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ORIGINAL ARTICLE

Two-period Colonel Blotto contest with cumulative investments over variable assets with resource constraints

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Abstract

Two resource constrained players compete by investing in two assets which may increase or decrease in value over two periods. A player's investment in period 1 carries over to period 2. If an asset is cheap in period 1, a player invests more in it in period 1, less in period 2, and does the opposite for the other asset. If an asset is cheap in period 2, a player invests more in it in period 2, less in period 1, and does the opposite for the other asset. If an asset increases in value, both players invest more in it in both periods, and less into the less valuable asset. An advantaged player may invest more into the less valuable asset than the least advantaged player. If an asset increases in value, both players invest more in it in period 2, until the advantaged player eventually ceases investment into the asset with low growth, to focus on the high-growth asset. Various intuitive and less intuitive effects are illustrated for how players strike balances across space (two assets) and time (two periods).

Keywords Contests · Two periods · Resource constraints · Growth · Discounting · Colonel Blotto

JEL Classification C6 · C7 · D72 · O

Introduction

This article provides a model that builds on four critical assumptions. The first is rent seeking. Each player competes for two assets (rents, prizes, etc.). The second is resource constraints. Instead of competing for one asset or several assets separately, each player competes in a Colonel Blotto game for two assets while being resource constrained across the two assets. The third is competition over two periods where each asset may appreciate or depreciate from period 1 to period 2. The fourth is that

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each player's investment in period 1 carries forward additively for the same asset to period 2.

Two players are analyzed with resource constraints competing over two assets which may increase or decrease in value over two periods. Resource constraints are often realistic, since players often utilize all their available resources. Resource constraints induce a linkage between the two assets by forcing each player to strike a balance between how to compete for the two assets. Without resource constraints, no linkage exists between the two assets, which means that each player competes in separate contests for each asset, which has been analyzed extensively in the literature. The two assets may be any asset class or subasset class, and may be of comparable or different values. The assets may be stocks, bonds, real estate, or rents commonly analyzed in the rent seeking literature, e.g., economic benefits, rights of various kinds, licenses, privileges, budgets, election opportunities, and Research & Development budgets. A player's competition may take the form of investment or effort to gain control or access to an asset, which succeeds depending on the other player's investment in the asset.

This conception is original. The phenomenon occurs in real life. Consider two investors allocating their resources into two assets at one point in time, and subsequently allocating their newly acquired resources into the same two assets at one subsequent point in time. Both assets may appreciate or depreciate in value between the two points in time. First, the assets may flow from two companies scheduled to yield returns on investment at two points in time. The assets are generally different at the two points in time, and are allocated according to the investors' investments. The investors may allocate their resources differently at the two points in time depending on the asset values, the unit costs of investment, how the assets appreciate or depreciate, and their time discount parameters. Second, investors may allocate their resources into shares or investment tokens for two athletes or athletic teams.[1] The shares or investment tokens represent stakes in athletic contracts which may increase or decrease in value. Third, the assets in the model can be R&D budgets, promotions, licenses, privileges, monopoly opportunities, election opportunities, struggles for government support between different industries, which are structured such that two points in time exist where something gets paid out as asset values, and investments carry forward from the first to the second point in time. How much each investor gets paid depends on how much it invests. The asset values at the two points in time must be quantifiable to enable calculating asset appreciation and depreciation.

The article shows how two players strike balances in resource allocation between two assets over two time periods. It is found that a player invests more in an asset in the period in which it is cheap, and invests less in that asset in the other period. An asset that increases in value incurs more investment in both periods. A resourceful player may invest more into the less valuable asset than a less resourceful player. If an asset appreciates substantially from period 1 to period 2, both players invest more

[1] https://alts.co/investing-in-athletes/, retrieved July 4, 2023.

in it in period 2, and the resourceful player may invest less in the asset with less appreciation.

For an overview of the rent seeking literature, see Congleton et al. (2008). Early developments are by Krueger (1974), Posner (1975), and Tullock (1980). Partly related to this article, Konrad (2018) considers best-of-three Colonel Blotto contests finding that if only the winner's resources carry over to subsequent contests, the leading player behave precautionarily, while the player lagging behind chooses all-in behavior. Hausken and Levitin (2009) and Hausken and Levitin (2010) similarly assume that players distribute their constrained resources optimally across two periods. Hausken and Levitin (2010) assume that both agents' resources are expendable and last only one period. They find that the defender protects equally in both periods when the attacker distributes resources unevenly between two attacks and can observe the defender's resource distribution in period 1. Hausken and Levitin (2009) assume that the attacker's resources are expendable and last only one attack, while, for the defender, two cases are analyzed. First, assets that survive the first attack keep their protection in the second attack. Second, assets that are not attacked in the first attack keep their protection in the second attack, while assets that are attacked but survive, do not keep their protection. They find that the attacker prefers that assets not attacked in the first attack keep their protection in the second attack, rather than that assets that survive the first attack keep their protection in the second attack, and vice versa for the defender.

Klumpp et al. (2019) consider a sequential Colonel Blotto game where each contest entered before one of the players wins a majority of the contests is allocated the same amount of resources from the player's overall available resources. Then, a player's probability of winning any one contest does not depend on that contest and how many earlier contests the player has won. That contrasts with the results from sequential majoritarian non-Blotto games. Anbarci et al. (2022) find for an n-player Colonel Blotto game where players have asymmetric resources that they allocate their resources proportionally to the asset values for every history. This result is supported by the current article where players also invest more in assets they value highly. Arbatskaya and Mialon (2012) find that players' effort expenditures are lower in a sequential multi-activity contest than in a simultaneous multi-activity contest.

The rent seeking literature with multiple contests typically does not assume that the players are resource constrained across the contests. Instead, each player incurs a cost of competing in each contest. Clark and Konrad (2007) analyze multiple contests, some of which have to be won to obtain the rent. To maximize effort in the contest per dimension and totally, they find a preference for running small contests with few dimensions. Dickson et al. (2018) analyze share contests where scarcer rents may become more contested. They relax the assumption of constant marginal rates of substitution where higher rents require more investment. They find that assets may become more hotly contested when they become scarcer. The current article does not analyze scarce assets, but since scarce assets tend to become more valuable, players may invest more resources into them. Hausken (2020) contrasts a player's multiple additive efforts, where one effort is sufficient to impact the probability of winning a contest, with a player's multiple multiplicative efforts, where all efforts must be positive to impact the probability of winning a contest.

Vesperoni and Yildizparlak (2019) axiomatize single-winner contests with different rents for a win and a draw. Lu and Wang (2015) axiomatize lottery contests assuming multiple rents. Assuming one rent, axiomatizations are provided by Skaperdas (1996), Arbatskaya and Mialon (2010), and Hausken (2021) for multi-activity contests, and Rai and Sarin (2009) for multiple types of investments. Hausken (2008, 2010) analyzes as contests whether to attack one asset in two periods, where the asset may appreciate or depreciate in value from period 1 to period 2. Assuming that the probability that an attack on an asset succeeds depends only on the defensive resource allocation, and not on the attacker's offensive resource allocation, Powell (2009) finds for Colonel Blotto games that the attacker moving second can be deterred. In contrast, applying the ratio contest success function impacted by both players' resource allocation, Hausken (2012) finds that the second mover cannot be deterred. These results relate partly to findings in the current article where an advantaged player may cease investing in an asset with low growth.

Hart (2008) and Kovenock and Roberson (2021) compare General Blotto games where resource constraints hold with probability one, with General Lotto games where resource constraints are satisfied in expectation. The latter may be interpreted as games with random returns. Another example of a game with random returns is Bell and Cover (1980). They consider two competitive investors each investing one unit of capital across a fixed number of stocks with the objective of earning more money than the other investor. They find that the optimal investment is the same as one that obtains the maximum capital growth rate in repeated independent investments. The current article does not consider random returns in the sense expressed by Hart (2008), Kovenock and Roberson (2021), and Bell and Cover (1980). The resource constraints are not random, but hold with probability one. The article assumes a contest success function, which is also common elsewhere in the General Blotto literature. A contest may be interpreted, so that a player wins an asset with a certain probability, hence the application of an expected utility, or so that the player is guaranteed to win a fraction of the asset as determined by the contest success probability. Both interpretations are combined with fixed resource constraints.

Article organization

The next section presents the model. The third section analyzes the model. The fourth section illustrates the solution. The fifth section summarizes key results. Conclusion is drawn in the last section.

The model

Consider two players. In period 1, player 1 invests x_1 at unit cost a_1, $a_1 > 0$, and player 2 invests X_1 at unit cost A_1, $A_1 > 0$, in competition for an asset valued as V, $V \geq 0$. Also, in period 1, player 1 invests y_1 at unit cost b_1, $b_1 > 0$, and player 2 invests Y_1 at unit cost B_1, $B_1 > 0$, in competition for an asset valued as W, $W \geq 0$. The asset valued as V in period 1 is valued as αV, $\alpha \geq 0$, in period 2, where α is a growth parameter. The asset valued as W in period 1 is valued as βW, $\beta \geq 0$, in period 2, where β is a growth parameter. In period 2, player 1 invests x_2 at unit cost a_2, $a_2 > 0$, and player 2 invests X_2 at unit cost A_2, $A_2 > 0$, in competition for the asset valued as αV. Also, in period 2, player 1 invests y_2 at unit cost b_2, $b_2 > 0$, and player 2 invests Y_2 at unit cost B_2, $B_2 > 0$, in competition for the asset valued as βW. The players are resource constrained. Player 1 has resources r_t, $r_t > 0$, in period $t, t = 1, 2$. Player 2 has resources R_t, $R_t > 0$, in period $t, t = 1, 2$. Hence

$$r_t = a_t x_t + b_t y_t, R_t = A_t X_t + B_t Y_t, t = 1, 2. \tag{1}$$

Player 1 has two free choice variables, i.e., x_t in period $t, t = 1, 2$. Player 2 has two free choice variables, i.e., X_t in period $t, t = 1, 2$. The four variables y_t and Y_t are dependent variables determined by (1). Applying the common ratio from contest success function (Skaperdas 1996; Tullock 1980), in period 1, player 1's expected utility is $u_1 = \frac{x_1 V}{x_1+X_1} + \frac{y_1 W}{y_1+Y_1}$, from contests $\frac{x_1 V}{x_1+X_1}$ and $\frac{y_1 W}{y_1+Y_1}$ over assets V and W, respectively. The utilities in this article are referred to as expected, since they are received with the probabilities expressed by the contest success functions. An alternative interpretation is that each contest does not specify a probability of winning a contest, but a deterministic fraction (between 0 and 1) of the asset to be received. Analogously, in period 1, player 2's expected utility is $U_1 = \frac{X_1 V}{x_1+X_1} + \frac{Y_1 W}{y_1+Y_1}$, from contests over assets V and W, respectively. Each player's investment in period 1 carries forward additively to period 2. Hence, in period 2, player 1 invests $x_1 + x_2$ into the contest over asset αV, invests $y_1 + y_2$ into the contest over asset βW, and earns expected utility $u_2 = \frac{(x_1+x_2)\alpha V}{x_1+x_2+X_1+X_2} + \frac{(y_1+y_2)\beta W}{y_1+y_2+Y_1+Y_2}$. Analogously, in period 2, player 2 invests $X_1 + X_2$ into the contest over asset αV, invests $Y_1 + Y_2$ into the contest over asset βW, and earns expected utility $U_2 = \frac{(X_1+X_2)\alpha V}{x_1+x_2+X_1+X_2} + \frac{(Y_1+Y_2)\beta W}{y_1+y_2+Y_1+Y_2}$. Each player may assess the two periods differently against each other. Hence, the players have different time discount parameters δ and Δ, respectively, $\delta \geq 0$, $\Delta \geq 0$, for period 2. Hence, the players expected utilities u and U, respectively, are

$$\begin{aligned} u = u_1 + \delta u_2 &= \frac{x_1 V}{x_1 + X_1} + \frac{y_1 W}{y_1 + Y_1} + \delta\left(\frac{(x_1 + x_2)\alpha V}{x_1 + x_2 + X_1 + X_2} + \frac{(y_1 + y_2)\beta W}{y_1 + y_2 + Y_1 + Y_2}\right), \\ U = U_1 + \Delta U_2 &= \frac{X_1 V}{x_1 + X_1} + \frac{Y_1 W}{y_1 + Y_1} + \Delta\left(\frac{(X_1 + X_2)\alpha V}{x_1 + x_2 + X_1 + X_2} + \frac{(Y_1 + Y_2)\beta W}{y_1 + y_2 + Y_1 + Y_2}\right), \end{aligned} \tag{2}$$

where x_t and X_t are the players' free choice variables in period $t, t = 1, 2$, and y_t and Y_t are given by (1). All parameters are common knowledge.

Analyzing the model

Interior solution

The game is solved by backward induction starting with period 2. Inserting y_t and Y_t from (1) into (2), and differentiating the players' expected utilities u_2 and U_2 in period 2 in (2) with respect to the players' period 2 strategic choice variables x_2 and X_2, and equating with zero, it gives

$$\frac{\partial u_2}{\partial x_2} = \frac{\left(X_1+X_2\right)\alpha V}{\left(x_1+x_2+X_1+X_2\right)^2} - \frac{\left(\frac{R_1-A_1X_1}{B_1}+\frac{R_2-A_2X_2}{B_2}\right)\frac{a_2}{b_2}\beta W}{\left(\frac{r_1-a_1x_1}{b_1}+\frac{r_2-a_2x_2}{b_2}+\frac{R_1-A_1X_1}{B_1}+\frac{R_2-A_2X_2}{B_2}\right)^2}=0,$$
$$\frac{\partial U_2}{\partial X_2} = \frac{\left(x_1+x_2\right)\alpha V}{\left(x_1+x_2+X_1+X_2\right)^2} - \frac{\left(\frac{r_1-a_1x_1}{b_1}+\frac{r_2-a_2x_2}{b_2}\right)\frac{A_2}{B_2}\beta W}{\left(\frac{r_1-a_1x_1}{b_1}+\frac{r_2-a_2x_2}{b_2}+\frac{R_1-A_1X_1}{B_1}+\frac{R_2-A_2X_2}{B_2}\right)^2}=0, \tag{3}$$

which are solved to yield

$$x_2 = \frac{A_2B_1\left(b_1r_2+b_2\left(r_1-a_1x_1\right)\right)\left(X_1+X_2\right)-a_2b_1x_1\left(B_2\left(R_1-A_1X_1\right)+B_1\left(R_2-A_2X_2\right)\right)}{a_2b_1\left(B_2\left(R_1-A_1X_1\right)+B_1\left(R_2+A_2X_1\right)\right)}, \tag{4}$$

$$X_2 = \frac{a_2b_1\left(B_1R_2+B_2\left(R_1-A_1X_1\right)\right)\left(x_1+x_2\right)-A_2B_1X_1\left(b_2\left(r_1-a_1x_1\right)+b_1\left(r_2-a_2x_2\right)\right)}{A_2B_1\left(b_2\left(r_1-a_1x_1\right)+b_1\left(r_2+a_2x_1\right)\right)}, \tag{5}$$

which uniquely express x_2 as a best reply to X_2, and X_2 as a best reply to x_2. The unique analytical solution for x_2 and X_2 is shown in (A1) and (A2) in Online Appendix A, which is inserted into the players' expected utilities u and U in (2) to give the players' period 1 expected utilities

$$\begin{aligned}
u = {} & \frac{x_1V}{x_1+X_1}+\frac{B_1W\left(r_1-a_1x_1\right)}{B_1\left(r_1-a_1x_1\right)+b_1\left(R_1-A_1X_1\right)} \\
& + \frac{A_2B_1V(b_2(r_1-a_1x_1)+b_1(r_2+a_2x_1))\alpha\delta}{a_2b_1(B_1R_2+B_2\left(R_1-A_1X_1\right))+A_2B_1(b_2(r_1-a_1x_1)+b_1(r_2+a_2(x_1+X_1)))} \\
& + \frac{B_1B_2W\left(b_2(r_1-a_1x_1)+b_1(r_2+a_2x_1)\right)\beta\delta}{b_1B_1B_2(r_2+a_2x_1)+b_1b_2B_2(R_1-A_1X_1)+B_1b_2(B_2(r_1-a_1x_1)+b_1(R_2+A_2X_1))}, \\
U = {} & \frac{X_1V}{x_1+X_1}+\frac{b_1W\left(R_1-A_1X_1\right)}{B_1\left(r_1-a_1x_1\right)+b_1\left(R_1-A_1X_1\right)} \\
& + \frac{a_2b_1V\left(B_2(R_1-A_1X_1)+B_1(R_2+A_2X_1)\right)\alpha\Delta}{a_2b_1(B_1R_2+B_2(R_1-A_1X_1))+A_2B_1(b_2(r_1-a_1x_1)+b_1(r_2+a_2(x_1+X_1)))} \\
& + \frac{b_1b_2W\left(B_2(R_1-A_1X_1)+B_1(R_2+A_2X_1)\right)\beta\Delta}{b_1B_1B_2(r_2+a_2x_1)+b_1b_2B_2(R_1-A_1X_1)+B_1b_2(B_2(r_1-a_1x_1)+b_1(R_2+A_2X_1))}.
\end{aligned} \tag{6}$$

Differentiating the players' period 1 expected utilities u and U in (6) with respect to the players' period 1 strategic choice variables x_1 and X_1, and equating with zero, it gives

$$\begin{aligned}
\frac{\partial u}{\partial x_1} &= \frac{VX_1}{(x_1+X_1)^2} - \frac{b_1B_1Wa_1\left(R_1-A_1X_1\right)}{\left(B_1\left(r_1-a_1x_1\right)+b_1\left(R_1-A_1X_1\right)\right)^2} \\
&+ \frac{a_2A_2b_1B_1(a_2b_1-a_1b_2)V(B_2(R_1-A_1X_1)+B_1(R_2+A_2X_1))\alpha\delta}{(a_2b_1(B_2R_1+B_1R_2-A_1B_2X_1)+A_2B_1(b_2(r_1-a_1x_1)+b_1(r_2+a_2(x_1+X_1))))^2} \\
&+ \frac{b_1B_1Wb_2B_2\left(a_2b_1-a_1b_2\right)\left(B_2(R_1-A_1X_1)+B_1(R_2+A_2X_1)\right)\beta\delta}{\left(b_1B_1B_2\left(r_2+a_2x_1\right)+b_1b_2B_2(R_1-A_1X_1)+B_1b_2\left(B_2\left(r_1-a_1x_1\right)+b_1(R_2+A_2X_1)\right)\right)^2} = 0, \\
\frac{\partial U}{\partial X_1} &= \frac{Vx_1}{(x_1+X_1)^2} - \frac{b_1B_1WA_1\left(r_1-a_1x_1\right)}{\left(B_1\left(r_1-a_1x_1\right)+b_1\left(R_1-A_1X_1\right)\right)^2} \\
&+ \frac{a_2A_2b_1B_1\left(A_2B_1-A_1B_2\right)V\left(b_2\left(r_1-a_1x_1\right)+b_1\left(r_2+a_2x_1\right)\right)\alpha\Delta}{\left(a_2b_1\left(B_2R_1+B_1R_2-A_1B_2X_1\right)+A_2B_1\left(b_2\left(r_1-a_1x_1\right)+b_1\left(r_2+a_2\left(x_1+X_1\right)\right)\right)\right)^2} \\
&+ \frac{b_1B_1Wb_2B_2\left(A_2B_1-A_1B_2\right)\left(b_2\left(r_1-a_1x_1\right)+b_1\left(r_2+a_2x_1\right)\right)\beta\Delta}{\left(b_1B_1B_2\left(r_2+a_2x_1\right)+b_1b_2B_2\left(R_1-A_1X_1\right)+B_1b_2\left(B_2\left(r_1-a_1x_1\right)+b_1\left(R_2+A_2X_1\right)\right)\right)^2} = 0.
\end{aligned} \tag{7}$$

Equation (7) contains two equations with two unknown variables x_1 and X_1 for the players' period 1 strategic choices. These are not generally analytically solvable.[2] They are determined numerically and interpreted over the next sections, and inserted into (A1) and (A2) in Appendix A to determine the players' period 2 strategic choices, and inserted into (2) to determine the players' expected utilities u and U.

Corner solutions

In addition to the interior solution where $(x_1, x_2, X_1, X_2) = (0 < x_1 < r_1/a_1, 0 < x_2 < r_2/a_2, 0 < X_1 < R_1/A_1, 0 < X_2 < R_2/A_2)$, if z, $z = 1, 2, 3, 4$, of the four variables x_1, x_2, X_1, X_2 have corner solutions which can be minimum 0 or maximum r_i/a_i or R_i/A_i, this can occur in $2^z\begin{pmatrix}4\\z\end{pmatrix}$ possible ways. Summing up gives $\sum_{z=1}^{4} 2^z\begin{pmatrix}4\\z\end{pmatrix} = 80$ corner solutions. Four interesting corner solutions are when one player in period 2 chooses minimum 0 or maximum r_2/a_2 or R_2/A_2 investment, i.e., $(x_2, X_2) = (x_2, 0)$, $(x_2, X_2) = (x_2, R_2/A_2)$, $(x_2, X_2) = (0, X_2)$, $(x_2, X_2) = (r_2/a_2, X_2)$. First, when player 2 withdraws from contesting asset αV with X_2 in period 2, i.e., $X_2 = 0$, player 1's best reply x_2 is found from inserting $X_2 = 0$ into (4). As in the previous section, x_2 from (4) with $X_2 = 0$, and $X_2 = 0$, are inserted into the players' period 1 expected utilities u and U in (2), causing the same result as in (6). More generally, inserting

[2] The first equation in (7) contains x_1 and not X_1 in the numerators. The second equation in (7) contains X_1 and not x_1 in the numerators. The four denominators in each of the two equations in (7) contain x_1^2 and X_1^2. Multiplying with the four denominators to eliminate them gives two equations with x_1^8 and X_1^8. Hence, the first equation in (7) is rewritten to contain x_1^9 and X_1^8. The second equation in (7) is rewritten to contain x_1^8 and X_1^9.

these four period 2 corner solutions into (2) gives the same period 1 expected utilities u and U as for the interior solution in (6). Second, when player 2 withdraws from contesting asset βW with Y_2 in period 2, i.e., $Y_2 = 0 \iff X_2 = R_2/A_2$, player 1's best reply x_2 is found from inserting $X_2 = R_2/A_2$ into (4). Third, when player 1 withdraws from contesting asset αV with x_2 in period 2, i.e., $x_2 = 0$, player 2's best reply X_2 is found from inserting $x_2 = 0$ into (5). Fourth, when player 1 withdraws from contesting asset βW with y_2 in period 2, i.e., $y_2 = 0 \iff x_2 = r_2/a_2$, player 2's best reply X_2 is found from inserting $x_2 = r_2/a_2$ into (4). Additional corner solutions are illustrated in the next section.

Illustrating the solution

The first subsection of this section illustrates the solution with a symmetric benchmark and the second subsection with an asymmetric benchmark. The model has 18 parameter values $a_i, A_i, b_i, B_i, r_i, R_i, \delta, \Delta, V, W, \alpha, \beta, i = 1, 2$. Because of symmetry, it suffices to analyze the dependence of the six parameter values $a_1, a_2, r_i, \delta, V, \alpha$.

Symmetric benchmark

This subsection assumes the 18 plausible symmetric unitary benchmark parameter values $a_i = A_i = b_i = B_i = r_i = R_i = \delta = \Delta = \mathrm{V} = W = \alpha = \beta = 1$. The unique benchmark solution found in Appendix B is $x_i = y_i = X_i = Y_i = 1/2$, $u_i = u/2 = U_i = U/2 = 1$. The FindRoot command in the Mathematica 13 software (www.wolfram.com) is used to confirm the benchmark solution. In Fig. 1, each of the six parameter values is altered from its benchmark, while the other 17 parameter values are kept at their benchmarks. Altering each parameter value minimally above and below the benchmark enables using the FindRoot command to determine a continuous unique solution through the benchmark. Division of $u/2$ and $U/2$ with 2 is for scaling purposes.

In Fig. 1aa', as player 1's period 1 unit investment cost a_1 into asset V increases above the benchmark $a_1 = 1$, player 1 intuitively chooses lower investment x_1 into asset V in period 1, since the investment becomes more expensive, $\lim_{a_1\to\infty} x_1 = 0$. All limit values are determined numerically. Player 1 invests increasingly into asset W, $\lim_{a_1\to\infty} y_1 = 1$. Player 2's corresponding investment X_1 is inverse U shaped and eventually decreases, $\lim_{a_1\to\infty} X_1 = 0$. Thus, player 2 also invests increasingly into asset W, $\lim_{a_1\to\infty} Y_1 = 1$. Player 2's advantaged and comparatively lower unit investment cost $A_1 = 1 < a_1$ enables it to exploit player 1 in the competition for asset V. In period 2, this gets reversed. Player 1 increases its investment x_2, applying its unit investment cost $a_2 = 1$ to compete more successfully for asset V than in period 1, $\lim_{a_1\to\infty} x_2 = 1$ and $\lim_{a_1\to\infty} y_2 = 0$. Player 2 increases its period 2 investment X_2 more moderately, benefitting from its more successful competition for asset V in period 1, $\lim_{a_1\to\infty} X_2 = 1$ and $\lim_{a_1\to\infty} Y_2 = 0$. Player 1's expected utilities u_i and $u/2$ decrease, while player 2's expected utilities U_i and $U/2$ increase. As a_1 decreases

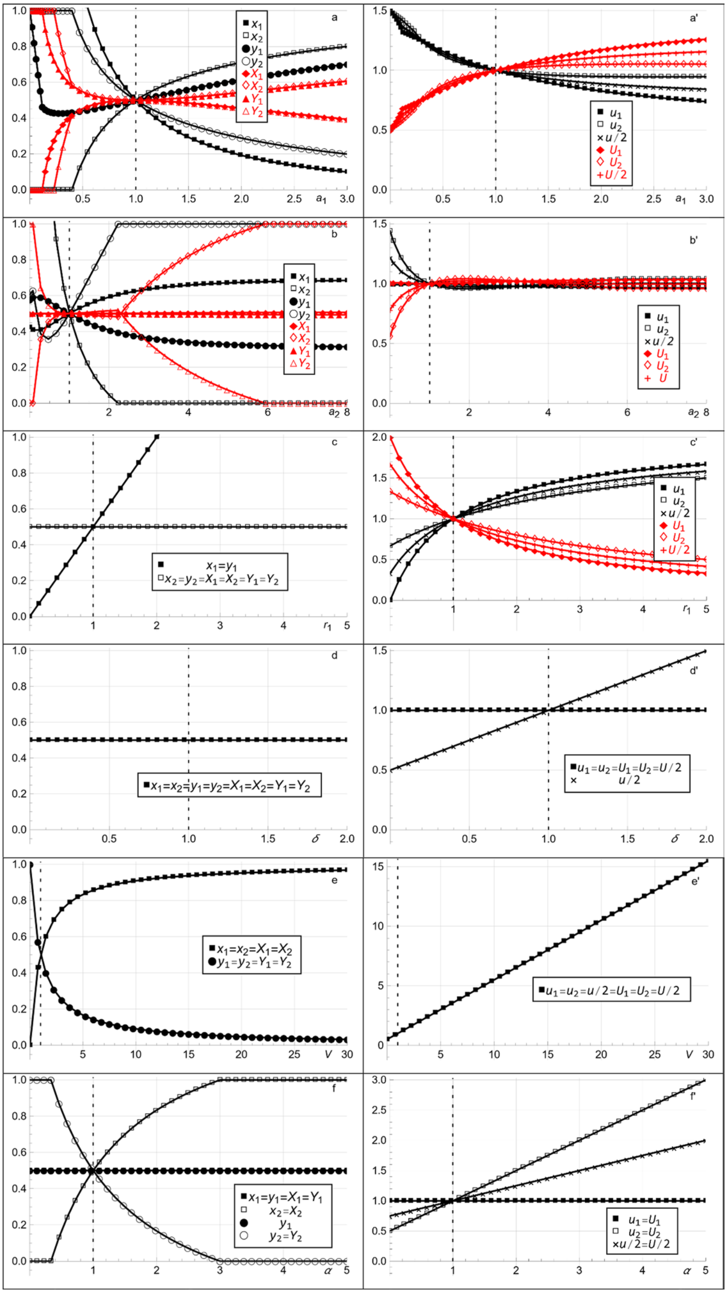

Fig. 1 The players' four strategies choice variables x_i and X_i, four dependent variables y_i and Y_i, and six expected utilities u_i, U_i, $u/2$, $U/2$, $i = 1, 2$, as functions of the six parameter values a_1, a_2, r_i, δ, V, α relative to the benchmark parameter values $a_i = A_i = b_i = B_i = r_i = R_i = \delta = \Delta = \mathrm{V} = W = \alpha = \beta = 1$. Division of $u/2$ and $U/2$ with 2 is for scaling purposes

below $a_1 = 1$, player 1 benefits by investing more x_1 in period 1 into asset V, and eventually more y_1 also into asset W. Player 1 invests less in period 2 into asset V, since the period 1 investment was so cheap, eventually ceasing investment $x_2 = 0$ when $a_1 \leq 0.40$. Player 1 instead invests more y_2 into asset W. That deters player 2 which eventually ceases investing into asset W in period 2, $Y_2 = 0$ when $a_1 \leq 0.23$. Decreasing a_1 further causes player 1 to be so superior for asset V in period 1 that player 2 eventually gives up asset V in period 1, $X_1 = 0$ when $a_1 \leq 0.12$.

In Fig. 1bb', as player 1's period 2 unit investment cost a_2 into asset V increases above the benchmark $a_2 = 1$, player 1 intuitively chooses lower investment x_2 into asset V in period 2, since the investment becomes more expensive, $x_2 = 0$ when $a_2 \geq 2.23$. Player 1 invests increasingly into asset W in period 2, $y_2 = 1$ when $a_2 \geq 2.23$. That deters player 2 from investing into asset W in period 2, $Y_2 = 0$ when $a_2 \geq 5.94$. Player 1 compensates in period 1 by investing more into asset V, $\lim_{a_2 \to \infty} x_1 = 0.64$, and less into asset W, $\lim_{a_2 \to \infty} y_1 = 0.36$. These limit values are intermediate, since period 1 investments carry over to period 2, and both assets V and W are interesting for player 1 in period 2. As a_2 decreases below $a_2 = 1$, player 1 benefits by investing more x_2 in period 2 into asset V. That deters player 2 which eventually ceases investing into asset V in period 2, $X_2 = 0$ when $a_2 \leq 0.08$. Player 2 instead invests increasingly into asset W in period 2, $Y_2 = 1$ when $a_2 \leq 0.08$. Player 1 responds in period 2 with the conventional U shaped investment y_2 into asset W, with a minimum at $y_2 = 0.36$ at $a_2 = 0.43$. As a_2 decreases below $a_2 = 0.43$, player 2 overall increases its investment y_2, since player 2 increases its investment Y_2.

In Fig. 1cc', as player 1's period 1 resources r_1 increase, player 1's period 1 investments $x_1 = y_1 = 0.5r_1$ into both assets V and W increase, while all the other investments are constant, $x_2 = y_2 = X_i = Y_i = 0.5$. Player 1's expected utilities u_i and $u/2$ increase, $\lim_{r_1 \to \infty} u_i = \lim_{r_1 \to \infty} u/2 = 1$, while player 2's expected utilities U_i and $U/2$ decrease, $\lim_{r_1 \to \infty} U_i = \lim_{r_1 \to \infty} U/2 = 1$.

In Fig. 1dd', as player 1's time discount parameter δ increases, all investments remain constant, $x_i = y_i = X_i = Y_i = 1$. This follows from each player having separate resources available in the two periods, i.e., $r_1 = 1$ and $r_2 = 1$ for player 1 and $R_1 = 1$ and $R_2 = 1$ for player 2. The option to roll over resources from one period to the next period gives one additional free choice variable for each player, which can be analyzed in future research. All the expected utilities remain constant, $u_i = U_i = U/2 = 1$, except player 1's expected utility $u/2 = 0.5\delta + 0.5$ over the two periods which increases in δ.

In Fig. 1ee', as asset V increases in value, both players increase their investments into asset V concavely in both periods, $\lim_{V \to \infty} x_i = \lim_{V \to \infty} X_i = 1$, and decrease their investments into asset W convexly in both periods, $\lim_{V \to \infty} x_i = \lim_{V \to \infty} X_i = 1 y_i = \lim_{V \to \infty} x_i = \lim_{V \to \infty} X_i = 1 Y_i = 0$. Both players' expected utilities $u_i = U_i = u/2 = U/2 = 0.5V$ increase.

In Fig. 1ff', as the growth parameter α for asset V increases, no players change their period 1 investments $x_1 = y_1 = X_1 = Y_1 = 0.5$ which remain constant at their benchmarks. In period 2, both players increase their investments $x_2 = X_2$ into asset V concavely until their maxima $x_2 = X_2 = 1$ when $\alpha \geq 3.00$, and decrease their investments $y_2 = Y_2$ into asset W convexly until they both cease investment, i.e., $y_2 = Y_2 = 0$, when $\alpha \geq 3.00$. Although asset W has value $W = 1$, the growth $\alpha \geq 3.00$ of asset V is so substantial that the players cease investing in it in period 2, instead relying on their period 1 investments being forwarded to period 2. As α decreases, the opposite occurs in favor of asset W. Both players increase their investments $y_2 = Y_2$ into asset W convexly until their maxima $y_2 = Y_2 = 1$ when $\alpha \leq 0.33$, and decrease their investments $x_2 = X_2$ into asset V concavely until they both cease investment, i.e., $x_2 = X_2 = 0$, when $\alpha \leq 0.33$. The players' period 1 expected utilities are constant, $u_1 = U_1 = 1$. The players' period 2 expected utilities increase as $u_2 = U_2 = 0.5\alpha + 0.5$. The players' expected utilities over both periods increase as $u/2 = U/2 = \alpha/4 + 3/4$.

Asymmetric benchmark

This subsection assumes, with two exceptions, the same benchmark parameter values as in the previous subsection labelled "Symmetric benchmark", i.e., $a_i = A_i = b_i = B_i = r_2 = R_i = \delta = \Delta = W = \alpha = \beta = 1, r_1 = 2, \quad V = 3$. The two exceptions are introduced to illustrate asymmetries common in practice, and proceed beyond the symmetric solutions. The first exception is that player 1 is assumed to possess twice as much resources $r_1 = 2$ at the benchmark in period 1 compared with its resources $r_2 = 1$ in period 2, and thus also twice as much resources $r_1 = 2$ in period 1 compared with player 2's resources $R_i = 1$ in both periods. The second exception is that the first asset $V = 3$ is assumed to be three times as valuable at the benchmark as the second asset valued at the unitary value $W = 1$ at the benchmark. The unique benchmark solution found in Appendix C is $x_1 = 3/2, x_2 = X_i = 3/4$, $y_1 = 1/2$, $y_2 = Y_i = 1/4$, $u_1 = 8/3$, $u_2 = 12/5$, $u/2 = 38/15$, $U_1 = 4/3$, $U_2 = 8/5$, $U/2 = 22/15$. The FindRoot command in the Mathematica 13 software (www.wolfram.com) is used to confirm the benchmark solution. In Fig. 2, each of the six parameter values is altered from its benchmark, while the other 17 parameter values are kept at their benchmarks. Altering each parameter value minimally above and below the benchmark enables using the FindRoot command to determine a continuous unique solution through the benchmark. Division of $u/2$ and $U/2$ with 2 is for scaling purposes.

In Fig. 2aa', as player 1's period 1 unit investment cost a_1 into asset V increases above the benchmark $a_1 = 1$, player 1's period 1 investment into asset V intuitively decreases, $\lim_{a_1 \to \infty} x_1 = 0$, and conversely for asset W, $\lim_{a_1 \to \infty} y_1 = r_1/b_1 = 2$. Player 2 responds with the conventional inverse U shaped investment V_1 into asset V, which eventually decreases due to player 2 being advantaged with a lower unit investment cost $A_1 = 1 < a_1$, $\lim_{a_1 \to \infty} X_1 = 0$, and conversely for asset W, $\lim_{a_1 \to \infty} Y_1 = 1$. As a_1 increases, the lower value $W = 1 < V = 3$ of asset W induces player 1 to decrease its period 2 investment into asset W more quickly

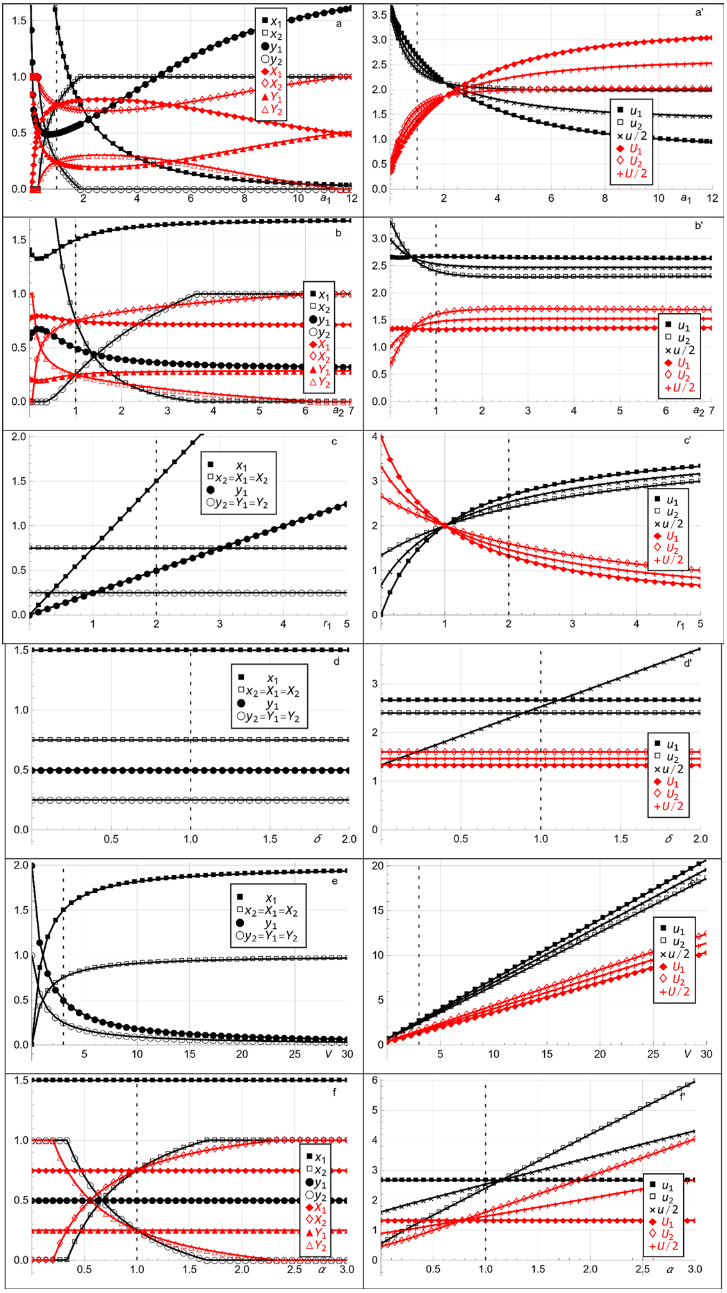

Fig. 2 The players' four strategies choice variables x_i and X_i, four dependent variables y_i and Y_i, and six expected utilities u_i, U_i, $u/2$, $U/2$, $i = 1, 2$, as functions of the six parameter values a_1, a_2, r_i, δ, V, α relative to the benchmark parameter values $a_i = A_i = b_i = B_i = r_2 = R_i = \delta = \Delta = W = \alpha = \beta = 1$, $r_1 = 2$, $V = 3$. Division of $u/2$ and $U/2$ with 2 is for scaling purposes

than in Fig. 1aa', to $y_2 = 0$ when $a_1 \geq 1.89$. Hence, player 1's period 2 investment into the valuable asset V increases more quickly than in Fig. 1aa', to $x_2 = 1$ when $a_1 \geq 1.89$. That induces player 2's period 2 investment into asset W to be inverse U shaped and decrease to $Y_2 = 0$ when $a_1 \geq 11.57$, which also occurs more quickly than in Fig. 1aa'. Hence, player 2's period 2 investment into the valuable asset V increases to $X_2 = 1$ when $a_1 \geq 11.57$. As a_1 decreases below $a_1 = 1$, the players invest similarly to Fig. 1aa', but more heavily into the more valuable asset V. Player 1 ceases period 2 investment into asset V, and player 2 ceases period 2 investment into asset W, when $a_1 \leq 0.31$. That value is lower than $a_1 \leq 0.40$ in Fig. 1aa', since asset $V = 3$ is more valuable in Fig. 2aa'.

In Fig. 2bb', as player 1's period 2 unit investment cost a_2 into asset V increases above the benchmark $a_2 = 1$, player 1 invests more into the valuable asset $V = 3$ than in Fig. 1bb', but eventually ceases to do so in period 2, $x_2 = 0$ when $a_2 \geq 3.64$, which is higher than $a_2 \geq 2.23$ in Fig. 1bb'. Player 1 instead invests into the less valuable asset $W = 1$ in period 2, $y_2 = 0$ when $a_2 \geq 3.64$. That overwhelms player 2 which eventually ceases period 2 investment into asset W, $Y_2 = 0$ when $a_2 \geq 6.28$. In period 1 with unit cost $a_1 = 1$, player 1's resourcefulness $r_1 = 2$ induces it to invest more heavily into the valuable asset V than in Fig. 1bb', $\lim_{a_2 \to \infty} x_1 = 1.69$, and less into asset W, $\lim_{a_1 \to \infty} y_1 = 0.31$. The less resourceful player 2 with $R_1 = 1$ also invests more into asset V, $\lim_{a_1 \to \infty} X_1 = 0.72$ versus $\lim_{a_1 \to \infty} Y_1 = 0.28$. As a_2 decreases below the benchmark $a_2 = 1$, player 1 in period 2 decreases investing in the less valuable asset W more quickly than in Fig. 1bb', and eventually ceases to do so, $y_2 = 0$ when $a_2 \leq 0.36$. Player 1 instead invests heavily with x_2 into the more valuable asset V. That eventually deters player 2 which ceases investing into asset V in period 2, $X_2 = 0$ when $a_2 \leq 0.05$.

In Fig. 2cc', as player 1's period 1 resources r_1 increase, the players' two investment curves in Fig. 1c get split into four curves. Since asset V is three times more valuable than asset W, player 1's investment $x_1 = 3r_1/4$ into asset V increases three times faster than $y_1 = r_1/4$ into asset W. The other investments are constant, i.e., $x_2 = X_i = 0.75$ for asset V and $y_2 = Y_i = 0.25$ for asset W. The players' expected utilities u_i, $u/2$, U_i, $U/2$ have the same structure as in Fig. 1c', but are higher because of the more valuable asset V.

In Fig. 2dd', as player 1's time discount parameter δ increases, the players' one investment curve in Fig. 1d gets split into four curves. Since asset V is three times more valuable than asset W, in period 1 when player 1 has resources $r_1 = 2$, its investment $x_1 = 3/2$ into asset V is three times higher than $y_1 = 1/2$ into asset W. In period 2 when both players are equally resourceful $r_i = 2$, their investments $x_1 = X_i = 3/4$ into asset V are three times higher than $y_1 = Y_i = 1/4$ into asset W. The players' expected utilities u_i, U_i, $U/2$ are constant, while player 1's utility $u/2 = 4/3 + 6\delta/5$ increases linearly in δ.

In Fig. 2ee', as the high-value asset V increases in value, the players' two investment curves in Fig. 1c get split into four curves. Player 1's period 1 investment x_1 increases to approach $\lim_{V \to \infty} x_1 = 2$ which is twice that of Fig. 1e, while $\lim_{V \to \infty} y_i = \lim_{V \to \infty} Y_i = 0$. Player 1's period 2 investment $x_2 = X_i$ increases as in

Fig. 1e to approach $\lim_{V\to\infty} x_2 = \lim_{V\to\infty} x_2 = 1$. Summing up, increasing asset value V or W causes both players to invest more in the valuable asset in both periods, and less into the less valuable asset. The advantaged player 1 ensures investing more into the less valuable asset than the least advantaged player 2.

In Fig. 2ff', as the growth parameter α for asset V increases above the benchmark $\alpha = 1$, no players change their period 1 investments x_1, y_1, X_1, Y_1 which remain constant at their benchmarks. In period 2, both players increase their investments x_2 and X_2 into asset V concavely, and decrease their investments y_2 and Y_2 into asset W convexly. Player 1 does so until $\alpha \geq 1.67$ after which $x_2 = 1$ and $y_2 = 0$. Player 2 does so until $\alpha \geq 2.33$ after which $X_2 = 1$ and $Y_2 = 0$. As α decreases, player 1 ceases investment into asset V, i.e., $x_2 = 0$ and $y_2 = 1$, when $\alpha \leq 0.33$. Player 2 ceases investment into asset V, i.e., $X_2 = 0$ and $Y_2 = 1$, when $\alpha \leq 0.20$. Summing up, high-growth parameter α for asset V or β for asset W causes both players to invest more into the valuable asset in period 2, until first the advantaged player 1 and thereafter player 2 eventually cease investment into the asset with low growth parameter. Low growth parameter α for asset V or β for asset W causes both players to invest less into the less valuable asset in period 2, until first the advantaged player 1 and thereafter player 2 eventually cease investment into the asset with low growth parameter.

Corner solutions

Table 1 presents the eight corner solutions in Figs. 1 and 2.

Additional corner solutions, left for the reader or future research, follow from permuting players 1 and 2, and by assuming combinations of extremely high and extremely low parameter values. For example, $(x_1, x_2, X_1, R_2/A_2)$ follows from permuting players 1 and 2 in $(x_1, r_2/a_2, X_1, X_2)$ in row 4 in Table 1.

Summarizing key results

The model provides six interesting results for how competing players allocate resources across two assets over two periods.

First, when player 1's period 1 unit cost a_1 of investing into the high-value asset V is low, player 1 invests substantially $x_1/2$ into the cheap asset V in period 1. That carries over to period 2 when player 1 does not need to invest, i.e., $x_2/2 = 0$, into the then more expensive asset V. An indirect effect is that player 1 invests substantially y_2 in asset W in period 2. Player 1 chooses the opposite strategy when a_1 is high.

Table 1 The eight corner solutions in Figs. 1 and 2

x_1, x_2, X_1, X_2	Figures	Explanation
$x_1, x_2, X_1, 0$	Figure 1b	Player 2 is disadvantaged for asset V in period 2 due to player 1 being advantaged with very low a_2
$x_1, 0, X_1, X_2$	Figure 1ab Figure 2abf	Player 1 is disadvantaged for asset V in period 2 due to high a_2, or advantaged with low a_1 in period 1, or disadvantaged with low α for asset V in Fig. 2f
$x_1, r_2/a_2, X_1, X_2$	Figure 2abf	Player 1 is advantaged with low a_2 for asset V period 2, disadvantaged with high a_1 in period 1, or advantaged with high α in Fig. 2f
$x_1, 0, X_1, 0$	Figure 1f Figure 2f	Both players are disadvantaged with low α for asset V
$x_1, 0, X_1, R_2/A_2$	Figure 1ab Figure 2ab	Player 1 is disadvantaged for asset V in period 2 due to high a_2, or advantaged with low a_1 in period 1, causing player 2 to be advantaged for asset V
$x_1, r_2/a_2, X_1, 0$	Figure 2b	Player 1 is advantaged for asset V in period 2 due to low a_2, causing player 2 to be disadvantaged for asset V
$x_1, r_2/a_2, X_1, R_2/A_2$	Figure 1f Figure 2af	Player 1 is disadvantaged with high a_1 for asset V period 1, causing advantage for both players for asset V in period 2. Both players are advantaged with high α for asset V
$x_1, 0, 0, R_2/A_2$	Figure 1a Figure 2a	Player 1 is advantaged with low a_1 in period 1, causing player 2 to be disadvantaged for asset V in period 1, causing player 1 to be disadvantaged for asset V in period 2, and player 2 to be advantaged for asset V in period 2

Second, when player 1's period 2 unit cost a_2 of investing into the high-value asset V is low, player 1 invests marginally $x_1/2$ in asset V in period 1, instead waiting to invest substantially $x_2/2$ in asset V in period 2. An indirect effect is that player 1 invests substantially y_1 in asset W in period 1. Player 1 chooses the opposite strategy when a_2 is high.

Third, when player 1's period 1 unit cost b_1 of investing into the low-value asset W is low, player 1 invests substantially y_1 into the cheap asset W in period 1. That carries over to period 2 when player 1 does not need to invest, i.e., $y_2 = 0$, into the then more expensive asset W. An indirect effect is that player 1 invests substantially $x_2/2$ in asset V in period 2. Player 1 chooses the opposite strategy when b_1 is high.

Fourth, when player 1's period 2 unit cost b_2 of investing into the low-value asset W is low, player 1 invests less y_1 in asset W in period 1, instead waiting to invest substantially y_2 in asset W in period 2. An indirect effect is that player 1 invests more $x_1/2$ in asset V in period 1. Player 1 chooses the opposite strategy when b_2 is high.

Fifth, when the asset value V or W increases, both players invest more in the valuable asset in both periods, and less into the less valuable asset. When player 1 is advantaged with more abundant period 1 resources $r_1 = 2$, it invests more into the less valuable asset than the least advantaged player 2.

Sixth, when the growth parameter α for asset V or β for asset W increases, both players invest more into the valuable asset in period 2, until the advantaged player 1 with more abundant period 1 resources $r_1 = 2$ eventually ceases investment into the asset with low growth parameter.

Conclusion

The article considers two resource constrained players in a Colonel Blotto game competing through investments over two assets which may increase or decrease in value over two periods. This conceptualization purifies some basic strategic concerns of investors facing multiple investment opportunities over time. A player's investment in period 1 carries over to period 2 when the player may add to its investment according to its new resource constraint in period 2. Intuitive and less intuitive results are interpreted.

When a player can invest cheaply in an asset in period 1, it does so, and invests less in the asset in period 2. The player accordingly invests less in the other asset in period 1, and more in the other asset in period 2.

When a player can invest cheaply in an asset in period 2, it does so, and invests less in the asset in period 1, in anticipation of period 2. The player accordingly invests more in the other asset in period 1, and less in the other asset in period 2.

When an asset value increases, both players invest more in it in both periods, and less into the less valuable asset. An advantaged player with more abundant period 1 resources may invest more into the less valuable asset than the least advantaged player.

When the growth parameter of an asset increases, both players invest more into it in period 2, until the advantaged player with more abundant period 1 resources eventually ceases investment into the asset with low growth parameter, to exploit the opportunities of the high-growth asset.

One policy implication of the model is that to the extent, some actor can structure, design, or impact asset valuations and investment costs through time, the actor should be cognizant of how investors allocate resources across assets depending on how assets appreciate or depreciate through time, how investment costs may change, and how investors may discount time.

Future research may generalize to more than two players, more than two assets, more than two periods, and more sophisticated growth and deterioration developments for each asset. Although players often prefer to or are required to utilize their resources in specified periods, sometimes under the threat that they will otherwise lose the resources, the players may be allowed to roll over parts of their available resources to subsequent periods. Players which are not resource constrained, in the sense that they do not use all their available resources in one or both periods, may be analyzed. For the model in this article that gives one additional free choice variable for each player in each period. The model can be tested empirically against real investors' strategies, identifying their portfolios, resource allocation, and unit investment costs for two assets at two points in time, determining the two asset values at the two points in time, and estimating the two investors' time discount parameters.

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Declarations

Conflict of interest No conflict of interest exists.

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