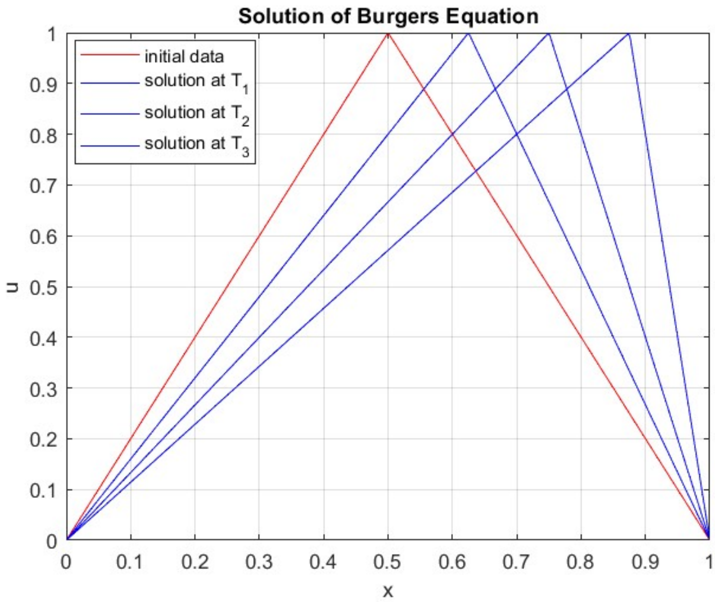

Figure 2.1: Solution of Example 2.1 at three different times

The resultant figure, shown in Figure 2.1, depicts how Burgers' equation's solution has changed over time. A shock wave is created as the shock steepens and the features converge with time. The characteristics of the burger's equation are shown as follows in Figure 2.2,

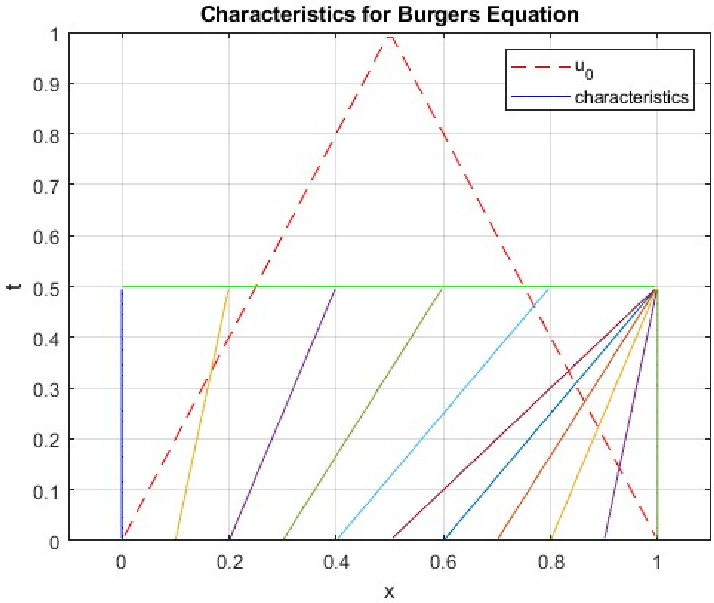

Figure 2.2: Characterstics for the Example 2.1

The pathways along which the solution is constant are referred to as characteristic lines or curves when it comes to the solution of partial differential equations (PDEs). In the system that the equation depicts, they resemble information highways.

When characteristics begin to diverge or spread apart, it's often a sign that the solution is behaving smoothly and reliably in those areas. The beginning or boundary circumstances of the ensuing solution inside these regions frequently have a substantial impact on the solution.

On the other hand, characteristics can ultimately cross when they converge or approach one another. This circumstance may bring about complications. Generally speaking, an intersection of characteristics denotes the occurrence of a discontinuity or shock in the solution. Hyperbolic PDEs, such as those used in fluid dynamics, aerodynamics, and electrodynamics, frequently exhibit this phenomena. The PDE's solution may become ill-defined at certain intersections, needing the use of specialized techniques like the method of weak solutions in order to understand it.

Based upon the above information if we Look at the graph in Figure 9, we can observe how the solution evolves over time for the given initial state. The solution is behaving smoothly until $t < 1/2$. As the t approaches to $1/2$ the characteristics begin to intersect at a specific point. The coming together of the characteristics symbolizes the formation of shock waves in the solution.

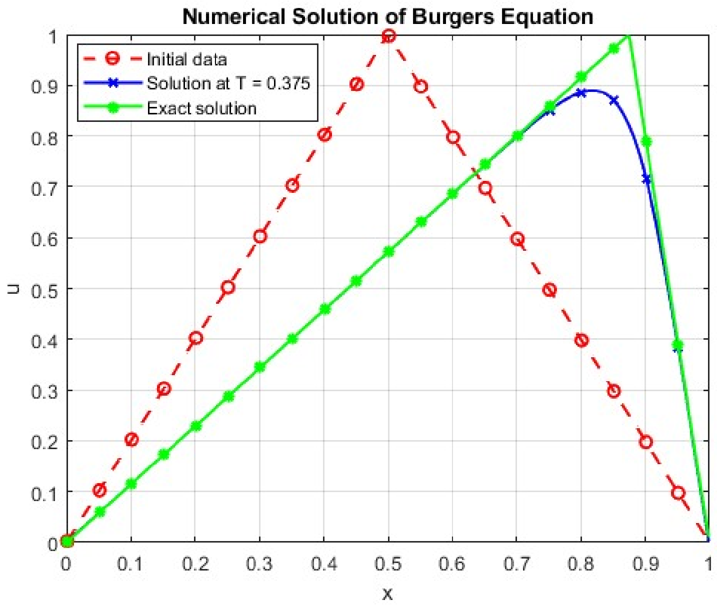

Figure 2.3: Graph of Example 2.1 by using its initial data (2.45)(Red line), exact solution(2.53)(Green line) and the Numerical solution(Blue line)

The Matlab code utilized for the Figure 2.3 employs a numerical grid with 400 grid points that spans from 0 to 1 in the x-direction, resulting in a grid spacing dx of 0.0025. For a finite volume technique, grid points are displaced by half the grid spacing $dx/2$ to guarantee they are positioned in the center of each cell. Finally, at $T_{end} = 0.375$, the code depicts the initial state and the solution. For clarity, just the twentieth point is highlighted on the plot. The exact solution has also been represented in order to check the comparison between numerical and exact solution. The graph illustrates how the nonlinear advection factor in the Burgers' equation causes the initial data set to change over time. The profile steepens in the solution, which is comparable to the evolution of a shock wave in more intricate fluid dynamic systems.

In Fig 2.3, we have implemented the Lax-Friedrichs scheme for solving the non-linear burger equation.

LeVeque (2002) claims that the graphical depiction shows how the initial data changes over time as a result of the nonlinear advection factor in the Burgers equation and the emergence of a shock wave in more intricate fluid dynamic systems. [5]

2.3.2 Checking the point at which the Chracterstics meet

we have the solution in equation (2.53), rewriting it we get,

$$u(x,t) = \begin{cases} \frac{2x}{1+2t}, 0 \leq x \leq 1/2 + t \\ \frac{2(1-x)}{1-2t}, 1/2 + t < x \leq 1 \end{cases} \tag{2.54}$$

The slope of the characteristic lines may be used to determine the moment when the characteristics first meet. We know that the solution's derivative with respect to x provides the slope. we are going to discuss the slopes of two different cases $0 \leq x \leq 1/2 + t$ and $1/2 + t < x \leq 1$ respectively.

$$u_x = \frac{\partial}{\partial x}\left(\frac{2x}{1+2t}\right) = \frac{2}{1+2t} \tag{2.55}$$

$$u_x = \frac{\partial}{\partial x}\left(\frac{2(1-x)}{1-2t}\right) = -\frac{2}{1-2t} \tag{2.56}$$

By combining these two states we get,

$$u_x = \begin{cases} \frac{2}{1+2t}, 0 \leq x \leq 1/2 + t \\ \frac{-2}{1-2t}, 1/2 + t < x \leq 1 \end{cases} \tag{2.57}$$

Which is the slope of the $u(x,t)$, it will provide us with information of a nonlinear nature, in the example we have provided, there will be a blow up as t approaches $\frac{1}{2}$, which will eventually lead to crossing the characteristics. The solution becomes non-unique when characteristics overlap, and the MOC(Method of characteristics) might not offer a clear solution in the overlapped area. This results from the PDE's nonlinearity and frequently causes shocks or discontinuities in the solution.

2.4 Condition for a Shock Formation

Lets generate a general expression for u_x for equation (2.12), we can do this by recalling the equation (2.30) that can be written as,

$$u_x = u_0'(w) \cdot (1 - f''(u(x,t)) \cdot \frac{\partial}{\partial x}u(x,t) \cdot t) \tag{2.58}$$

Where as,

$$w = x - f'(u(x,t))t \tag{2.59}$$

Now it can be written as follows,

$$u_x = u_0'(w) \cdot (1 - f''(u(x,t)) \cdot u_x(x,t) \cdot t) \tag{2.60}$$

By doing some calculations we get,

$$u_x = \frac{u_0^{'}(w)}{1 + u_0^{'}(w) f''(u)t} \tag{2.61}$$

From equation (2.61) we can deduce that $1 + u_0^{'}(w)f^{''}(u)t \neq 0$, because if $1 + u_0^{'}(w)f^{''}(u)t = 0$, then we will have the following conditions,

- If we divide by zero, we get an indeterminate condition in mathematics, then the value for u_x will goes to infinity which is providing us the condition for which the Jump or Shock wave is formed.
- It is also providing us the information about $u_0^{'}(w)$, $f^{''}(u)$ and t that all of them are decreasing if $1 + u_0^{'}(w)f^{''}(u)t = 0$, which is pretty unusual for the case of t which is time, therefore we will take any finite time wheever discussing the waves.

This discussion also leads us towards the direction to identify that why $1 + u_0^{'}(w)f^{''}(u)t \neq 0$ in equation (2.37). The conclusion of our discussion is that whenever we find the solution of any equation there will be a jump formation for the finite times.
Let us relate our discussion with the equation (2.57), in which we are observing the shock formation at $t = 1/2$, because at $t = 1/2$, u_x becomes undefined in equation (189).

2.5 Rankine-Hugoniot condition

In the context of conservation laws or hyperbolic partial differential equations (PDEs), a shock wave is often connected with the formation of a jump (or discontinuity) in a solution. The Rankine-Hugoniot criteria, which define the relationship between the states of the conserved quantity on each side of the discontinuity and the speed of the shock wave, may be used to compute the solution over the jump.
In a conservation law of the kind given in the below equation, the connection between a shock wave's initial and ultimate states is described by the Rankine-Hugoniot condition.

$$u_t + f(u)_x = 0 \tag{2.62}$$

where $f(u)$ is the flow function and $u(x,t)$ is the conserved quantity. The PDE is integrated across a moving control volume that occupies the shock wave to derive the Rankine-Hugoniot condition from the conservation law.
We need to define the rectangle R in the space-time domain before we can integrate the conservation law $u_t + f(u)_x = 0$ across it. The rectangle R with the vertices (x_1, t_1), (x_1, t_2), (x_2, t_1), and (x_2, t_2) should be considered. The conservation law is now integrated across the rectangle R,

$$\iint_R (u_t + f(u)_x) dxdt = 0 \tag{2.63}$$

The integral may be divided into two independent integrals as follows,

$$\iint_R u_t dxdt + \iint_R f(u)_x dxdt = 0 \tag{2.64}$$

The Leibniz rule will now be applied to each integral. Regarding the initial integral,

$$\frac{\partial}{\partial t} \iint_R u dx = \int (u(x,t_2) - u(x,t_1)) dx \tag{2.65}$$

For the second integral,

$$\frac{\partial}{\partial x} \iint_R f(u) dt = \int (f(u(x_2,t)) - f(u(x_1,t))) dt \tag{2.66}$$

Now, we have,

$$\int (u(x,t_2) - u(x,t_1)) dx + \int (f(u(x_2,t)) - f(u(x_1,t))) dt = 0 \tag{2.67}$$

In the space-time domain, this equation describes the conservation rule integrated across the rectangle R. The conserved quantity u is shown as having a net change inside the rectangle R and having a net flow across its edges on the left side of the equation.
Consider a shock wave moving through a medium at a speed s, with u_L and u_R standing for the beginning and end states of the conserved quantity to the shock wave's left and right, respectively.The shock wave is thought to go from (x_1, t_1) to (x_2, t_2). The shock wave goes over the rectangle in two time and space dimensions: $\Delta t = t_2 - t_1$ and $\Delta x = x_2 - x_1$, respectively.
We can approximate the integrals using the initial and final states,

$$\Delta t \int (u_R - u_L) dx + \Delta x \int (f(u_R) - f(u_L)) dt \approx 0 \tag{2.68}$$

Dividing both sides by $\Delta t \Delta x$, we have,

$$(u_R - u_L) + \frac{(f(u_R) - f(u_L))}{\Delta t/\Delta x} \approx 0 \tag{2.69}$$

In the limit as Δt and Δx go to zero, the ratio $\Delta t/\Delta x$ approaches the shock speed s,

$$(u_R - u_L) + \frac{(f(u_R) - f(u_L))}{s} = 0 \tag{2.70}$$

Now, rearrange the terms to get the Rankine-Hugoniot condition,

$$s(u_R - u_L) = f(u_R) - f(u_L) \tag{2.71}$$

The conservation rule across the rectangle R in the space-time domain was integrated to get the Rankine-Hugoniot condition, which is represented by this equation. It links the preserved quantity's initial and end states throughout the shock wave and the shock speed.[5]

2.6 Riemann problem

When studying hyperbolic partial differential equations (PDEs), notably in the context of fluid dynamics, electromagnetism, and other disciplines of physics, Riemann problems are a particular kind of boundary value problem. The mathematical form of the Riemann problem for a scalar conservation law is as,

$$u_t + f(u)_x = 0 \tag{2.72}$$

Where $f(u)$ is the flux function, subscripts t and x imply partial differentiation, and u is the unknown function of space x and time t.
The initial data is as follows,

$$u(x,0) = \begin{cases} u_l, & \text{if } x < 0 \\ u_r, & \text{if } x > 0 \end{cases} \tag{2.73}$$

where the constant states to the left and right of the origin, respectively, are denoted by u_l and u_r.

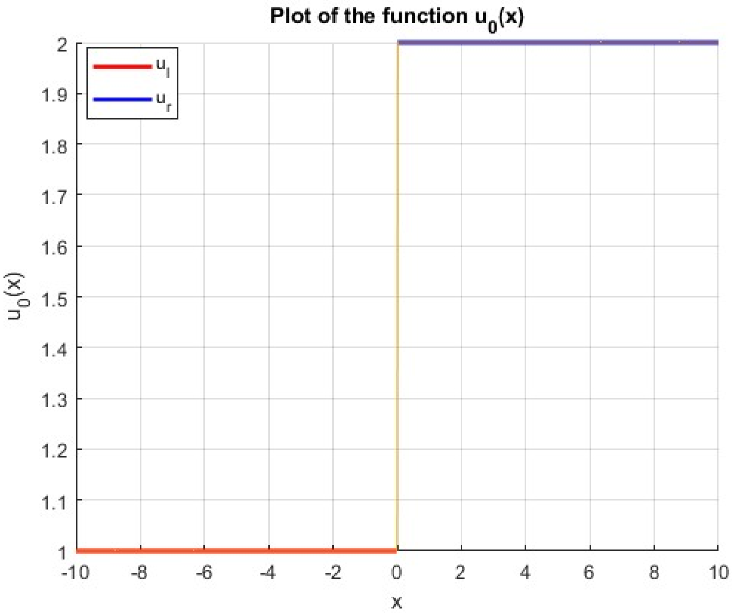

Figure 2.4: Graph of Riemann Problem (2.73)

The Figure 2.4 represents that how a Riemann Problem may look like in the graphical interpretation from. This problem is named for German mathematician Bernhard Riemann, one of the pioneers in studying it in the setting of gas dynamics.
The Riemann problem contains precise solutions for many significant equations, making it the simplest nontrivial issue for hyperbolic PDEs. It is especially helpful in the development and evaluation of numerical techniques for hyperbolic PDEs, where the accuracy of the method's Riemann problem solution is frequently checked.
Depending on the properties of the equation and the initial circumstances, the Riemann problem might have several kinds of solutions. Among them are shock waves, which are discontinuous solutions that move at a specific pace, and rarefaction waves, which are smooth solutions that disperse with time. A stationary discontinuity in the state variable known as a contact discontinuity may also be present in the solution. The Riemann issue can also incorporate waves from many families, each linked with a distinct characteristic speed, in more intricate systems of conservation rules. These waves can interact and result in more complicated structures, such as interactions between shock waves or shock waves and rarefaction.[11]
In general, we employ the method of characteristics and the integral form of the conservation law to build the Riemann problem solution. The Rankine-Hugoniot condition, which connects the change in the state variable through-

out the shock to the shock speed, and the entropy condition, which chooses the physically sound solution from among all feasible weak solutions, are also necessary for shock waves.[5]

2.7 Construction of similarity solution ($f^{'}(u_l) < f^{'}(u_r)$)

One can get the continuous solution to a Riemann problem by resolving a set of ordinary differential equations that connects u_l and u_r with a rarefaction wave. This is distinct from a shock solution, in which a discontinuity connects u_l and u_r.
The following is the Riemann problem for scalar conservation law.

$$u_t + f(u)_x = 0 \tag{2.74}$$

The initial condition is as,

$$u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases} \tag{2.75}$$

When $u_l < u_r$, a rarefaction wave occurs in the case of a scalar conservation law with convex flux function $f(u)$.
Let's suppose that $u(x,t) = v(x/t)$ then, using the chain rule for differentiation, we can insert the assumed form of the solution into the PDE (207) and convert the PDE into an ordinary differential equation (ODE) for v.
we are going to find the value u_t by using $u(x,t) = v(x/t)$

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} = -\frac{x}{t^2}v'(\xi) \tag{2.76}$$

where as $\xi = x/t$, and $v'(\xi)$ signifies the derivative of v with respect to ξ.
finding $f(u)_x$

$$f(u)_x = f'(v(\xi))\frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{t}f'(v(\xi))v'(\xi) \tag{2.77}$$

where $f'(v(\xi))$ is the derivative of f with respect to v, calculated at $v(\xi)$.
Substitute these derivatives into the PDE (2.74).

$$-\frac{x}{t^2}v'(\xi) + \frac{1}{t}f'(v(\xi))v'(\xi) = 0 \tag{2.78}$$

In simple form we can write as,

$$-\frac{x}{t}v'(\xi) + f'(v(\xi))v'(\xi) = 0 \tag{2.79}$$

The equation (2.79) can also be written as follows,

$$-\xi v'(\xi) + f'(v(\xi))v'(\xi) = 0 \tag{2.80}$$

Simplified version will be as follows,

$$v'(\xi)(-\xi + f'(v(\xi))) \tag{2.81}$$

Here we have the two case as follows,
Case 1: $v'(\xi) = 0$
If we consider $v'(\xi) = 0$ then as a result, the function $v(\xi)$ can be a constant function. This is equivalent to a stationary solution to the Riemann problem, meaning there is no jump in the initial condition and the initial states u_l and $u_r$are the same.
In this instance, the common initial state serves as the constant value in the Riemann problem solution, i.e $u(x,t) = v(\xi) = \text{const}$.
Case 2: $v'(\xi) \neq 0$:
In this case, we can divide the ODE by $v'(\xi)$ to get

$$-\xi + f'(v(\xi)) = 0 \tag{2.82}$$

which simplifies to

$$f'(v(\xi)) = \xi \tag{2.83}$$

This is a first order nonlinear ODE for $v(\xi)$ that, unless the flux function f(u)) has a simple form, often requires numerical techniques to solve.
The Riemann problem's left and right states are connected by the function $v(\xi)$ that is provided by the solution to this ODE, which defines the wave's propagation. The specific form of the flow function $f(u)$ and the Riemann problem's boundary conditions will determine the precise shape of the solution.The flow function$f(u)$ in many physical systems governed by conservation principles is constructed so that its derivative $f'(u)$ is an increasing function of u. This is referred to as the flow function's condition of convexity. The existence and uniqueness of entropy solutions, which are the physically sound solutions in the presence of shocks or other discontinuities, are guaranteed by this crucial characteristic in the theory of conservation laws.
The fact that $f(u)$ is a convex function is the mathematical explanation for why $f'(u)$ is increasing. If a function's second derivative in calculus is not negative, the function is said to be convex. To put it another way, a function $f(u)$ is convex if and only if $f''(u) \geq 0$ for every u. This indicates that the function's slope, which is $f'(u)$(an increasing function of u).
Let's now explore the ODE for $v(\xi)$ under the assumption that $f'(u)$ is increasing,

$$f'(v(\xi)) = \xi \tag{2.84}$$

We got our desired function $v(\xi)$ that will verify our ODE, f'(v(ξ)) $= \xi$ This provides us with the derivative of $v(\xi)$ at each pointξ, which is equivalent to providing us with the slope of v at each point.

This formula (2.85) depicts a continuous curve (a rarefaction wave) that joins the values of u_l and u_r in the Riemann problem solution. The shape of the flow function $f(u)$ and the precise values of u_l and u_r determine the precise form of this curve. The solution $u(x,t)$ of the Riemann problem can be written in the form of a piecewise function as

$$u(x,t) = \begin{cases} u_l & \text{if } x/t < f'(u_l) \\ v(x/t) & \text{if } f'(u_l) \leq x/t \leq f'(u_r) \\ u_r & \text{if } x/t > f'(u_r) \end{cases} \tag{2.85}$$

Here, v is a decreasing function that connects u_l and u_r. It satisfies the ODE

$$v'(\xi) = \frac{1}{f'(v(\xi))}, \quad \xi = x/t \tag{2.86}$$

The requirement that $u(x,t)$ must be a weak solution of the PDE determines this function v. As a result, the integral form of the conservation law is satisfied. In this method, the solution to the Riemann problem for a scalar conservation law with a convex flux function is produced. The structure of the rarefaction wave, which connects the left and right states, is governed by the shape of the flux function and the initial conditions. This method offers a thorough knowledge of the behavior of conservation law solutions, particularly when discontinuities are present.[5]

2.8 Construction of shock wave solution which is entropy consistent $(f^{'}(u_l) > f^{'}(u_r))$

The solution to a Riemann problem can also involve a shock wave, which is a discontinuity that propagates at a certain speed. In order to choose the physically correct solution from among all weak solutions that satisfy the conservation law, the construction of a shock wave solution for a scalar conservation law needs an additional principle called the entropy condition.
The shock wave is used since the point where the characteristics overlap would result in the solution having numerous values, which is not physically possible. As a result, a shock or discontinuity is produced in order to retain a unique solution value. This shock travels in such a way that this intersection of characteristics is effectively resolved, guaranteeing that the solution

stays single-valued. The shock wave is therefore a way for the system to deal with abrupt changes in the starting circumstances while maintaining physical consistency. Let's take a look at the scalar conservation law as a Riemann problem.

$$u_t + f(u)_x = 0 \tag{2.87}$$

With initial data,

$$u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases} \tag{2.88}$$

When $u_l > u_r$, a shock wave occurs in the case of a scalar conservation rule with a convex flux function. The shock wave is a discontinuous function linking u_l and u_r, which is the Riemann problem's solution in this instance. The shock wave solution has the form,

$$u(x,t) = \begin{cases} u_l, & x/t < s \\ u_r, & x/t > s \end{cases} \tag{2.89}$$

where s is the speed of the shock, which is to be determined.

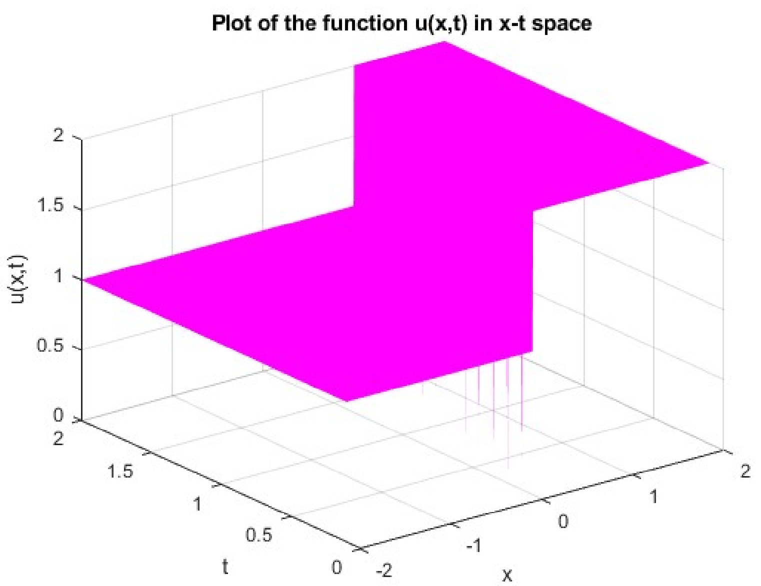

Figure 2.5: Representation of equation(2.89)

The Rankine-Hugoniot condition, which is derived from the conservation law by integrating it over a control volume that contains the shock, may be used to determine the shock speed s. The following is the Rankine-Hugoniot requirement for the scalar conservation law,

$$s(u_r - u_l) = f(u_r) - f(u_l) \tag{2.90}$$

However, because the conservation rule permits numerous weak solutions as a result of the generation of shocks, this requirement does not specifically specify the shock speed. To choose the physically sound solution, we must apply the entropy condition.
It is common to express the entropy requirement for a scalar conservation rule in terms of the flux function $f(u)$. The entropy criterion for a convex flux function stipulates that the shock speed s must be between the characteristic speeds on the shock's left and right sides, i.e.,

$$f'(u_l) > s > f'(u_r) \tag{2.91}$$

This requirement ensures that the solution is entropy consistent, that is, that it satisfies an extra inequality that stands in for the entropy principle, which is the second law of thermodynamics. The entropy condition guarantees the physical correctness of the solution to the Riemann problem with a shock.
In conclusion, there are two basic processes in the creation of a shock wave solution for a scalar conservation law,

- Using the Rankine-Hugoniot condition to determine the shock speed
- Determining if the solution satisfies the entropy requirement

2.9 Example

$$u_t + \frac{1}{2}u^2 = 0 \tag{2.92}$$

with the initial data,

$$u(x,t) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases} \tag{2.93}$$

We are going to solve the problem for the refraction wave case, Since we are aware that $f'(u) = u$ is the derivative of the flux function, the slopes of the characteristic lines are u.
In the case of a rarefaction wave, the general solution is expressed as follows,

$$u(x,t) = \begin{cases} u_l, & \text{if } x < f'(u_l)t = 0 \\ v(x/t), & \text{if } f'(u_l)t \leq x \leq f'(u_r)t \\ u_r, & \text{if } x > f'(u_r)t = t \end{cases} \tag{2.94}$$

A rarefaction wave exists in the intermediate area and may be characterized by the function $v(\xi)$, where $\xi = x/t$. By noticing that the solution u is constant along each characteristic line $x = ut$, we may determine $v(\xi)$. Therefore, in the center area, $v(\xi) = \xi$. This results in the Riemann problem for the Burgers equation with the specified initial conditions being solved,

$$u(x,t) = \begin{cases} 0, & \text{if } x < 0 \\ x/t, & \text{if } 0 \leq x \leq t \\ 1, & \text{if } x > t \end{cases} \tag{2.95}$$

In the area $0 \leq x \leq t$, the solution changes linearly from $u_l = 0$ to $u_r = 1$, forming a rarefaction wave that propagates through time.
The graphical interpretation for the equation (2.95) in x-space will be as follows,

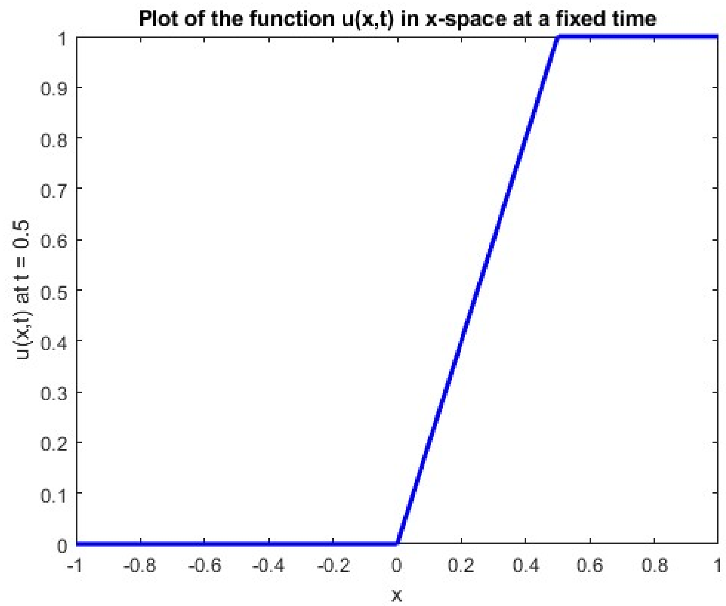

Figure 2.6: Depiction of the Rarefaction wave (2.95) in x space

The graphical interpretation for the equation (2.95) in xt-space will be as follows,

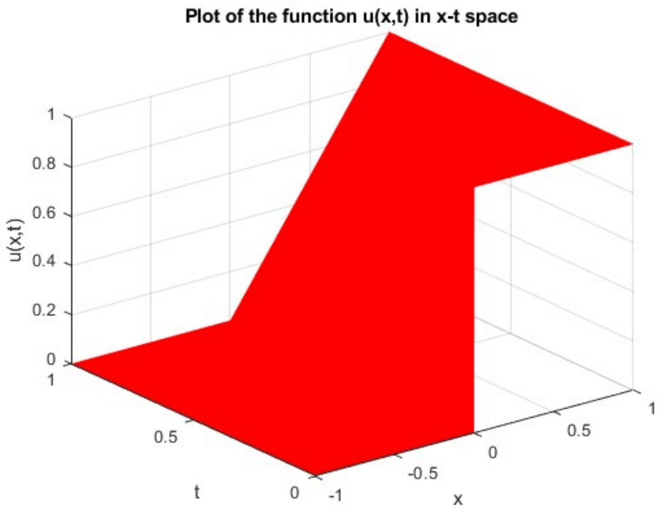

Figure 2.7: Depiction of the Rarefaction wave (2.95) in xt space

In the case of the crossing characteristics, the initial data will take the from,

$$u(x,t) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0 \end{cases} \tag{2.96}$$

If the starting condition is a step function with the left state being less than the right state ($u_l < u_r$), the characteristics in Burgers' equation will cross and generate a shock because the right characteristics move more quickly than the left ones. The features create a discontinuity or shock when they cross.

The Rankine-Hugoniot condition yields the shock speed, s, as $s = (f(u_r) - f(u_l))/(u_r - u_l)$. Given that $f(u) = u^2/2$ in this instance, $s = ((u_r^2 - u_l^2)/2)/(u_r - u_l) = (u_r + u_l)/2$.

In this instance, the generic form of the Burgers' equation solution is,

$$u(x,t) = \begin{cases} u_l, & \text{if } x < st \\ u_r, & \text{if } x > st \end{cases} \tag{2.97}$$

Accordingly, the shock moves to the right at a speed of s, and at the shock, the value of u changes from u_l to u_r.

The shock speed is $s = (0 + 1)/2 = 0.5$ for the particular initial condition, where $u_l = 0$ and $u_r = 1$, hence the answer is,

$$u(x,t) = \begin{cases} 0, & \text{if } x < 0.5t \\ 1, & \text{if } x > 0.5t \end{cases} \tag{2.98}$$

The shock goes to the right at a speed of 0.5, and the answer is 0 to the shock's left and 1 to its right.
The graphical interpretation for the equation (2.98) in x-space will be as follows,

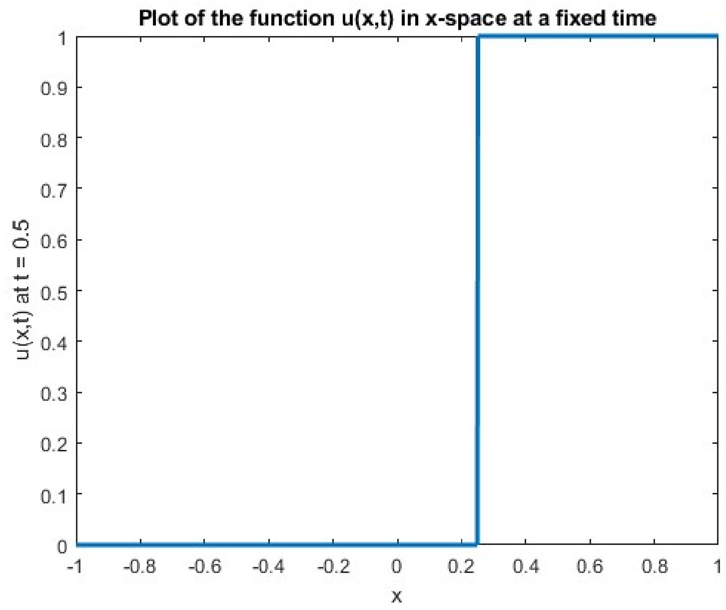

Figure 2.8: The graph of (2.98) in x space

The graphical interpretation for the equation (2.98) in xt-space will be as follows,

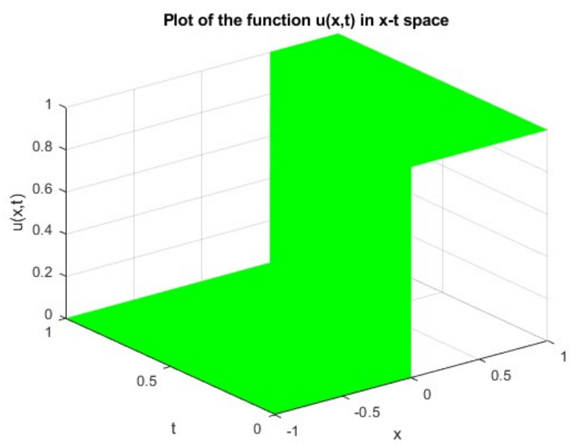

Figure 2.9: The graph of (2.98) in xt space

2.10 Comparison between Method of Characteristics and Finite Difference Technique

2.10.1 Method of Characteristics

- **Concept:** The partial differential equation (PDE) is converted into a collection of ordinary differential equations (ODEs) along characteristic curves using the method of characteristics (MOC). The original PDE's solution may be found by solving the ODEs.
- **Advantage:** Burger's problem may be analytically solved using the MOC, resulting to a precise representation of the answer without numerical mistakes. It provides information on the solution's physical characteristics and behavior, including shock development and wave propagation.
- **Limitations:** The MOC may not be simply extended to systems of PDEs since it is primarily relevant to scalar PDEs. Additionally, it necessitates the use of characteristic curves, which can be challenging for equations or geometries with more complex structures. When boundary conditions are complicated or when features meet to provide multivalued solutions, the MOC might not be appropriate.

2.10.2 Finite Difference Technique

- **Concept:** The finite difference method discretizes the temporal and spatial domains into a grid and uses finite differences to approximate the PDE derivatives. As a consequence, an algebraic equation system that can be solved numerically is created.
- **Advantage:** In addition to scalar PDEs and systems of PDEs, problems with complicated geometries or boundary conditions can also be solved using finite difference methods. They are ideal for large-scale simulations and are simple to build with the aid of contemporary computing power. To strike a balance between precision, stability, and processing cost, a number of finite difference techniques, including explicit, implicit, and Crank-Nicolson, can be used.
- **Limitations:** Numerical mistakes introduced by finite difference methods, such as discretization and truncation errors, can reduce the precision of the solution. For reliable and consistent results, it is essential

to select a suitable grid size and time step. Finite difference techniques may have trouble correctly capturing discontinuities like shocks or rarefaction waves, which can result in erroneous oscillations or numerical diffusion.

In conclusion, the finite difference technique is a flexible numerical approach that can handle a wide range of situations but introduces numerical inaccuracies, whereas the method of characteristics is an analytical approach that offers precise solutions to Burger's equation but may be limited in applicability. The individual situation, the required level of precision, and the available processing resources will determine which of these two approaches is best.

2.11 Difference between convex and concave flux

2.11.1 Convex Flux

If a flux function's second derivative is not negative, or if $f^{''}(u) \geq 0$ for all u in the domain, then the flux function is said to be convex. Convex fluxes are related to the following scalar conservation laws,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \tag{2.99}$$

Convex flux functions produce initial value problems that are well-posed, which means that they have solutions that are both unique and continually reliant on initial data. The existence of entropy solutions and the Lax-Oleinik formula, which may be used to explicitly design solutions, are both guaranteed by the system's convexity condition.[5]

2.11.2 Concave Flux

If the second derivative of a flux function $f(u)$ is negative, or if $f^{''}(u) \leq 0$ for all u in the domain, the flux function is concave. Concave fluxes, as opposed to convex ones, might result in initial value issues that may not have solutions, may not have unique solutions, or may not have solutions that depend constantly on the initial data.[5]

2.12 Implications of flux for Conservation Laws

The behavior of solutions to conservation laws is significantly affected by the flow function's curvature,

- Strong solutions are guaranteed to exist and be unique in problems with convex flux functions. These solutions are easier to examine analytically and are more likely to have physical meaning.
- Determining the existence, uniqueness, or stability of solutions can be difficult in cases when concave flux functions lead to ill-posed problems. Finding physically sound solutions and creating numerical approximations may become challenging as a result.
- The properties of the conservation law are influenced by the convexity or concavity of the flow function. Characteristics do not cross in the case of a convex flow, but they may do so in the case of a concave flux, which might result in the production of shocks or other discontinuities in the solution.

For the examination of conservation laws, it is essential to comprehend the characteristics of convex and concave fluxes because they shed light on how well-posed the issue is, how the characteristics behave, and how discontinuities emerge.[12]

2.13 Significance of the Riemann Problem

The Riemann problem holds a central position in the study of scalar conservation laws due to the following reasons,

- When the starting data consists of two constant states separated by a discontinuity, it is the most simple version of an initial value problem for a conservation law. The solutions to the Riemann problems provide a good foundation for comprehending a wide range of challenges in domains such as fluid dynamics, gas dynamics, and traffic flow.[12]
- The Riemann problem has had a significant influence on the development of numerical approaches for solving conservation laws, such as finite volume and Godunov-type schemes. These approaches seek to approximate the conservation law's solution while keeping critical qualities such as shock capture, conservation, and suitable wave propagation.[12]
- The Riemann problem elucidates the behavior of shock and rarefaction waves under conservation laws. Exploration of the generation, interaction, and propagation of these waves is critical for understanding complicated physical processes associated to nonlinear conservation

principles. Examining the answers to the Riemann problem can help you get this insight.[12]

- It is used to test the stability and well-posedness of conservation rules. The features of Riemann problem solutions, such as existence, uniqueness, and continuous dependency on beginning data, can give a deeper understanding of the conservation law's general behavior and its usefulness for diverse applications.[12]

To summarize, the Riemann problem is important in the context of scalar conservation laws because of its simplicity, role in the evolution of numerical techniques, contribution to understanding shock and rarefaction wave behavior, and insights into the stability and well-posedness of conservation laws.

2.14 Weak Solution for conservation law

A typical transport equation might be presented as follows,

$$u_t + uu_x = 0 \tag{2.100}$$

This equation (2.100) can be converted into a conservation form, yielding,

$$u_t + (\frac{u^2}{2})_x = 0 \tag{2.101}$$

It is often useful to express the conservation form in a more general configuration,

$$u_t + f(u)_x = 0 \tag{2.102}$$

When applying formal differentiation with respect to x to the initial equation, one derives the following result,

$$v_t + uv_x = -v^2 \tag{2.103}$$

where as v=u_x the variation in v can be outlined using the following conditions,

$$x\prime(t) = u(x(t), t), x(0) = x_0 \tag{2.104}$$

The fluctuation in $v+$ can be represented as,

$$\frac{d}{dt}v(x(t), t) = -v^2(x(t), t) \tag{2.105}$$

The nonlinearity of this Ordinary Differential Equation (ODE) (2.105) is quadratic. It is commonly assumed that the solution v to this ODE (2.105) can become unbounded in a limited period. As a result, even if the initial derivative is tiny, the spatial derivative of the solution to the original equation might rise sharply. This rise in the derivative indicates that simple solutions to the initial problem may be difficult to find.
The above example indicates that simple or conventional answers to the conservation law are improbable. However, because these models are physics-based, they must allow for solutions in some form or another. These solutions are sometimes referred to as "weak solutions".[6]

2.15 The entropy condition

The investigation of Partial Differential Equations (PDEs) frequently requires the use of a viscosity parameter, which is subsequently set to zero following analysis. This may be accomplished using the viscous approximation of the scalar conservation rule, as shown below,

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon \tag{2.106}$$

where $\epsilon > 0$ is a minor parameter. The second-order term u_{xx} serves as the diffusion term, which transforms the conservation law into a convection-diffusion equation of the parabolic category upon introduction (Suli Mayers, 2003) [8]
Let $\eta : R---> R$ is any convex function, and $q : R-----> R$ be a constructed function (Evans, 2010)[1],

$$q(u) = \int_0^u f^{'}(s)\eta^{'}(s)ds \tag{2.107}$$

where as the relation between η and q is as follows,

$$q^{'} = n^{'}f^{'} \tag{2.108}$$

multiplying both sides of (2.106) by $\eta^{'}(u)$ and using the chain rule and the relation of (2.108) we get,

$$\eta(u^\epsilon)_t + q(u^\epsilon)_x = \epsilon\eta(u^\epsilon)_{xx} - \epsilon\eta^{''}(u^\epsilon)(u_x^\epsilon)^2 \tag{2.109}$$

Since η is a convex function and the second term on the right-hand side is non-positive therefore we obtain,

$$\eta(u^\epsilon)_t + q(u^\epsilon)_x \leq \epsilon\eta(u^\epsilon)_{xx} \tag{2.110}$$

Therefore, any vanishing viscosity solution $u = \lim_{\epsilon\to 0} u^{\epsilon}$ satisfies,

$$\eta(u)_t + q(u)_x \leq 0 \tag{2.111}$$

As usual, this expression must be interpreted in the sense of distributions: For all test functions $\phi \in C_c^1(R \times [0, \infty])$ with $\phi \geq 0$ and U satisfies,

$$\int_{R^+}\int_R \eta(u(x,t))\phi_t(x,t) + q(u(x,t))\phi_x(x,t)dxdt + \int_R \eta(u_0(x))\phi(x,0)dx \geq 0 \tag{2.112}$$

The equivalent function q is known as an entropy flow, and the function η is known as an entropy function. An entropy pair is represented by the pair $(\eta; q)$. The entropy condition, often known as the inequality (2.112), is true for each entropy pair $(\eta; q)$. Take note that given a scalar conservation law, any convex function produces an entropy pair. Contrast this to conservation law systems, where there is often just one entropy pair. [5][6]

Chapter 3

Solution to the Riemann Problem a More Complex Case

In this chapter, we will investigate scalar conservation laws, with a focus on Burgers' equation. These laws are extremely important in the area of partial differential equations and are required for modeling a variety of physical systems, including fluid dynamics, vehicle traffic, and gas dynamics, among others.
The occurrence of abrupt transitions, or "shock waves," in the solutions of scalar conservation laws as time progresses is a fascinating feature of their solutions. The complexity associated with these discontinuities are an important topic of research.
We begin with Burgers' equation, a nonlinear, first-order partial differential equation in which the impact of features is critical. The solutions might manifest as shock waves or rarefaction waves depending on the initial circumstances and velocities of the features. The study of these phenomena is inextricably linked to the notion of entropy solutions, which allows us to appropriately capture shock waves and so produce physically meaningful solutions.
This chapter digs into the characteristics technique, a powerful tool for solving partial differential equations. We rigorously examine the transmission of features and identify the conditions that result in the creation of shock waves or rarefaction waves. We focus on Riemann problems with piece wise constant initial conditions.
The chapter includes a thorough assessment of various initial conditions and their related results. Under these conditions, the development of shock and rarefaction waves is thoroughly investigated. Theoretical and numerical solutions are investigated in order to improve our knowledge of scalar conservation laws.

3.1 Example

In this section we are going to solve the burger equation,

$$u_t + (1/2u^2)_x = 0 \tag{3.1}$$

for the following initial data,

$$u_0(x) = \begin{cases} 1, & x < -1 \\ 0, & -1 < x < 1 \\ -1, & x > 1 \end{cases} \tag{3.2}$$

Riemann Problem 1: Here we are going to divide the initial data into two parts as follows,
for $x = -1$, we have

$$u_0(x) = \begin{cases} 1, & x < -1 \\ 0, & -1 \leq x \end{cases} \tag{3.3}$$

Here in this data we take $u_l = 1$ and $u_r = 0$ then we are going to identify that whether it will form a Shock wave or the rarefaction wave?
As for equation (3.1) $f(u) = \frac{1}{2}u^2$ then $f^{'}(u) = u$, so from here we can deduce that $f^{'}(u_l) = 1$ and $f^{'}(u_r) = 0$, which is clearly given us the direction that $f^{'}(u_l) > f^{'}(u_r)$. As a result this Riemann problem will form a shock wave. By utilizing the Rankine-Hugoniot Condition (2.90) can be utilized as follows,

$$s(u_r - u_l) = f(u_r) - f(u_l) \tag{3.4}$$

By using the values of u_r,u_l,$f(u_r)$ and $f(u_l)$ we will get the value of s,i.e.

$$s = \frac{1}{2} \tag{3.5}$$

So the solution for (3.3), will be as follows using the shock condition of equation (2.97), will adopt the form,

$$u(x,t) = \begin{cases} 1 & x + 1 \leq \frac{1}{2}t \\ 0 & x + 1 > \frac{1}{2}t \end{cases} \tag{3.6}$$

Similarly for $x = 1$, we will do the fallowing calculations.
Riemann Problem 2: Here the Initial data will take the form,

$$v_0(x) = \begin{cases} 0, & x < 1 \\ -1, & 1 \leq x \end{cases} \tag{3.7}$$

Here in this data we take $v_l = 0$ and $v_r = -1$ then we are going to identify that whether it will form a Shock wave or the rarefaction wave?
As for equation (3.1) $f(u) = \frac{1}{2}u^2$, which can be written in terms of v as,$f(v) = \frac{1}{2}v^2$ then $f^{'}(v) = v$, so from here we can deduce that $f^{'}(v_l) = 0$ and $f^{'}(v_r) = -1$, which is clearly given us the direction that $f^{'}(v_l) > f^{'}(v_r)$. As a result this Riemann problem will form a shock wave. By utilizing the Rankine-Hugoniot Condition (2.90) can be utilized as follows,

$$s(v_r - v_l) = f(v_r) - f(v_l) \tag{3.8}$$

By using the values of v_r,v_l,$f(v_r)$ and $f(v_l)$ we will get the value of s,i.e.

$$s = -\frac{1}{2} \tag{3.9}$$

So the solution for (3.7), will be as follows using the shock condition of equation (2.97), will adopt the form,

$$u(x,t) = \begin{cases} 0 & x - 1 \leq -\frac{1}{2}t \\ -1 & x - 1 > -\frac{1}{2}t \end{cases} \tag{3.10}$$

The final solution will be as follows,

$$u(x,t) = \begin{cases} 1 & x \leq -1 + \frac{1}{2}t \\ 0 & -1 + \frac{1}{2}t < x \leq 1 - \frac{1}{2}t \\ -1 & x > 1 - \frac{1}{2}t \end{cases} \tag{3.11}$$

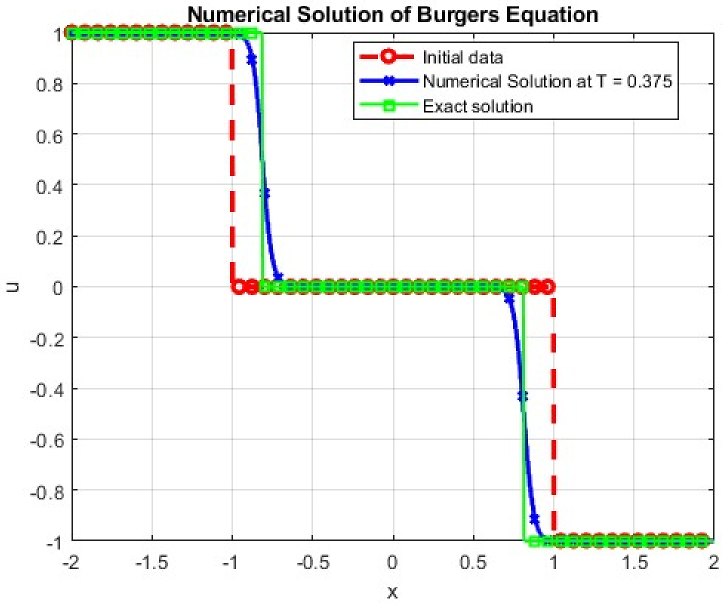

Figure 3.1: Depiction of the initial data(3.2)(red line),Exact solution(3.11)(green line) and The numerical solution(blue line)

This Figure 3.1 show the evolution of a Burger's equation solution by using the 1000 grid cells. It illustrates wave propagation and, with time, can create shocks (discontinuities).

In the graph the red line depicts the initial data (3.2), The Blue line is representing the numerical solution at time 0.375. For numerical solution, The Lax-Friedrichs method, which is a simple and robust numerical method for solving hyperbolic PDEs, was used to obtain this solution. This method works by averaging neighboring solutions and then correcting for the solution's average slope over time. The graph clearly shows that the solution has evolved from the initial condition, and the shock is developing.

The third graph depicts the precise solution to Burger's equation at time 0.375. This graphic serves as a checkpoint for the numerical method's correctness and convergence. If the numerical method is implemented correctly, the numerical solution should approach the exact solution as we refine the grid.

Time at which the two Shocks will collide:

In order to calculate the time at which the two shock collide in the solution of equation (3.11), we can perform the following calculations,

$$-1 + \frac{1}{2}t = 1 - \frac{1}{2}t \tag{3.12}$$

Then we have,

$$t = 2 \tag{3.13}$$

Now, we will calculate the solution after this time. For that purpose we have the criteria as follows. For $t = 2$, the equation (3.11) will adopt the form,

$$u(x, t = 2) = \begin{cases} 1 & x \leq 0 \\ -1 & x > 0 \end{cases} \tag{3.14}$$

Which is another Riemann problem so we will solve it again by considering that $w_l = 1$ and $w_r = -1$, then $f^{'}(w_l) = 1$ and $f^{'}(w_r) = -1$ indicating that $f^{'}(w_l) > f^{'}(w_r)$ so the shock wave will form. Remember that we have utilized $f(w) = \frac{1}{2}(w^2)$. Now, we can calculate the shock speed s by using the equation,

$$s(u_r - u_l) = f(u_r) - f(u_l) \tag{3.15}$$

Then,

$$s = 0 \tag{3.16}$$

Hence the final solution for (3.14) is,

$$u(x, t) = \begin{cases} 1 & x \leq 0 \\ -1 & x > 0 \end{cases} \tag{3.17}$$

As we have calculated that the time at which the two shock's collide is $t = 2$, therefore we are going to plot our solution for $t = 2$ that will give us the following graphical interpretation,

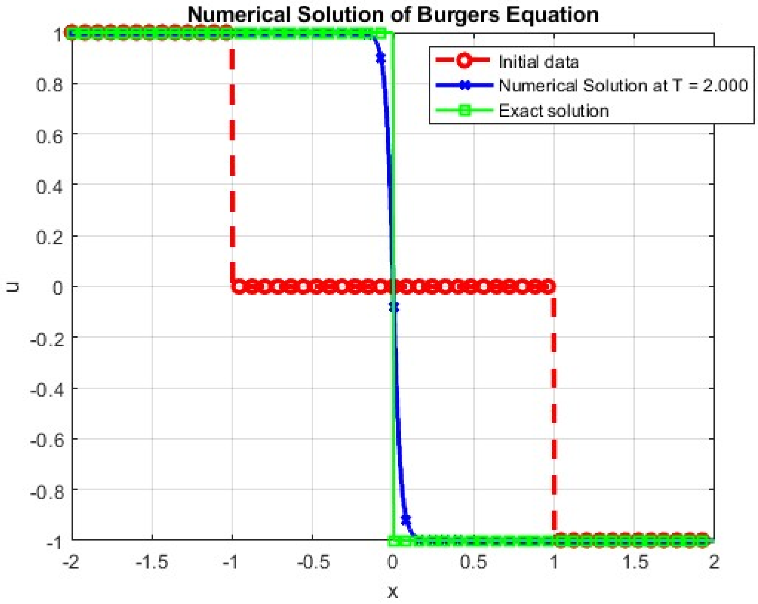

Figure 3.2: Numerical solution and the exact solution at T=2.0 using the Lax-Friedrichs scheme

3.1.1 Burger Equation With the initial condition forming Rarefaction

The Burger's equation,

$$u_t + (1/2u^2)_x = 0 \tag{3.18}$$

With the initial data,

$$u_0(x) = \begin{cases} -1, & x < -1 \\ 0, & -1 < x < 1 \\ 1, & x > 1 \end{cases} \tag{3.19}$$

Riemann Problem 1: Here we are also going to the follow the same process that what kind of wave will be formed(Shock or rarefaction). for $x = -1$, we have

$$u_0(x) = \begin{cases} -1, & x < -1 \\ 0, & -1 \geq x \end{cases} \tag{3.20}$$

Here in this data we take $u_l = 1$ and $u_r = 0$ then we are going to identify that whether it will form a Shock wave or the rarefaction wave?
As for equation (3.18) $f(u) = \frac{1}{2}u^2$ then $f^{'}(u) = u$, so from here we can deduce that $f^{'}(u_l) = -1$ and $f^{'}(u_r) = 0$, which is clearly given us the direction that

$f^{'}(u_l) < f^{'}(u_r)$. As a result we have the situation of the rarefaction wave, so the solution will be of the from (226), keeping in mind that at $x = -1$ we are facing a first jump in our solution,

$$u(x,t) = \begin{cases} u_l, & \text{if } \frac{x+1}{t} \leq f'(u_l) \\ (f^{'})^{-1}(\frac{x+1}{t}), & \text{if } f'(u_l) < \frac{x+1}{t} < f'(u_r) \\ u_r, & \text{if } \frac{x+1}{t} \geq f'(u_r) \end{cases} \tag{3.21}$$

Utilize the value of $u_l = -1$ and $u_r = 0$, we have,

$$u(x,t) = \begin{cases} -1, & \text{if } \frac{x+1}{t} \leq -1 \\ \frac{x+1}{t}, & \text{if } -1 < \frac{x+1}{t} < 0 \\ 0, & \text{if } \frac{x+1}{t} \geq 0 \end{cases} \tag{3.22}$$

Riemann Problem 2: Similarly when we have $x = 1$, then the initial data will adopt the form as follows,

$$v_0(x) = \begin{cases} 0, & x < 1 \\ 1, & x \geq 1 \end{cases} \tag{3.23}$$

Here in this data we take $v_l = 1$ and $v_r = 0$ then we are going to identify that whether it will form a Shock wave or the rarefaction wave?
As for equation (3.18) $f(v) = \frac{1}{2}v^2$ then $f^{'}(v) = v$, so from here we can deduce that $f^{'}(v_l) = 0$ and $f^{'}(v_r) = 1$, which is clearly given us the direction that $f^{'}(v_l) < f^{'}(v_r)$. As a result we have the situation of the rarefaction wave, so the solution will be of the from (2.94), keeping in mind that at $x = 1$ we are facing a first jump in our solution,

$$v(x,t) = \begin{cases} v_l, & \text{if } \frac{x-1}{t} \leq f'(v_l) \\ (f^{'})^{-1}(\frac{x-1}{t}), & \text{if } f'(v_l) < \frac{x-1}{t} < f'(v_r) \\ v_r, & \text{if } \frac{x-1}{t} \geq f'(v_r) \end{cases} \tag{3.24}$$

Utilize the value of $v_l = 0$ and $v_r = 0$, we have,

$$v(x,t) = \begin{cases} 0, & \text{if } \frac{x-1}{t} \leq 1 \\ \frac{x-1}{t}, & \text{if } 1 < \frac{x-1}{t} < 0 \\ 1, & \text{if } \frac{x-1}{t} \geq 0 \end{cases} \tag{3.25}$$

Then the final solution will be obtained by joining the solutions of equation (3.22) and (3.25),

$$u(x,t) = \begin{cases} -1, & \text{if } x \leq -1-t \\ \frac{x+1}{t}, & \text{if } -1-t < x < -1 \\ 0, & \text{if } -1 \leq x \leq 1 \\ \frac{x-1}{t}, & \text{if } 1 < x < t+1 \\ 1, & \text{if } x \geq t+1 \end{cases} \tag{3.26}$$

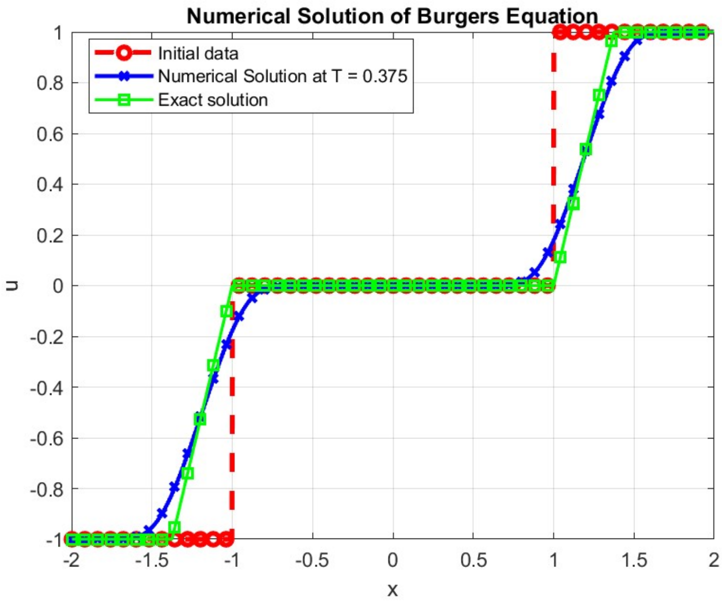

Figure 3.3: Representation of the initial data (3.19)(Red line), Exact solution(3.26)(Green line) and Th numerical solution(Blue line)

This Figure 3.3 show the evolution of a Burger's equation solution by using the 1000 grid cells. It illustrates wave propagation and, with time, can create shocks (discontinuities).
In the graph the red line depicts the initial data (3.19), The Blue line is representing the numerical solution at time 0.375. For numerical solution, The Lax-Friedrichs method, which is a simple and robust numerical method for solving hyperbolic PDEs, was used to obtain this solution. This method works by averaging neighboring solutions and then correcting for the solution's average slope over time. The graph clearly shows that the solution has evolved from the initial condition, and the shock is developing.

The third graph depicts the precise solution to Burger's equation at time 0.375. This graphic serves as a checkpoint for the numerical method's correctness and convergence. If the numerical method is implemented correctly, the numerical solution should approach the exact solution as we refine the grid.

Chapter 4

Conclusion

This thesis makes an important contribution to the research and application of numerical solutions to nonlinear conservation laws. It demonstrates the durability of the characteristics approach and the Lax-Friedrichs scheme via MATLAB implementations. The comparison of various numerical approaches, as well as the evaluation of more challenging scenarios, broadens the scope of this study.
Future research might include extending these ideas to other types of conservation laws, as well as developing other numerical algorithms and enhancing the MATLAB implementation for greater efficiency and variety. We believe that our findings will lay a firm basis for future research and development in this field.

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