Conceptions of teaching and justice as pivotal to mathematics teacher educators' thinking about mathematical knowledge for teaching

Mark Hoover[1]
University of Michigan School of Education
Ann Arbor, MI, USA
mhoover@umich.edu (corresponding author)

Matthew Dahlgren[1]
University of Michigan School of Education
Ann Arbor, MI, USA

Reidar Mosvold[2]
University of Stavanger Faculty of Arts and Education
Stavanger, Norway

Imani Goffney[3]
University of Maryland School of Education
Silver Springs, MD, USA

Abstract
Recent scholarship has explored mathematical demands faced by mathematics teacher educators and ways to support their development, but little attention has been given to the basic question of how mathematics teacher educators think about content knowledge for teaching. Knowing what they think could inform efforts to support them. Our analysis reveals that some think about mathematical knowledge for teaching as an independent, abstracted resource to be taught and learned in relative isolation from teaching, while others think about it as dynamic, situated work. We argue that this key difference matters for how they work with teachers. Further, our analysis reveals that their thinking about both teaching and justice interacts with their thinking about mathematical knowledge for teaching and that their thinking in these other two domains can be a resource for supporting their mathematical development. [4]

Keywords Mathematics teacher educators · Mathematical knowledge for teaching · Teaching · Justice · Equity

The mathematical education of teachers is key to improving students' mathematics learning, yet providing adequate mathematical education for teachers at scale has not been straightforward (Blömeke et al., 2014). One reason for this is that the mathematics needed is substantial (CBMS, 2001; Ma, 1999). Beyond knowing the mathematics being taught, teachers need to know how to use representations, what counts as a mathematical explanation, how to hear students' mathematical ideas, and much more. As the field has come to understand the mathematical demands of teaching and turned attention to supporting teachers in learning such mathematics, scholars have raised questions about the demands for mathematics teacher educators (Krainer et al., 2021). What do mathematics teacher educators need to know and be able to do? At first glance, this may seem excessive self-contemplation. After all, mathematics teacher educators are established professionals, often researchers themselves, often with more expertise than any

[1] University of Michigan School of Education, Ann Arbor, MI, USA. mhoover@umich.edu (✉)
[2] University of Stavanger Faculty of Arts and Education, Stavanger, Norway.
[3] University of Maryland School of Education, Silver Springs, MD, USA.
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other group. But the group is heterogenous (Goos & Beswick, 2021), expertise within it differs (Castro Superfine & Pitvorec, 2021), and overlooking support for them often undermines teaching and learning improvement efforts (Borko et al., 2014; Elliott et al., 2009). Indeed, the *mathematics teacher educators* (MTEs) responsible for educating prospective or practicing teachers include mathematicians, teacher educators, professional developers, lead teachers, and others. They may have extensive background in mathematics or not. They may be skillful elementary or secondary teachers or not. They may have studied teaching or not. They may know much about the historical, political, economic, and social contexts of schooling or not. In addition to needing expertise in all these areas, MTEs need a mix of meta and integrated knowledge and skill beyond immediate domain knowledge.

Reviewing the growing research on developing mathematics teacher educators, we find two compelling insights: (i) the mathematical demands of mathematics teacher education are significant; and (ii) central to these demands is skill in connecting such knowledge to the practice it is meant to serve. Just as teacher knowledge subsumes the mathematics students must learn, scholars who study MTEs and their development have viewed MTEs' expertise as meta-knowledge-and-skill, subsuming levels within it (Beswick & Goos, 2018). The nested nature of this meta-knowledge is illustrated in Prediger et al.'s (2019) three-tetrahedron model, which starts with the didactical triangle as a foundation for understanding the content-demands of teaching and extends this to nested tetrahedrons for MTEs and their development.

The focus in this literature, though, has been on the mathematics MTEs need to know and approaches to supporting their development (Appova et al., 2020; Goos & Beswick, 2021; Krainer et al., 2021). More basic questions, about MTEs themselves, have received much less attention. Some effort has been given to investigating who MTEs are (Masingila et al., 2012), but which features of their background deserve attention remains unclear. In addition, Li and Castro Superfine (2018) have investigated MTEs' perspectives on their design of content courses, including their goals, approaches to achieving those goals, and challenges faced. We know little, though, about MTEs' conceptions of mathematical knowledge for teaching (MKT).[5] Given its meta character and the many ways researchers have sought to conceptualize it (e.g., Ball et al., 2008; Carrillo-Yañez et al., 2018; Davis & Renert, 2013; McCrory et al., 2012; Rowland et al., 2005; Wasserman, 2018), it would hardly be surprising to find that MTEs think about MKT in fundamentally different ways and that these differences shape what MTEs do with teachers and lead to a system of mathematics teacher education that lacks coherence and effectiveness. Knowing how they think about such knowledge could inform approaches to supporting their development.

In addition, if the mathematical knowledge that teachers and MTEs need is fundamentally connected to practice, then MTEs' conceptions of teaching are necessarily crucial and likely differ in unknown yet potentially significant ways. For as Lortie (1975) has argued, individualized notions about teaching are the norm and we lack technical knowledge of teaching. One further addition is that if teaching is truly in service of students, communities, and a democratic society, then conceptions of justice are also crucial and similarly may differ in unknown yet potentially significant ways. The importance of the social and political environment for the work of mathematics teaching has been noted by other scholars as well (e.g. Ball, 2018; Jaworski & Potari, 2009.)

In this paper, we investigate the following question:

> *How do MTEs conceptualize MKT and what shapes their thinking in ways that might matter for their professional growth?*

To understand how MTEs differ in their thinking and how these matter, we interviewed MTEs who attended four-day workshops on MKT offered by our larger project team. It should be noted that we are not studying the workshops, but rather are using our extended time and experience with these MTEs to inform our interpretations of how they understand, think about, take up, and work with teachers on MKT.

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[5] We use the phrase "mathematical knowledge for teaching" and its acronym to refer broadly to knowledge important for teaching.

Background for the study

This study grew from a project supporting MTEs' skill in developing instructional tasks addressing MKT. The larger project team conducted a dozen workshops of approximately 30 participants each. MTEs applied as teams of individuals with different professional roles in their local area. We began workshops by introducing MKT and task writing as characterized by Ball (2017). In addition to developing tasks addressing familiar mathematical work of teaching, such as evaluating mathematical representations or choosing an example to make a mathematical point, we introduced four ways teaching can advance a more just society:

- By seeing one's identity and role as part of a broader system
- By deliberately disrupting patterns of marginalization, exclusion, or oppression
- By seeing individual students — their strengths and their mathematical work
- By skillfully opening up "content" and possibilities for students to connect with and do mathematics

For each of these, we considered associated mathematical demands and tasks. Participants used videos of teaching to discuss the mathematical work of teaching and get ideas for tasks. Working in whole-group and small-groups, they engaged with tools for generating and reviewing tasks.

For this study, we interviewed 12 participants, with different demographics, professional affiliation, and experience (Table 1). Sampling was purposeful across roles and experiences. Extensive pre-workshop applications and post-workshop surveys were also available, including responses to specialized content knowledge tasks with participants' detailed explanations. Following Patton (1990), a guided interview protocol served multiple purposes: to gather feedback on the workshop, gain insight into MTEs' understanding of MKT, and inform future development efforts. Interviews were conducted within a week or two of a workshop, via video conferencing, recorded, and transcribed.

Table 1: Interviewees' background information.[6]

Name	*Gender*	*Race*	*Workshops Attended*	*Primary Affiliation*	*Degree*	*Elementary or Secondary*	*School Teaching Exp*	*Primarily Teacher Ed or Prof Dev*	*Courses Taught*
Adi	M	Asian	3	Educ Dept	Math ed (doctorate)	Both	10+ yrs.	TE	Both
Andy	M	White	2	K-12 school	Math (masters)	Sec	2 yrs.	TE	Content
Claire	F	White	2	Math Dept	Math ed (masters)	Sec	3 yrs.	TE	Methods
Gwen	F	White	2	Math Dept	Math ed (doctorate)	Elem	2 yrs.	TE	Content
Kiara	F	Black	1	Educ Dept	Math ed (doctorate)	Both	10+ yrs.	TE	Both
Liz	F	White	2	Math Dept	Math ed (doctorate)	Both	10+ yrs.	TE	Both
Meg	F	White	2	Educ Dept	Math ed (doctorate)	Both	5 yrs.	TE	Content
Naima	F	Black	1	K-12 school	Education (masters)	Both	6 yrs.	PD	Both
Rose	F	White	2	K-12 school	Science ed (masters)	Sec	10 yrs.	PD	Content
Sarah	F	White	2	State Ed Dept	Education (Masters)	Both	10+ yrs.	PD	Both
Teresa	F	Latinx	1	K-12 school	Math ed (bachelors)	Sec	10+ yrs.	PD	Both
Walt	M	White	3	Educ Dept	Math (doctorate)	Both	0 yrs.	TE	Both

Interviews unfolded in four phases:

1. Phase 1: Background and reasons for applying
2. Phase 2: What interviewees learned and found useful
3. Phase 3: Perspectives on a videoclip of a workshop discussion

[6] Names are pseudonyms, and other information is self-identified; K-12 are the U.S. primary and secondary school years.

4. Phase 4: Thoughts about how to support other MTEs' engagement with MKT.

Each phase included several questions, each with a series of follow-up prompts. As an example, for phase 2, specifically for participants who attended multiple workshops, there was a series of four questions:

- Can you tell me a bit about your experience with the first workshop you attended? Why did you apply to be a part of the workshop? What was it like? And what did you take with you from your first workshop?
- How many workshops have you attended? Why did you reapply each time?
- I'd like to get a sense of your experience at this most recent workshop. We'll start pretty broadly. Could you describe your experience at the workshop?
- Do you think your experience at this recent workshop will impact your own professional work, perhaps with teachers, in courses, or with colleagues? If so, in what ways?

The probes for the last of these were:

- Has your perception of SCK[7] changed?
- Has your thinking about issues of equity and social justice in mathematics teaching changed?
- Do you have plans for the tasks you created?
- What about plans for future collaborations with cross-professional groups?
- What supports would be most helpful to facilitate a more meaningful integration of SCK into your work?

During the third phase of the interview, we asked participants to watch two short video clips and respond to some questions about them. We first showed a 3 ½ minute video from a 5th grade class that we had all watched together at the workshop. In this video, students discussed the fraction associated with a mark on a number line. We then showed a 6 ½ minute video of a discussion workshop participants had about student thinking and mathematical demands of teaching implicated in the 5th grade class video. During a second viewing, we would stop after a significant contribution and ask what the interviewee heard the person in the clip saying, what they made of what was said, and what stood out to them. Another set of questions addressed the discussion as a whole. The many layers of meaning in this situation helped us see what MTEs were attending to and not able to attend to and provided invaluable insight into their thinking. The interview was not designed to directly ask how MTEs viewed MKT, teaching, or justice, but each phase of the interview surfaced these in different ways.

Interpretive method

Our analytic approach was empirically grounded and interpretive. We worked inductively, moving iteratively between data and relevant theoretical and practice perspectives (Erickson, 1986; Geertz, 1973; Gerring, 2001). Erickson (1986) distinguishes interpretive approaches as "using as a basic validity criterion the *immediate and local meanings of actions*, as defined from the actors' point of view" (p. 119). Our analysis coordinated different perspectives, practical concerns, and logical consideration, driven by local meaning and an assumption that what people say is sensible from their perspective.

We conducted a thematic analysis, with initial themes informed by narratives written for a preliminary analysis (Dahlgren et al., 2019). Themes were then refined during coding of three interviews, at which point we coded all interviews (including the three previously coded). We identified salient differences in MTEs' thinking in three domains: MKT, teaching, and justice. Our unit of analysis was excerpts of text extended enough to allow interpretation of what is said, yet selective enough to indicate how the talk serves as evidence of thinking. Directly

[7] Specialized content knowledge (SCK) is a domain of MKT characterized by Ball et al. (2008) and often used at the workshop.

viewing video recordings, we individually nominated excerpts, coded excepts, and wrote explanatory memos. We then worked together to establish agreement on interpretations and codes and wrote a collective memo. When uncertainty or inconsistency arose, we re-watched, explored different interpretations, imagined from the interviewee's perspective, and set aside to revisit later or to consult with an outside party.

For instance, one of the most challenging interviews was conducted in a noisy coffee shop by a Korean researcher with a black woman from the southeastern United States. The three men analyzing the video (a white European, a white American, and a biracial Latino American) found it hard to hear and hard to understand, with some initial concern that the interview was not being conducted seriously enough to be of value. Instead of removing the interview as unusable, we listened repeatedly, over time, trying out different interpretations. What initially appeared to be perfunctory or even incoherent exchange, eventually became thoughtful, cogent, and consistent, both questions and comments. As our ability to hear grew, interpretation became consistent throughout the interview and among us. We checked our interpretation with other data sources: the interviewee's application, her post-workshop survey, video of her participation in the workshop, and had another black, female MTE watch and check our interpretation.

We generated initial codes directly from video because we found that interviewees' meanings were easier to interpret from the richer medium. We recorded excerpts and codes in transcripts and used transcripts for analysis once robust interpretations were firmly established. We then developed meta-codes that distinguished the salient character of more- and less-developed thinking in the domain. Exploratory data analysis (Tukey, 1977) provided consolidated means for representing patterns in the codes and testing them against our interpretations. The first three authors participated in all aspects of the analyses, with three other project staff, and the last author contributed to the analysis of thinking about justice.

As a window on the interview data, we begin with compressed narratives for two contrasting MTEs. We then report on differences in MTEs' conceptions in each domain and examine alignment in thinking across the domains and its relationship to background variables.

Teresa

Teresa identifies as Latina. She has a bachelor's degree in mathematics education and more than 10 years of teaching experience in K–12 settings, with aspirations of pursuing an advanced degree if opportunity arises. Having worked primarily with teachers in professional development contexts, Teresa considered participating in the workshops to be "an opportunity to hit some of the goals that I've had in learning how to really push teachers' thinking and helping them think about what they're doing in a structured way." Teresa was familiar with literature on MKT before attending the workshops and had used released MKT assessment tasks in reform work with teachers. When asked about what she hoped to gain from the workshop, Teresa said she hoped to learn to work with teachers in ways she was already able to do with children:

> *So, in my classroom, in hearing what students were saying, I was able to interpret what would be the next best step for them and so I was able to select a task to put it in front of them to move their thinking forward.*

She then continued to explain that this was the kind of knowledge she was hoping to help teachers learn. When asked about how her thinking about MKT had changed during the workshop, she talked about a new regard for social justice. Using a metaphor of an equalizer board in music, she explained that teachers occasionally must "amplify somebody's voice." She talked about how she now considered attention to social justice to be entailed in MKT. In addition to being able to "look at the mathematical work of students" and "build from what they're saying," she had come to realize that attending to students' identities is crucial and that she needs to foreground it in her work with teachers.

> *And now I found it seems like a simple solution: tell the whole story. Paint the picture, tell who this student is so that it is part of how they are making their decisions when they have this much time to think about it versus in the classroom we have a split second to make that decision. That, I think, is my biggest aha!*

Later in the interview, when asked to comment on the video of the workshop discussion in phase 3, she first remarked how her focus is typically on what is going on mathematically in a lesson, and "helping students make those connections and merge those ideas." She then reflected on how using videos of teaching in tasks might highlight the real-time nature of the work. Discussing such videos in the workshop had influenced her own thinking, as she had realized the importance of attending to students who might routinely be dismissed as disruptive. These discussions prompted her to interrogate her own practice: "How did I position them to be valued for their contributions and not dismissed?" Furthermore, the discussion pushed Teresa's thinking about how she could help teachers "notice that moment that they have the opportunity to capitalize on students' thinking," and, at the same time, attending to "all these other aspects of social justice or context and what makes them [students], them."

While Teresa's thinking about MKT used to emphasize attention to students' mathematical thinking, she has now come to realize that student identities and consideration for "who this student is" has been missing even though it matters for the mathematical work. Toward the end of the interview, Teresa explained, "Miriam's contribution just kept highlighting to me how I tend to focus on the math, and how I want to be able to keep my vision open, like I want to promote attending to students' thinking as we attend to the whole child." Teresa also reflected on how knowing math is not enough and suggested that there is a need to develop a shared understanding of what is involved in "making this real and helping students." After the workshops, her increased attention to student identities, positioning, and access has prompted development of her thinking of MKT as well as of teaching. She used to think about MKT mostly in terms of listening to, and interpreting students' mathematical thinking, with an accompanying reform-oriented approach to teaching that emphasized "orchestrating productive discussions" instead of "direct explicit instruction," and she has developed clear and robust language to talk about this. Her thinking about MKT is now developing to include attention to issues of social justice, and her thinking about teaching is now developing to include attention to student identities and considerations of access.

Adi

Adi identifies as Asian and has studied and worked internationally, with more than 10 years of experience teaching K–12. He has a PhD in mathematics education and currently works in a school of education in the United States where he teaches both content and methods courses. His interview began with his reflections on experiences at the two workshops he attended. Referencing articles sent in preparation, Adi expressed appreciation for "the idea that teachers are not born, they should be able to be taught." He then described a task he developed during the first workshop, where the addition of two fractions, in different word problems, takes on different meanings. Another task combined addition and division of fractions. With such tasks, Adi tried to help prospective teachers understand definitions and "understand that the context of a word problem matters."

His focus in the first workshop was on developing tasks that he could use in his own courses, but at the second his attention shifted. "I would say that the first thing that struck me — and this may be the first time I really listened — is the importance of social justice." He valued this shift, and it led him to a new awareness of teaching. In his interview, he specifically pointed to the practice of being intentional when selecting students to contribute mathematically as eye-opening. When prompted about what he found most interesting at the workshops, he replied, "the videos, the discussions, the feedback, and how social justice was interwoven with the mathematical work of teaching."

Despite his enthusiasm, however, Adi spoke of being uncertain about core terminology. He described his struggle to understand how MKT was different from the mathematical work of teaching, and he expressed that, after conversation with a fellow participant, he started to "think of SCK as a noun, and the mathematical work as a teacher as a verb." This helped him identify that "SCK is a foundation, the knowledge that you have to have" in order to teach mathematics. This talk about MKT as something static, which is separated from the more dynamic work of teaching, was repeated by Adi throughout the interview.

In phase 3 of the interview, when reflecting on the 5th grade video and the workshop discussion about it, Adi remarked that he that he "wouldn't have thought it was an intentional choice" to invite this particular student to

present, because her answer was "quite a way out off base." He described how the discussion about the 5th grade video made him realize that calling on "African American students", who are often "considered disruptive" in class, can be an intentional pedagogical move. He acknowledges that he "would never have known that" but that he has a new regard for being deliberate about such matters. The last part of the interview was devoted to questions about future directions, where he repeated that learning a process for developing tasks has been useful to him. He was intrigued by the idea of having a repository of SCK tasks for use in teacher education and would like to contribute to the development himself, although he feels like he is "still learning" and is not sure how much he can contribute.

In his interview, Adi's language conveys an image that treats MKT as distinct from the "mathematical work of teaching." The former is a "noun", while the latter is a "verb". Whereas other participants who attended more than one workshop commented on social justice being more prominent at the most recent workshop, Adi's attention was as much on noticing teaching in new ways. He repeatedly spoke of how invaluable the discussions about the videos of teaching had been for him and the ways they opened his eyes to the many perspectives that bear on decision-making in teaching. And he carried his new-found capacity to see teaching into his distinction between "mathematical knowledge" and "mathematical work," the former being mathematics and the latter being teaching. In contrast, other participants often used these two phrases as part and parcel of a single phenomenon, or even interchangeably.

Not only did Adi's language distinguish MKT from mathematical work of teaching, but he also used this distinction in his reasoning. For instance, he repeatedly made the point that MKT is a prerequisite for mathematics teaching. "That is why I am saying that SCK is just kind of a necessary condition but not sufficient for the mathematical task of teaching, which I consider … is the enaction." This thinking of mathematical knowledge as a prerequisite is visible in his comments about pre-service teachers needing to be able to solve fraction problems for themselves and in his reflections on the 5th grade classroom video, where he explains that simply being able to identify a student response as wrong is not sufficient to teach well.

This separation of MKT from teaching is also visible in the tasks he generated. Although task development in the workshops focused on developing tasks bound up in the mathematical demands of teaching, Adi's tasks were more familiar mathematics problems, often tasks that children might well be expected to solve or tasks that emphasized content-based ideas instead of pedagogically based issues. For instance, in the fraction problems mentioned above, Adi was concerned with addressing mathematically distinguishable meanings of operations, where different fraction problems give different results even though they look quite similar. He did not design the problems to surface mathematical demands that arise in teaching, such as recognizing common ways students think about this content or how to check student understanding.

Salient differences in MTEs' thinking

To understand better how MTEs think about MKT and how their conceptions of teaching and justice might play a role, we looked for differences in their thinking indicative of professional growth and capacity for writing high-quality tasks.

Resource-based versus practice-based conceptions of MKT

The theme that emerged in our analysis of MTEs' conceptions of MKT was a difference between viewing it as *appropriately removed from, or abstracted out of, direct connections to teaching (resource-based)* or as *necessarily situated in specific instructional contexts (practice-based)*. We coded for this distinction as it was evident in relation to (i) mathematics curriculum and content, (ii) students and their thinking, (iii) environment (context and concern for justice), and (iv) in general. Different types of evidence established thinking as practice-based: reference to artifacts of practice such as student work or curricular materials; use of labels such as "for teaching", "practice-based", "specialized", "what teachers need to know"; and talking explicitly about tasks of teaching when talking about MKT. We distinguished strong and weak forms of these codes as well.

In looking back at our codes, we noticed that the reasoning in excerpts coded as resource-based were often grounded in disciplinary mathematics. For example, when talking about what she learned about MKT, Gwen expresses her new insights in mathematical terms, with little reference to instructional context. She says she now sees and values the additional mathematical work of connecting actions with models to ways of recording, step by step.

> *One of the things that actually struck me at this workshop that I have never really thought about before was the difference between solving a problem by using a model and explaining an algorithm, how the algorithm comes out in the model. And so, what I often do is I have my students* [teachers] *use a model to solve a problem and then explain the model, but what I don't really do is relate that model to the actual algorithm.*

She might value this mathematical work because connecting the physical actions to the symbolic manipulations is key to supporting children's learning, but the surrounding discussion gives the impression that she instead sees this as a new mathematical insight that is worth attending to but not particularly situated in teaching. Similarly, Meg talks about the value of MKT tasks based on mathematical appreciation, absent pedagogical appreciation.

> *They* [prospective teachers] *get very, very excited when they see the correlation between the bar graph and the pie chart. It's just, they've never thought of it, and it's one of those aha moments for them … They love the activity, and they just get excited about seeing the connections that they never think of making.*

Meg recognizes that the referenced task is contextualized in the pedagogical work of helping children identify connections among representations, but as with Gwen, her enthusiasm, and that of her students is for the mathematics without reference to pedagogical work.

In addition to talking about mathematical merit without reference to pedagogical work, some, such as Adi, talked about MKT as a required resource for teaching, where tasks, even when inspired by a video of teaching, extracted mathematics out of its pedagogical context.

In contrast, comments in other interviews referred to specific pedagogical work and contextually relevant aspects of teaching. For instance, Naima talks explicitly about the need to consider mathematical work in situ.

> *Teachers need this content knowledge, but they need like broad knowledge of content, but they also need a really specific knowledge of mathematics, the math they need to have in order to like teach and to see what kids are doing, to see how kids are accessing problems, and the work is actually then what they're doing with that knowledge and what they're doing with the data they're gathering in that moment. Like how are they facilitating a classroom environment with that understanding?*

She describes MKT as mathematics that arises inside the work, as knowledge involved in seeing, hearing, and understanding children. Where Adi developed tasks that abstracted MKT out of and away from teaching, Naima viewed MKT as residing in teaching and wrote tasks consistent with this view.

Similarly, Sarah talks about MKT as subtle and arising in the moment-to-moment "intricacies" of teaching. She describes the specialized mathematical demands that arise in interpreting mathematical meaning, "I did not even notice the change in his [the student's] voice and even think about the impact that had on his meaning and the direction that the conversation really needed to go." Here Sarah expresses how mathematical interpretation can depend on intonation. She is not identifying this as a mathematical topic of interest within the discipline, but as a mathematical skill specialized to teaching.

When MTEs based their thinking about MKT in reference to the discipline and emphasized a distinction between knowledge for teaching and the act of teaching, the implied relationship to teaching was seen as straightforward and left to teachers to figure out. In this thinking, knowledge was viewed as a resource that, once learned, would be used as needed. When MTEs based their thinking in reference to teaching and used teaching situations to characterize the knowledge, the implied relationship was one of interdependence. In this thinking, knowledge was viewed as inseparable from the practice where it operates.

We found that most MTEs exhibited resource-based or practice-based thinking exclusively (Figure 1).

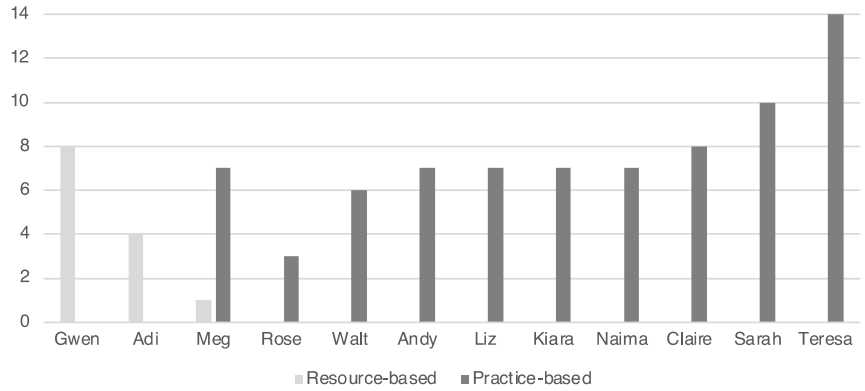

Fig. 1 Number of resource-based and practice-based codes for each MTE

Straightforward/limited versus mutual conceptions of teaching

The theme that emerged in our analysis of MTEs' conceptions of teaching was a difference between viewing it as *straightforward, where one gives attention to an aspect of teaching or to the coordination of mathematical goals and student thinking, but with little regard for other aspects*, to views of it as *involved, where, while giving attention to an aspect of teaching, one maintains regard for other aspects and the whole, with a sense of mutuality*. By *aspect* we mean a focused slice of teaching and learning. By a sense of *mutuality*, we mean regard for the comprehensive interaction of teaching and learning, including the influence of broader social environments and the need to contend with complexity by specifying practice at a finer grain size (Ball & Forzani, 2009; Grossman & McDonald, 2008), while also considering its character as dilemma management (Ball, 1993; Lampert, 1985).

In analyzing MTEs' thinking about teaching, we found that they often expressed different views of what to attend to and how to manage complexity in teaching. Claire, for example, describes interpreting and classifying student thinking as best done in isolation from other considerations. She indicates that students' backgrounds or identities are extraneous and suggests that considering them can be a distraction. She acknowledges that teaching is complex but believes that teachers manage complexity by focusing on "narrow pieces" of teaching.

> *Instead of thinking about like okay there are forty individual approaches in this classroom, there are reasonably like three or four ways my students might be thinking about that, and … based on these* [students'] *comments, we can try to put students into this model that we already have of how a student is thinking.*

Claire trusts that care and good intentions will sufficiently address other aspects of teaching. She has an abiding regard for mathematics and teaches prospective teachers how, for example, to set clear objectives, yet treats this and other pedagogical tasks as separate matters that do not come into play when classifying student thinking and vice versa. For Claire, the pedagogical tasks that make up teaching can each be learned relatively independently and used straightforwardly.

Other comments reflected thinking about teaching as the coordination of mathematical goals and student thinking, viewed as mutually interdependent. For example, Teresa describes Stein et al.'s (2008) five practices for leading a productive discussion as the essence of teaching, where the goal is to "use students' words to tell the story." For her, when teaching focuses on mathematical issues, student thinking must be near at hand, and when teaching scrutinizes student thinking, mathematical goals must be kept in mind.

Similar thinking is evident in Andy's interview. He remarks how challenging it is to hear the parts of student contributions that are mathematically correct and to see how to leverage these to advance the mathematics. For Andy, this is the heart of skillful teaching. He describes the need for mutual, coordinated regard in the work of bridging student thinking and mathematical goals. He also refers to behavior issues and other things teachers must manage, but his comments imply that these are tasks to be dealt with separately rather than coordinated with the

central work of bridging student thinking and mathematical goals. For example, when discussing how teachers might respond to unconventional student solutions, Andy fails to consider how the real-time nature of teaching shapes the responses that might be available to a teacher and how particular responses to behavior might exclude certain children from the mathematical work, both relevant in this instance. Such evidence suggests a view of teaching as coordinated attention to mathematical goals and student thinking, but without mutual regard for other aspects, such as for students' identities, classroom culture, materials available, pacing, or instructional design.

Still other comments revealed thinking that conveys a fuller coordination of mutual concerns. Naima's reflections on the workshop discussion also focus on interpreting the same two students' contributions as Andy (above) and using them together to advance instruction, but she incorporates regard for other aspects of the situation as well. As she discusses the mathematical content and the collective trajectory of the class, she folds in comments about needing to choose who should speak, with reference to each student's strengths and growth, how students are positioned, and the overall classroom culture. She attends to all parts of the instructional triad. Throughout her comments Naima routinely focused on specific concerns yet maintained a sensibility for teaching as complex interactional work, inserting brief asides to other aspects of teaching and offering examples that situate her specific point in an overall picture.

As visible in Figure 3, a prominent difference in MTEs' thinking about teaching is between thinking that attends to a single aspect or the coordination of mathematical goals and student thinking without regard for other aspects and thinking that may attend to an aspect yet maintains regard for other aspects and the whole, with a sense of mutuality.

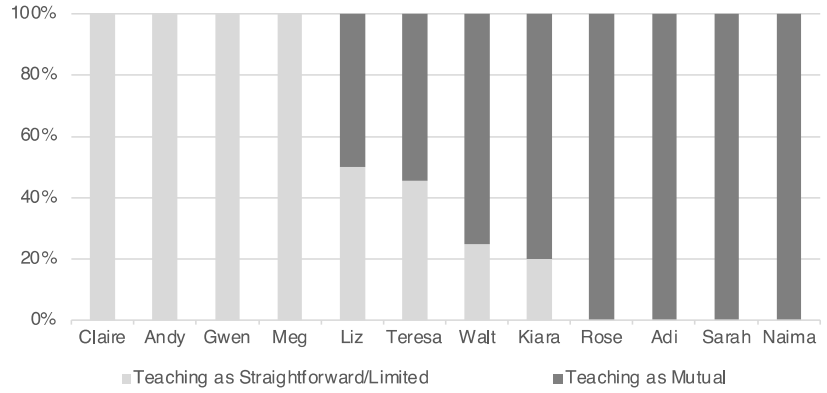

Fig. 3 Percents of each MTE's coded excerpts about teaching as straightforward/limited and mutual thinking

The middle four MTEs were mixed in their thinking. For instance, Teresa realizes she has been leaving significant parts of teaching implicit in her work with teachers. She reflects that, in her focus on the five practices for orchestrating a productive discussion, "the characters, for lack of a better word, the little people, the kids have been absent."

> *I didn't realize how powerful it is to actually paint that context and paint the picture of who is that student with that voice. Like, as a classroom teacher I think I did attend to that, like I understood who kinda- like when you're working with an equalizer board in music, there are times where I need a tone, somebody's voice, and I was- I felt like I was in tune with that. But I didn't realize how to make that visible and explicit to teachers as they think about how they do that with their students.*

Teresa's insight surfaced as she began to consider issues of justice. For her, attention to justice issues became an inroad that extended her constrained thinking about teaching as navigation only between mathematical objectives and student thinking to navigation involving a much fuller set of human interactions. Like Teresa, the three other MTEs in the middle of Figure 3 all have frames that constrain their thinking about teaching (limited views of

mathematics, of what is interesting or important, of students, or of teaching), but also have experiences that allow them at times to take up pedagogical concerns with fuller mutual regard.

Topical/optional versus fundamental/pervasive conceptions of justice

The theme that emerged in our analysis of MTEs' conceptions of justice was a salient distinction between whether justice is viewed either as *topical and optional* or as *fundamental to and pervasive in the dynamics of teaching and learning*. Topical/optional thinking views justice as a special interest topic, without significant or pervasive impact on mathematics teaching and learning. This thinking often ignores identities and questions or rejects their relevance, takes up non-mathematical issues in narrow or colorblind ways, makes few meaningful connections to larger social issues, and deals with justice concerns separate from mathematical and pedagogical issues, if at all. Fundamental/pervasive thinking views justice issues as ubiquitous, drawing connections among student identities, teaching, and larger societal patterns of injustice and displaying awareness of one's own identity and its implications for professional work. This thinking often places justice issues in context, as pervasive and intractable concerns in a society with a long history of injustice, with many invested in the status quo. It also communicates a view of justice as intertwined with pedagogical or mathematical concerns, with implications beyond the classroom.

In the interviews, the distinction is perhaps most visible in MTEs' attentiveness to children's identities. Some showed care and sensitivity, recognizing how teaching and learning implicate children's identities, naming racialized or gendered dynamics of classroom interaction, or attending to assets rather than deficits. Attention to MTEs' own identities often accompanied attention to children's identities. For instance, in reflecting on a classroom interaction where a black boy attempted to ask a question of another student presenting at the board but then admitted he was not paying attention, Sarah drew out the racialized nature of her own perceptions (stereotypes that have led her, in the past, to see black-boy behavior as disruption).

> *I think there are kids in our classrooms whose identities are most visible in the ways they disrupt or act, as opposed to in the ways they learn or express mathematics … he was starting to integrate those two pieces of the mathematics, but it was so easy for us to see him as a disruption instead of seeing the actual beautiful thinking that was going on in his head.*

Other MTEs, however, demonstrated a lack of awareness concerning identity, talking as if ideas are independent of the students who produce them or suggesting that one can "take [students'] ideas at face value" without considering how larger social dynamics might shape perceptions of these ideas and how responses might shape whether students feel they belong in mathematics. These MTEs were more likely to voice deficit views of black and brown children and rarely displayed reflexivity regarding their own social identities.

Identifying this theme engaged us in a much longer process than for thinking about MKT or teaching. We first identified relevant excerpts and wrote memos, using the literature to inform our interpretations. We drew from Helms' (1994; 1997) frame for the development of white racial identity, from oblivious to conflicted, to intellectualized acceptance, to honest appraisal, to multicultural identity; DiAngelo's (2018) white fragility and initial stages of development, where guilt, defensiveness, denial, and other forms of emotionally laden resistance are common; Tatum's (1997/2017) critiques of colorblind meritocracy; Adiredja and Louie's (2020) analysis of deficit views of students and communities of color; and Hottinger's (2016) examination of the cultural neutrality of the discipline and its potential to be repressive. In our analysis, we interpret MTEs' discourse from these understandings while also listening for what distinguishes professional growth in relation to the work of mathematics teacher education. And while we consider justice broadly, with regard for its many forms (related to gender oppression, religious oppression, economic oppression, and so forth), given the U.S. context and a focus on racism in the workshops, racism is most prominent in our analysis.

Looking across our explanatory memos, we explored potential groupings and intermediate labels to characterize groups of quotes that were similar. Intermediate labels were then grouped into themes (Figure 4). Much of the literature can be characterized as identifying either problematic views and actions or constructive ones, consistent

with the two major phases of Helm's (1997) white identity development model. Intermediate labels were then grouped into less-developed views and more-developed views.

Less-developed and/or problematic views	*More-developed and/or constructive views*
Denial and avoidance	Deliberate attention
Affective resistance	In our own back yard
Irrelevance	Attending to children
Surprise and apology	Disrupting patterns of injustice
Excuses	Self-awareness
Mathematical neutrality	Mathematics and justice are intertwined
Simplistic colorblindness	Acknowledgement and commitment
Patronizing	Collective work

Fig. 4 Initial themes for less-developed and more-developed views evident in MTEs' thinking about justice

Although it may be tempting to infer a linear order or underlying dimensions for these themes, our analysis did not identify such structure and we do not make claims of this kind. Instead, our analysis uncovered a basic distinction in how MTEs viewed the relevance of justice for mathematics education. For example, the ideas that justice is an "extra piece" and not "an issue in my natural environment" suggest a view of justice as something one might choose to attend to but only as an add-on relevant in some contexts and not others. Statements associated with colorblind approaches and mathematical neutrality argue that injustice is a matter of personal prejudice and does not occur when race and other social differences are avoided. In contrast, statements associated with more-developed views convey an awareness of far-reaching relevance of justice into every moment of teaching. MTEs referenced fine-grained and yet impactful justice concerns that they "hadn't thought about." They asked questions about how to help teachers "notice moments" in teaching that connect practice with larger social issues, such as the "systemic removal of populations from the field of mathematics." (See the appendix for brief descriptions of these initial themes, intermediate labels, and example quotes.) We developed codes for these initial themes and used these to develop codes for an overall distinction between topical/optional and fundamental/pervasive conceptions of justice.

MTEs' reactions often coincided with their identities and backgrounds, and we observed patterns typical of privilege and whiteness.[8] Several were defensive or apologetic or minimized their role in reproducing injustice, saying that such matters "apply less to" them because of their specific context or professional interests. Such comments often privileged white norms and reflected deficit views of students and communities of color. Several invoked colorblind ideas of meritocracy and the cultural neutrality of mathematics. Others were self-reflective, acknowledging the complexity and scope of inequity and their personal and professional part in its reproduction. This was often coupled with a desire to learn and build new skills, but also with feelings of guilt, remorse, or regret, and with uncertainty about how to move forward. Those who recognized challenges often could not keep issues of injustice on the table when discussing specific mathematical or pedagogical issues. Alternatively, some became overly focused on justice concerns, losing track of mathematical and pedagogical concerns. Indeed, Rose, a white MTE, could focus on little else in her interview and her confusion and distress were palpable. In contrast, two black MTEs, each with different experiences and language for talking about injustice, exhibited none of these impediments and were able to incorporate what they were learning into their thinking and professional work.

In addition, we found differences in what MTEs saw as the scope of justice concerns. Some identified diversity of thought or experience as a universal good, ignoring issues of power and pervasive disparities. Others presumed that providing access to appropriate mathematics alone would suffice. Still others focused almost exclusively on student identity development, with an emphasis on positioning marginalized students as mathematical contributors. While

[8] We acknowledge that this statement and others in this section may be experienced by readers as emotionally charged or disparaging; however, we understand this statement to be descriptive — using concepts from the literature to describe the data. Many scholars suggest that defensiveness is to be expected.

comments occasionally connected individual empowerment to larger issues, often this thinking treated equity concerns as relevant only for certain groups in certain contexts. Other comments directly connected the work of justice-concerned teaching to larger issues of inequity and social change. Such thinking advanced a view of justice and mathematics teaching as intertwined, where deliberate and careful attention to content, pedagogy, and children are necessary for disrupting patterns of injustice. Several spoke about the work of choosing which student should speak next, with attention to disrupting default status hierarchies — such as supporting the class to hear and understand a black student who is making a mathematically crucial contribution.

The overarching categories of topical/optional versus fundamental/pervasive distinguish MTEs' thinking about justice (Figure 5). It is our sense that participants to the right in Figure 5 had more exposure to and fluency with justice concerns and more willingness to grapple with justice concerns than those to the left.

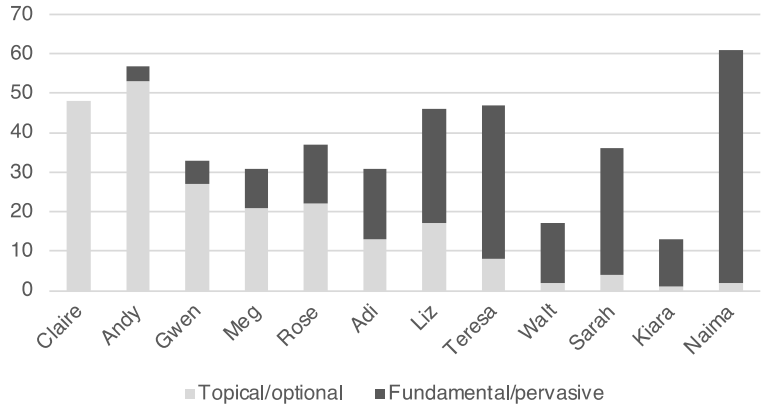

Fig. 5 Counts of topical/optional and fundamental/pervasive codes (with MTEs ordered by the ratio of topical/optional to fundamental/pervasive codes)

Alignment in thinking across the domains and its relationship to background

We set out to understand how MTEs differ in their thinking, with an eye for reasons for these differences and how thinking about teaching or justice might shape thinking about MKT. To investigate potential relationships across thinking, we developed indicators for the three domains from the analyses presented above.[9] Figure 6 shows these indicators for each participant. Perhaps most striking is the similar picture for the six participants to the right.

[9] For an indicator of thinking about mathematical knowledge for teaching (*MKT*), we converted raw counts for the resource-based and practice-based codes to scales from 0 to 3, made the resource-based scale negative, and added the two scales. To generate an indicator for thinking about teaching (*Teaching*), we negated scaled *Teaching-as-Straightforward* and *Teaching-as-Math-and-Student-Thinking* counts and added the scaled *Teaching-as-Mutual* counts. To generate an indicator of thinking (*Justice*), we negated the scaled *Topical/optional* counts and added the scaled *Fundamental/pervasive* counts. As we combined counts and scales, we standardized at each step.

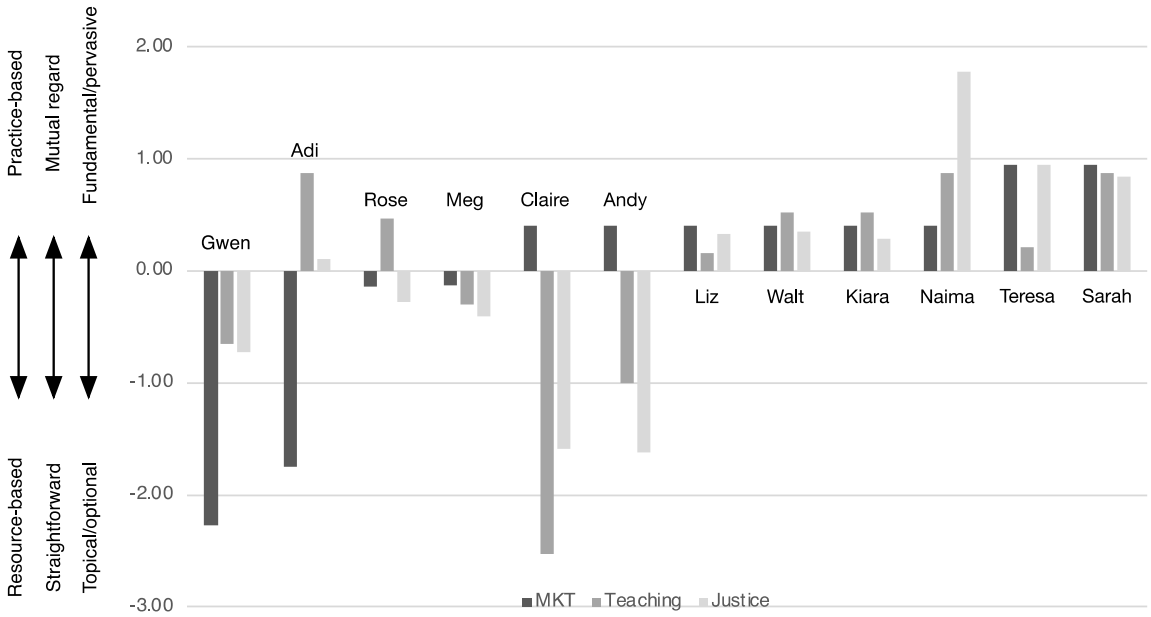

Fig. 6 Profiles of the extent to which MTEs think of MKT as practice-based, teaching as mutually involved, and justice as fundamental/pervasive

Indeed, alignment is strong, with a few notable discrepancies. Table 2 provides basic correlation coefficients. The correlation between *Teaching* (gray) and *Justice* (light gray) is 0.82. The correlation coefficient of 0.03 for *MKT* (black) and *Teaching* (gray) suggests no association, yet the multiple correlation coefficient for *MKT* with *Teaching* and *Justice* as independent factors (how well *MKT* can be predicted from a linear combination of *Teaching* and *Justice*) (0.45) suggests that the near-zero correlation between *MKT* and *Teaching* is hiding positive correlation for MTEs to the right (with positive *Justice* scores) and negative correlation for most MTEs to the left (with negative *Justice* scores) (see Figure 6). This supports earlier conjectures about the influence of thinking about teaching and justice on thinking about MKT. For instance, MTEs who think about justice as fundamental/pervasive are likely to think about MKT in relation to practice. Our sample is small and our statistical analysis exploratory, but these findings fit our interpretations.

Table 2 Bivariate and multiple correlation coefficients for thinking about *MKT*, *Teaching*, and *Justice*.

Bivariate correlation coefficient for *MKT* and *Teaching*	0.03
Bivariate correlation coefficient for *MKT* and *Justice*	0.28
Bivariate correlation coefficient for *Teaching* and *Justice*	0.82
Multiple correlation coefficient for *MKT* with *Teaching* and *Justice* independent:	0.45
Multiple correlation coefficient for *Teaching* with *MKT* and *Justice* independent:	0.85
Multiple correlation coefficient for *Justice* with *MKT* and *Teaching* independent:	0.86

Further examination of Figure 6 reveals several clusters of MTEs with distinctive characterizations. Half of the MTEs have all positive scores, and one has all negative scores. Andy and Claire have similar profiles, as do Rose and Adi. Combining these observations with our narratives of interviewees, we identify four orientations, with implications for the development of MTEs (Figure 7). We refer to these as orientations because we do not mean to essentialize MTEs. Nevertheless, we see these characterizations as potentially helpful in planning professional development for MTEs and making sense of their engagement.

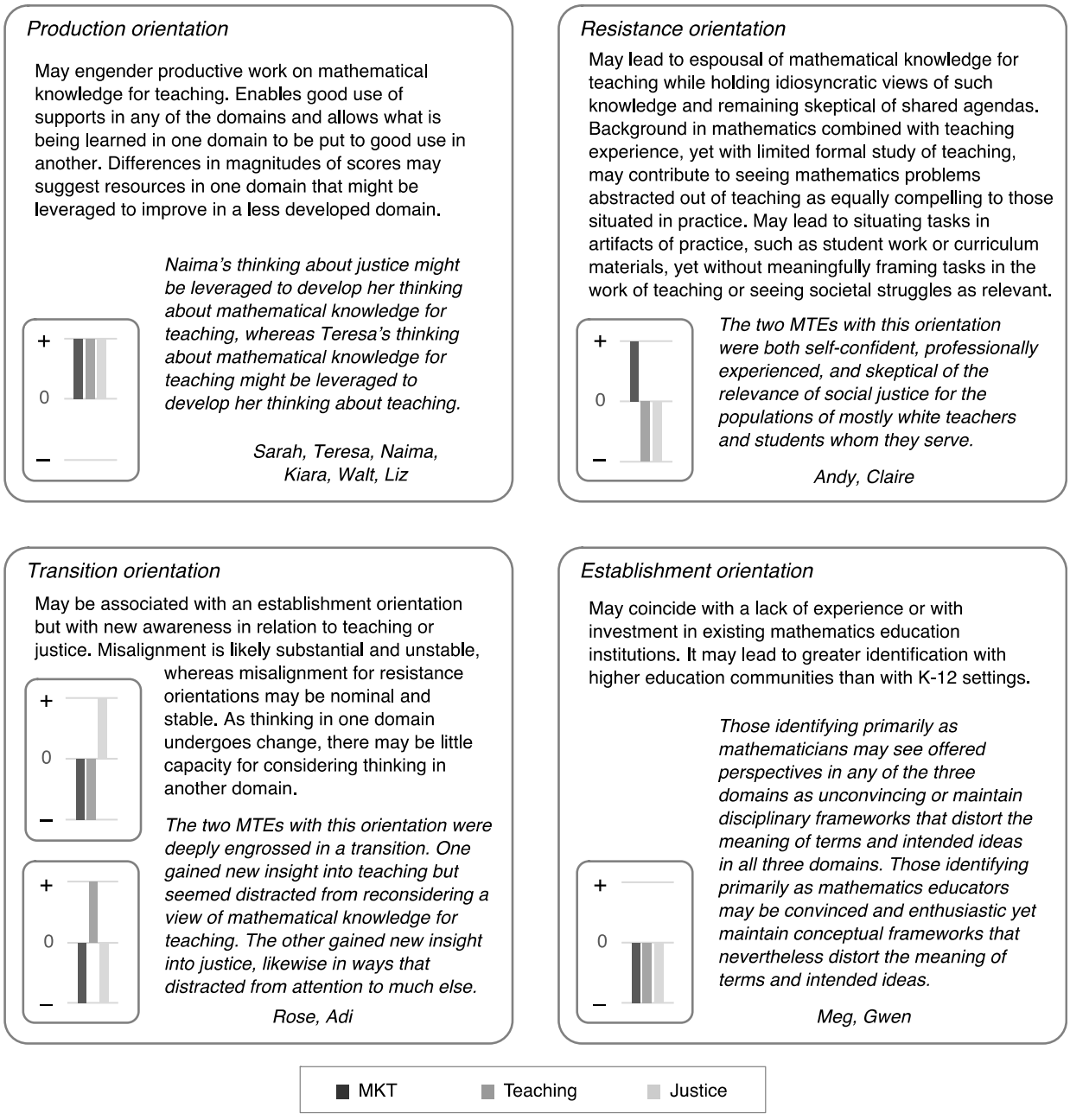

Fig. 7 Characterizations of four MTE orientations with icons indicating the directionality of relationships

Combining these results with our previous analyses suggests that the misalignment evident in transition orientations may indicate change, if not in the moment, perhaps soon or with appropriate support. To further examine how MTEs' background may shape their thinking and be leveraged for professional growth, we first consider correlations between background variables and indicators of *MKT*, *Teaching*, and *Justice* (Table 3).

Table 3 Correlations between background variables and the indicators *MKT*, *Teaching*, and *Justice*.[10]

Background variables	*MKT*	*Teaching*	*Justice*

[10] Sets of variables associated with professional role/title, highest degree, focus level of schooling, and years of experience as a MTE are not listed because correlation coefficients were low.

K-12 teaching experience of 5+ years	0.197	0.677**[a]	0.662**[a]
Affiliation with K-12 school	0.295	0.098	0.151
Affiliation with math department	-0.295	-0.606**[a]	-0.400
Affiliation with school of education	-0.199	0.298	0.061
Interest in or experience with PD rather than TE	0.397	0.445	0.605**[a]
Interest in or experience with content courses rather than methods courses	-0.397	-0.278	-0.561*[a]
Gender (female)	0.191	-0.076	0.235
Race (Black)	0.187	0.325	0.483
Race (White)	0.000	-0.457	-0.574*[b]
Graduate training in mathematics	0.187	-0.115	-0.296
Graduate training in mathematics education	-0.513*[c]	-0.335	-0.350
Graduate training in science education	-0.047	0.145	-0.088
Graduate training in education other than mathematics and science	0.315	0.407	0.611**[c]

* Significant at the 0.10 level. ** Significant at the 0.05 level.

[a] Proximity-to-practice theme.

[b] Social-identity theme.

[c] Social-and-political-conservatism theme.

Background variables alone do not appear to be good proxies for progress with *MKT*. However, given the alignment identified above, we explore three themes across these variables, suggested by the correlations, consistent with our interpretation of interviews.

First, we contend that *proximity to practice*, as a combination of background variables, may matter for MTE thinking.

- *Teaching* correlates positively with *K-12 teaching experience* (0.677) and negatively with *Affiliation with math department* (-0.606).
- *Justice* correlates positively with *K-12 teaching experience* (0.662) and *Interest in or experience with PD* (0.605) and negatively with *Interest in or experience with content courses* (-0.561).

Our interpretation of the second bullet is that proximity to practice can provide first-hand experiences of how systemic racism, oppression, and inequity take place in schools. Such experiences likely allow one to see influences on teaching that others might miss.

A second theme explores MTEs' *social identity*. Gender is weakly related, but race is more strongly related, in particular with *Justice*. This makes sense as people from oppressed groups are likely to have experiences that foster deeper understandings of issues of justice, which may help them see impacts on mathematics teaching and learning. For instance, Black and Latina MTEs in our study were more likely to express practice-based views and reference social contexts when talking about teaching than were white MTEs.

A final theme arises in considering graduate education. It is striking that *Graduate training in mathematics education* correlates negatively with all three domains and is statistically significant for *MKT*. In contrast, *Graduate training in education other than mathematics and science* correlates positively with all three. Namely, educators with training in mathematics or mathematics education are more likely to express resource-based views of MKT and limited views of teaching and justice. Maybe this should not be surprising. First, in many countries, mathematics education, like mathematics, is a white, male institutional space invested in maintaining existing forms of power and privilege (Battey & Leyva, 2016; Hoover, 2021; Martin, 2019). Second, it is our experience that many mathematics educators were initially educated in mathematics and gravitate to educational psychology as foundational, with cognitive models echoing mathematical models. Many who work in mathematics education enjoy mathematics and maintain its centrality. They see mathematics as standing above the fray of social issues and view its teaching as simply sequencing topics from unambiguous foundations. Taken together, white-establishment power and Platonist-informed thinking may contribute to an underlying *social and political conservatism*. Consider, for example, much of the guidance in the literature, such as Stein et al.'s (2008) five practices for leading a discussion, and its use by practitioners in the field. In focusing on moves for coordinating mathematics and student thinking, might we

routinely undermine mutual regard for other aspects of teaching, including much of the nuanced mathematical work of attending to justice in mathematics teaching (as characterized by Goffney & Hoover, 2023)?

Together, these three themes, as constellations of variables, help us consider how background may shape MTEs' thinking.

Implications for understanding and supporting MTEs

Consideration of background and thinking helps us understand how MTEs work with teachers and how they engage with professional development. These, in turn, allow us to reconsider MTEs' development and suggest ways of targeting support. We discuss implications of the findings above in the context of our own workshops but imagine they have implications for professional development efforts more broadly. For instance, Gwen is an experienced MTE with a PhD in mathematics education. She displays resource-based thinking about MKT and straightforward thinking about teaching. Her background likely contributes to her limited development: her proclivity for mathematics, employment in a mathematics department, inexperience in school classrooms, focus on content courses more than methods courses, and identity as a white woman grounded in the mathematics education establishment. In her interview, Gwen elaborated on how she incorporates MKT in her courses.

> *We don't just say okay, you know, here's how you subtract because, (a) they already know how to do that basically, and (b) they need to know more than that. So, we talk about different types of subtraction problems and what's the same and what's the difference and different ways to model them and different algorithms for subtraction, and why do they work. And so, thinking about what do I need to make sure to have in my course beyond just, here, add and subtract and multiply and divide.*

Gwen distinguishes the knowledge that a layperson needs from the knowledge that teachers need but identifies strictly mathematical topics for teachers related to mathematical topics for students. She does not provide connections to teaching or give a sense that the "types of subtraction problems" will need to be recognized when teaching and used for specific pedagogical purposes. Even though the "topics" she names could connect readily to teaching, it is likely that her inexperience in school settings gives her little to draw on, while conceptual frames from her training dominate her thinking. Her focus on mathematics and her privileged position limit her thinking — about the mathematical demands, interactional dynamics, and justice implications of teaching. Gwen's strong mathematical frames lead her to strip ideas about MKT of their integrated character. Instead of professional development that focuses directly on MKT, we suspect Gwen might profit from professional development that centers on specific issues in teaching or justice in ways that might compete with and disrupt her strong disciplinary orientation.

Another white MTE, Liz, also has a PhD in mathematics education, works in a mathematics department, and focuses on university-based teacher education but has extensive school teaching experience and teaches methods courses. These differences suggest why Liz displays more nuanced thinking about teaching and more practice-based views of mathematical knowledge. While Gwen brings important mathematical appreciation and knowledge to her work with teachers, Liz's thinking about teaching and justice, perhaps in tandem, helps her see different mathematics and make different mathematics available to teachers.

Knowledge of different MTEs' thinking and background (or a sense of their orientation) can suggest different ways of working with them. Consider the case of Rose, a white woman who works as a consultant in schools. She describes her developing understanding:

> *I used to think it was just- I treat all my students the same, right, with deference and kindness and all that. And then, I … suddenly understood that kids only have equity for learning math if they have access to instruction that is effective, so it's not just a matter of how you treat kids, it's what they're actually getting for opportunities to learn. Well that was horrifying because we don't have skilled people everywhere or school systems that support that so- and then to come to the March* [workshop] *was this other layer that is growing more visible at least to me, of, you know, thinking about these marginalized people and how the*

things that we do during instruction that feel natural, and feel beneficial, might be just reiterating those patterns that are keeping those people in those positions.

Although Rose's comments reflect limited regard for the dynamics of marginalization, the workshops brought her new awareness of how justice might be relevant for mathematics teaching. Her new awareness coincides with self-reflection on how her positionality might inform her understanding of interactions in classrooms and, consequently, the mathematical demands of teaching. Connecting micro-interactions of practice to larger social patterns has fostered a more practice-based understanding of MKT for her. Her mixed views, at one moment conveying that justice is fundamental to teaching and at another that it is optional (see Figure 5), together with her desire to learn more, support her personal narrative of growth and her newfound appreciation of complexity. Her views also support the idea that misalignment signals a transition and is an opportunity for growth. Nearly all her comments about teaching and MKT occur when she is talking about issues of justice. With continued press on her understanding of whiteness, while helping her process her distress, Rose, and other MTEs like her, might productively focus on the mathematical demands of teaching in relation to justice concerns.

Final thoughts

Figure 8 offers a visual representation of the relationship between the three domains of thinking together with an indication of how the salient distinction in each domain progresses developmentally. It also suggests that justice serves as a foundation for developing thinking about teaching, and in turn MKT.

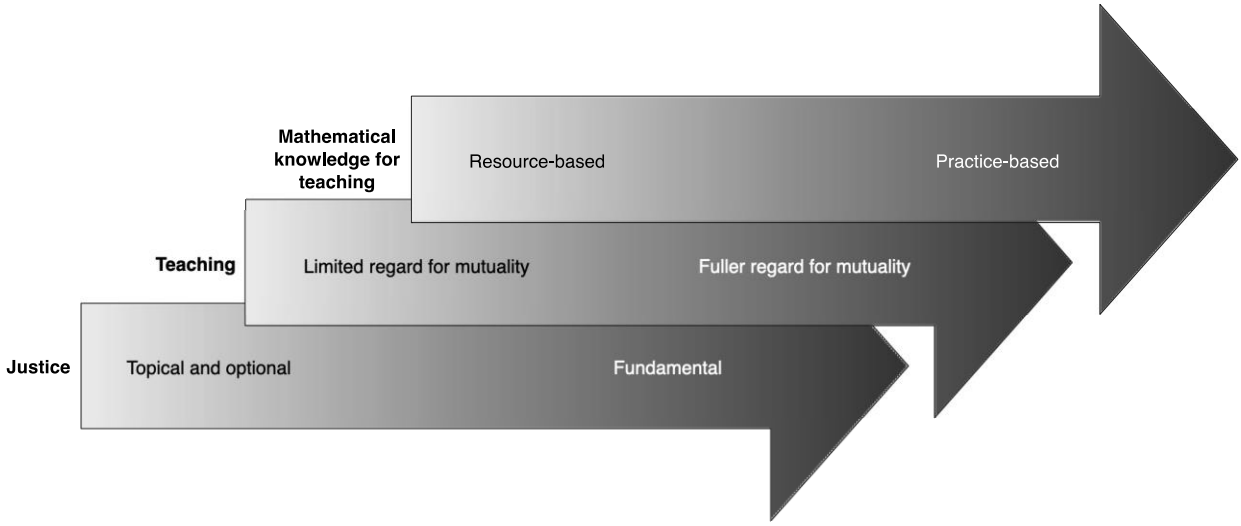

Fig. 8 Key components of the development of MTEs' thinking

We place teaching at the center of Figure 8 because it seems to play a central role in the dynamics of MTE development. Holding regard for other aspects of teaching while considering a focal aspect reflects something important for MTEs' development, including how they think about MKT, the mathematical tasks they write, and the mathematical opportunities they provide to teachers. We recognize that our elaboration of the concept of mutuality is limited, but it offers a starting point. It is our sense that the distinction between topical/optional and fundamental/pervasive interacts with thinking about teaching and, through that, thinking about MKT. It may serve as a helpful focus in working with MTEs, and perhaps working with teachers. For instance, a common approach in teacher education aims at socio-political education, with consciousness raising and learning about diversity and systemic oppression (Civitillo et al., 2018). We agree with such work, but it may be strategic to focus more on understanding how the macro plays out in the micro, how the micro shapes the macro, and how mathematical objectivity belies privilege and exclusion. As teachers and MTEs come to understand how pervasive the dynamics of injustice are, they may see its relevance in the nitty-gritty of their mathematical work.

In addition, our analysis suggests that the distinction between resource-based and practice-based thinking is key to the design and implementation of MTE-development efforts. Resource-based thinking is likely to preserve a system that promotes academic success but leaves teachers unable to bring mathematical knowledge to bear in teaching, much as Borko and her colleagues describe with Ms. Daniels (Borko et al., 1992). Attending to whether MTEs engage in resource-based and practice-based thinking may clarify where they are in their development and shed light on how to support them.

We have identified differences in MTEs' thinking that are not discussed in the field. We find that these differences tend to align and that misalignment signals change and opportunities for growth. Before engaging in this study, we would not have thought that developing thinking about justice would be an important tool for developing thinking about MKT. Now we do. Our results point to centering thinking about teaching and justice, not just because they are good additions, but because they are essential to practice-based understanding. Furthermore, our analysis suggests that individuals who understand the mutual character of teaching or the fundamental/pervasive character of justice are assets for collective work on MKT. Professional development efforts should draw on communities more likely to exhibit these understandings — communities of color (and other marginalized groups) and those who may lack status but bring skill in attending to mutuality.

As we initially discussed, attention to issues of justice is missing in research on MTEs' development, including in our own prior work. This absence may be due in part to the composition of the field. Indeed, dominant research is overwhelmingly carried out by relatively privileged members of society in white, European-colonial, middle-class, patriarchal institutions. Our current study suggests that, in presuming independence from concerns for justice, scholarship has overlooked key phenomena and perpetuated false notions that mathematics and its teaching are free from dynamics of power and privilege.

Many have stressed the importance of our individual and collective obligation to consider privilege in our lives and institutions. Justice is not a topic to add to a list or even raise up as a major domain of research but pervades all that educators and researchers do. Failing to attend to justice, at all times, in our practice and research, implicates us and implies that existing injustice is our continued and deliberate choice. As Martin (2019) argues, white benevolence and the appeal to white benevolence are insidious.

> *The appeal to white benevolence for equity and inclusion rests on the implied promise of not radically altering the status quo of white supremacy and antiblackness.* (p. 469)

He is clear about what needs to be done:

> *I suggest that those who believe in the humanity of Black people actively resist and reject mathematics education research that results in epistemological violence and mathematics reforms that perpetuate antiblackness.* (p. 471)

We see our work as a small step in responding to Martin's call. In opening our analysis to the possibility that views of justice might matter, we have come to see how research on MTE development has obscured injustices in our work as educators and researchers, and has, consciously or unconsciously, served to maintain existing dynamics of exploitation. We hope that others will take up our plea that these concerns deserve further attention.

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Appendix

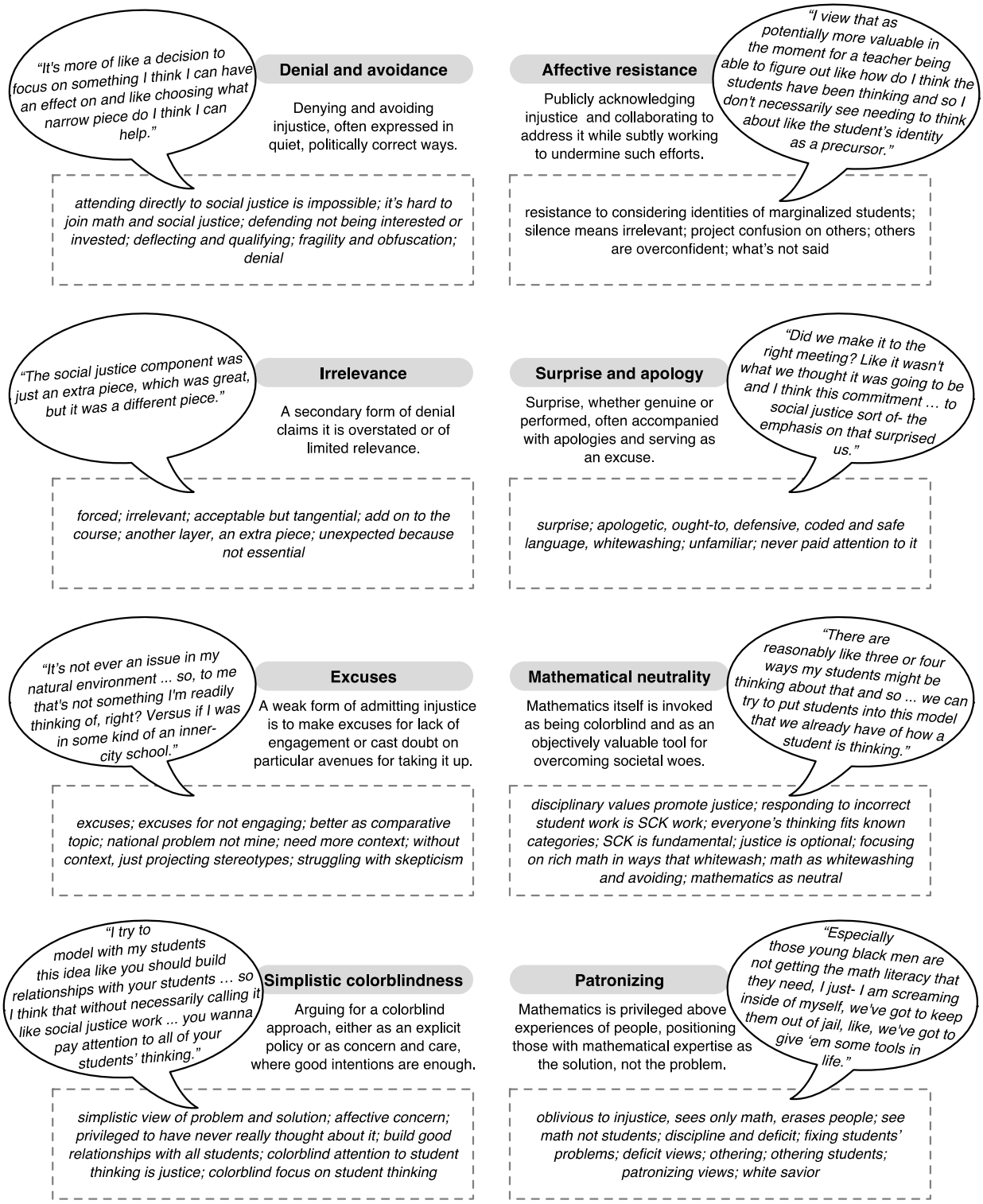

Fig. 9 A snapshot of analysis of less-developed and/or problematic views evident in MTEs' thinking about justice (whether intentional or unintentional), with themes (in bold), brief descriptions, intermediate labels of code instances (in dashed rectangles), and example quotes (in speech bubbles)

Fig. 10 A snapshot of analysis of more-developed and/or constructive views evident in MTEs' thinking about justice, with themes (in bold), brief descriptions, intermediate labels of code instances (in dashed rectangles), and example quotes (in speech bubbles)