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Design of switching state-feedback controllers for linear systems subject to asymmetric saturations ⋆

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Abstract: This paper considers the problem of controlling a linear system affected by asymmetrical input saturation. The proposed solution is based on using a linear matrix inequality (LMI)-based methodology to find the gains of a switching state-feedback controller. The main difference and contribution when compared to existing approaches is that the switching rule is chosen based on the closed-loop performance that each of the non-saturating controller gains can achieve when used with the current value of the state vector. Although the main focus of the paper is on time-invariant systems, the possible extension to linear parameter-varying (LPV) systems is discussed. An illustrative example is used to show the main features of the proposed approach.

Keywords: Linear systems, Saturation, Linear matrix inequalities (LMIs)

1. INTRODUCTION

Actuator saturation is likely one of the most researched nonlinearities in control theory over the last several decades, owing to the potential performance degradation or destabilization induced onto the closed-loop system (see e.g. Hu (2001); Tarbouriech et al. (2011) and references therein). In general, existing approaches are classified into two categories based on how the saturation is handled. The first approach entails directly addressing the input constraints in the controller design stage (Hu et al., 2002; Wu et al., 2007). On the other hand, the second approach is the anti-windup compensation, which involves pairing a pre-designed controller with a compensator that handles the saturation (Wu and Lu, 2004; Tarbouriech and Turner, 2009).

Despite the extensive literature, the majority of proposed approaches deal with symmetric actuator saturations. Nonetheless, asymmetric saturation may often occur in practical systems. The aforementioned approaches can be utilized to manage asymmetric saturation functions by treating this function as a symmetric saturation with the most restrictive saturation limits, thus introducing conservatism. For this reason, several researchers have recently addressed this topic in an attempt to reduce the conservatism. For example, Mariano et al. (2020); Braun et al. (2021) suggested a coordinate transformation in order to enlarge the estimated region of attraction. In Benzaouia et al. (2014), a linear matrix inequality (LMI)-based methodology has been proposed for stabilizing an asymmetrically saturated system by converting it into a saturated one with symmetric saturation limits and a bounded disturbance. Alternatively, a linear system with asymmetric saturations can be converted into a switched linear model with symmetric saturations, as demonstrated by Yuan and Wu (2015); Li and Lin (2018); Groff et al. (2019), who proposed different LMI-based methodologies for designing a switching controller.

The present work aims at proposing an LMI-based methodology for designing a switching state-feedback controller for a linear system subject to asymmetric input saturations. Although the idea of using a switched model is based on the idea of partitioning the state space into multiple regions as proposed by Yuan and Wu (2015), the main difference and contribution of this paper when compared to existing approaches is that the switching rule is designed based on the attainable closed-loop performance. In particular, the design conditions are obtained through the application of the theory of invariant sets and the use of a quadratic Lyapunov function (QLF), thus guaranteeing that the control action remains in the linearity region

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of the actuators. Furthermore, a closed-loop performance criterion is established by associating distinct decay rate values to the different possible controller modes, equipping the control system with the ability to change its closed-loop performance in the sense of guaranteed convergence speed.

The rest of this paper is organized as follows. The problem formulation is provided in Section 2. The main results are given in Section 3. The extension to the parameter-varying case is discussed in Section 4. Simulation results are presented in Section 5. Finally, the main conclusions and perspectives on future research are outlined in Section 6.

Notation: $\mathbb{R}$ and $\mathbb{R}_+$ represent the sets of real and positive real numbers, respectively. The $n\times m$ real matrices are denoted as $\mathbb{R}^{n\times m}$, and $\mathbb{S}^n$ corresponds to the set of $n\times n$ symmetric matrices. For $M \in \mathbb{S}^n$, $M \succ 0$ ($M \succeq 0$) and $M \prec 0$ ($M \preceq 0$) stand for a positive (semi-)definite matrix and for a negative (semi-)definite matrix, respectively. $M \in \mathbb{S}^n_+$ is used as a shorthand for positive-definite symmetric matrices. The symbol $\star$ denotes the block induced by symmetry in a matrix. $M = \operatorname{diag}\{M_1, M_2\}$ represents a block diagonal matrix. The shorthand $\operatorname{He}\{\cdot\} = (\cdot)+(\cdot)^\top$ is used in situations with limited space. $\mathcal{I}_{[a,b]}$ denotes the set of integers $\{a, a+1, \ldots, b\}$ with $a, b \in \mathbb{N}$ and $a \leq b$. The subscript $[i]$ represents the i^{th} row of a matrix. For a given Pólya's relaxation degree $d \in \mathbb{N}$, $d \geq 2$ and the number of vertices n_μ, the $d-$dimensional multi-index $\mathbf{k} \in \mathbb{N}^d$ is defined as $\mathbf{k} = (k_1, \ldots, k_d)$, and the sets $\mathbb{I}_{(d,n_\mu)}$ and $\mathbb{I}^+_{(d,n_\mu)}$ denote:

$$\mathbb{I}_{(d,n_\mu)} \triangleq \left\{\mathbf{k} \in \mathbb{N}^d : 1 \leq k_m \leq n_\mu,\ \forall m \in \mathcal{I}_{[1,d]}\right\},$$
$$\mathbb{I}^+_{(d,n_\mu)} \triangleq \left\{\mathbf{k} \in \mathbb{I}_{(d,n_\mu)} : k_m \leq k_{m+1},\ m \in \mathcal{I}_{[1,d-1]}\right\},$$

whereas $\mathcal{P}(\mathbf{k}) \subset \mathbb{I}_{(d,\mu)}$ corresponds to the set of permutations, with possible repeated elements, of the multi-index $\mathbf{k}$. Finally, the time dependency of variables is dropped to lighten the notation.

2. BACKGROUND AND PROBLEM STATEMENT

Let us consider the following continuous-time system:

$$\dot{x} = Ax + B\operatorname{sat}(u, \underline{u}, \overline{u}), \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector and $u \in \mathbb{R}^{n_u}$ denotes the control input vector. $A \in \mathbb{R}^{n_x\times n_x}$ and $B \in \mathbb{R}^{n_x\times n_u}$ represent the state and input matrices, respectively, and $\operatorname{sat}(u, \underline{u}, \overline{u}) : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ denotes the standard asymmetric saturation function defined as:

$$\operatorname{sat}(u_i, \underline{u}_i, \overline{u}_i) \triangleq \begin{cases} \overline{u}_i, & \text{if } u_i > \overline{u}_i \\ u_i, & \text{if } u_i \in [-\underline{u}_i,\ \overline{u}_i] \\ -\underline{u}_i, & \text{if } u_i < -\underline{u}_i \end{cases},\ i \in \mathcal{I}_{[1,n_u]}, \tag{2}$$

with known $\underline{u}_i, \overline{u}_i \in \mathbb{R}_+\setminus\{0\}$. Note that the standard symmetric saturation function is recovered if $\underline{u}_i = \overline{u}_i$ $\forall i \in \mathcal{I}_{[1,n_u]}$, which will be denoted by the notation $\operatorname{sat}(u_i, \overline{u}_i)$.

According to Yuan and Wu (2015), the asymmetric saturation (2) can be characterized by a combination of 2^{n_u} symmetric saturations, the limits of which are specified as:

$$\overline{\nu}_{(j)} = D_{(j)}\overline{u} + \hat{D}_{(j)}\underline{u},\ j \in \mathcal{I}_{[1,2^{n_u}]}, \tag{3}$$

where $\overline{\nu}_{(j)} \in \mathbb{R}^{n_u}_+\setminus\{0\}$ corresponds to the j^{th} saturation limit vector, $D_{(j)} \in \mathbb{R}^{n_u\times n_u}$ is a diagonal matrix whose elements take the value 0 or 1 and $\hat{D}_{(j)} = I_{n_u} - D_{(j)}$.

Then, the system (1) can be recast as a switched linear system subject to symmetric saturation as follows:

$$\dot{x} = Ax + B\operatorname{sat}(u, \overline{\nu}_{(j)}),\ \overline{\nu}_{(j)} \in \mathcal{V},\ j \in \mathcal{I}_{[1,2^{n_u}]}, \tag{4}$$

where the set $\mathcal{V} \triangleq \left\{\overline{\nu}_{(1)}, \ldots, \overline{\nu}_{(2^{n_u})}\right\}$ contains all the possible combinations of the saturation limit vectors $\underline{u}, \overline{u} \in \mathbb{R}^{n_u}_+\setminus\{0\}$.

2.1 Closed-loop performance criterion

To stabilize the system (4), let us define the following switched state-feedback control law:

$$u = K_{(j)}x, \quad j \in \mathcal{I}_{[1,2^{n_u}]}, \tag{5}$$

where $K_{(j)} \in \mathbb{R}^{n_u\times n_x}$ corresponds to the j^{th} controller gain matrix. In this way, the following closed-loop system is obtained:

$$\dot{x} = Ax + B\operatorname{sat}(K_{(j)}x, \overline{\nu}_{(j)}),\ \overline{\nu}_{(j)} \in \mathcal{V},\ j \in \mathcal{I}_{[1,2^{n_u}]}. \tag{6}$$

Let us consider the following QLF:

$$V(x) = x^\top P^{-1}x, \tag{7}$$

with $P \in \mathbb{S}^{n_x}_+$.

Thereupon, the closed-loop performance criterion considered throughout this work is defined by the following definition:

Definition 1. The closed-loop response (6) is said to have a desired *guaranteed switching decay rate* if:

$$\dot{V}(x) \leq -2\,\mathrm{d}_{\mathrm{R}(j)}\,V(x),\ \forall j \in \mathcal{I}_{[1,2^{n_u}]} \tag{8}$$

where $V(x)$ is a positive definite function and $\mathrm{d}_{\mathrm{R}(j)} \in \mathbb{R}_+$ corresponds to the j^{th} desired decay rate value associated to the controller gain $K_{(j)}$ in (5).

2.2 Region constraints

Let us define the asymmetric region of linearity of system (1) such that $\operatorname{sat}(u, \underline{u}, \overline{u}) = u$, as follows:

$$\mathcal{L}_a(u, \underline{u}, \overline{u}) \triangleq \left\{u \in \mathbb{R}^{n_u} : -\underline{u}_i \leq u_i \leq \overline{u}_i, i \in \mathcal{I}_{[1,n_u]}\right\}. \tag{9}$$

Due to the relationship between the input u and the state x given by the controller law (5), then for the active controller gain j, the region of linearity (9) is mapped onto a symmetric region, defined as:

$$\mathcal{L}_s(K_{(j)}, \overline{\nu}_{(j)}) \triangleq \left\{x \in \mathbb{R}^{n_x} : |K_{[i](j)}x| \leq \overline{\nu}_{i(j)}, i \in \mathcal{I}_{[1,n_u]}\right\}. \tag{10}$$

Then, with the objective of guaranteeing $\forall t \geq 0$ that $x(t) \in \mathcal{L}_s(K_{(j)}, \overline{\nu}_{(j)})$ and, as a consequence $u(t) \in \mathcal{L}_a(u, \underline{u}, \overline{u})$ $\forall t \geq 0$, the following set of inclusions is established $\forall j \in \mathcal{I}_{[1,2^{n_u}]}$:

$$\mathcal{X}_R \subset \mathcal{E}(P) \subset \mathcal{L}_s(K_{(j)}, \overline{\nu}_{(j)}), \tag{11}$$

where $\mathcal{X}_R$ and $\mathcal{E}(P)$ correspond, respectively, to the given known region of expected initial conditions described by the vertices $x_{(1)}, \ldots, x_{(n_v)}$:

$$\mathcal{X}_R \triangleq \operatorname{Co}\left\{x_{(1)}, x_{(2)}, \ldots, x_{(n_v)}\right\}, \tag{12}$$

and the unit Lyapunov level curve set of (7):

$$\mathcal{E}(P) \triangleq \{x \in \mathbb{R}^{n_x} : V(x) \leq 1\}. \tag{13}$$

Hence, as long as (11) holds, it can be ensured that if the initial condition $x_0 \in \mathcal{X}_R$, and hence $x_0 \in \mathcal{E}(P)$, $x(t) \in \mathcal{L}_s(K_{(j)}, \overline{\nu}_{(j)})$ $\forall t > 0$. Furthermore, it is also guaranteed that at least one of the 2^{n_u} computed controller gains $K_{(j)}$ will lead to an input signal (5) that lies in $\mathcal{L}_a(u, \underline{u}, \overline{u})$ defined in (9).

2.3 Problem definition

Based on Definition 1 and the region constraints (11), the problem under consideration in this work is formalized as follows:

Problem 1. For the continuous-time system (1) under the asymmetric saturation function (2), design a switching state-feedback controller (5) and a switching rule such that for any initial state $x_0 \in \mathcal{X}_R$ the closed-loop system response ensures the *guaranteed switching decay rate* (8).

3. MAIN RESULTS

Let us define the following theorem for designing a switching state-feedback controller (5) using the Lyapunov function candidate (7), Definition 1, and the set of region inclusions (11). Theorem 1 guarantees that $\mathcal{E}(P)$ is an invariant and contractive ellipsoidal set with respect to all trajectories of (6), forcing x to reside within this region where $\mathrm{sat}(u, \overline{\nu}_{(j)}) = u$ and, therefore, $u \in \mathcal{L}_a(u, \underline{u}, \overline{u})$.

Theorem 1. Given the regions (10), (12) and (13) with the known vertices $x_{(l)}$ and the desired *switching decay rate* values $\mathrm{d}_{\mathrm{R}(j)} \in \mathbb{R}_+$, let there exist decision matrices $P \in \mathbb{S}_+^{n_x}$ and $\Gamma_{(j)} \in \mathbb{R}^{n_u \times n_x}$ satisfying $\forall j \in \mathcal{I}_{[1,2^{n_u}]}$:

$$\mathrm{He}\left\{AP + B\Gamma_{(j)}\right\} + 2\,\mathrm{d}_{\mathrm{R}(j)}\,P \preceq 0, \tag{14}$$

$$\begin{bmatrix} P & x_{(l)} \\ \star & 1 \end{bmatrix} \succeq 0, \quad l \in \mathcal{I}_{[1,n_v]}, \tag{15}$$

$$\begin{bmatrix} \overline{\nu}_{i(j)}^2 & \Gamma_{[i](j)} \\ \star & P \end{bmatrix} \succeq 0,\ \overline{\nu}_{(j)} \in \mathcal{V},\ i \in \mathcal{I}_{[1,n_u]}. \tag{16}$$

Then, the closed-loop system response (6) with the switching state-feedback controller (5) computed as $K_{(j)} = \Gamma_{(j)} P^{-1}$ has the *guaranteed switching decay rate* performance expressed by (8). Furthermore, the convergence of $x(t) \to 0$ when $t \to \infty$ is ensured for any $x_0 \in \mathcal{X}_R$ such that $x(t) \in \mathcal{L}_s(K_{(j)}, \overline{\nu}_{(j)})$ and, hence, $u \in \mathcal{L}_a(u, \underline{u}, \overline{u})$.

Proof. The proof is split in two parts. The first part demonstrates how the performance criterion (8) is guaranteed by the LMI (14). The second one shows that any state trajectory $x(t)$ starting from an $x_0 \in \mathcal{X}_R \subset \mathcal{E}(P)$ belongs to $\mathcal{L}_s(K_{(j)}, \overline{\nu}_{(j)})$, $j \in \mathcal{I}_{[1,2^{n_u}]}$, implying that $x(t) \to 0$ for $t \to \infty$ with $u \in \mathcal{L}_a(u, \underline{u}, \overline{u})$.

Part 1: By assuming that $\mathrm{sat}(K_{(j)}x, \overline{\nu}_{(j)}) = K_{(j)}x$ $\forall j \in \mathcal{I}_{[1,2^{n_u}]}$, the closed-loop system response (6) can be rewritten as:

$$\dot{x} = \left(A + BK_{(j)}\right) x. \tag{17}$$

Then, the following set of bilinear matrix inequalities (BMI) can be obtained by considering (8):

$$\mathrm{He}\left\{P^{-1}A + P^{-1}BK_{(j)}\right\} + 2\,\mathrm{d}_{\mathrm{R}(j)}\,P^{-1} \preceq 0. \tag{18}$$

By pre- and post-multiplying (18) by P, one gets:

$$\mathrm{He}\left\{AP + BK_{(j)}P\right\} + 2\,\mathrm{d}_{\mathrm{R}(j)}\,P \preceq 0, \tag{19}$$

which can be transformed into the LMI (14) by means of the change of variable $\Gamma_{(j)} = K_{(j)}P$ $\forall j \in \mathcal{I}_{[1,2^{n_u}]}$. Therefore, the feasibility of the LMI (14) ensures that the closed-loop response of system (6) has the *switching guaranteed decay rate* performance (8).

Part 2: By using Schur's complement, the LMI (15) is equivalent to:

$$x_{(l)}^\top P^{-1} x_{(l)} \le 1, \quad l \in \mathcal{I}_{[1,n_v]}, \tag{20}$$

which leads to the inclusion $\mathcal{X}_R \subset \mathcal{E}(P)$ (see Hu (2001), Chap. 7). Next, the following condition is obtained by pre- and post-multiplying the LMI (16) by $\mathrm{diag}\left\{1, P^{-1}\right\}$:

$$\begin{bmatrix} \overline{\nu}_{i(j)}^2 & K_{[i](j)} \\ \star & P^{-1} \end{bmatrix} \succeq 0,\ \overline{\nu}_{(j)} \in \mathcal{V},\ \begin{array}{c} i \in \mathcal{I}_{[1,n_u]} \\ j \in \mathcal{I}_{[1,2^{n_u}]} \end{array}, \tag{21}$$

which by applying Schur's complement and pre- and post-multiplying by $x^\top$ and x, respectively, leads to:

$$x^\top \frac{K_{[i](j)}^\top K_{[i](j)}}{\overline{\nu}_{i(j)}^2} x \le x^\top P^{-1} x,$$

which yields $\mathcal{E}(P) \subset \mathcal{L}_s(K_{(j)}, \overline{\nu}_{(j)})$. Consequently, $x \in \mathcal{L}_s(K_{(j)}, \overline{\nu}_{(j)})$ is ensured for any trajectory $x \in \mathcal{E}(P)$ and, hence, $u \in \mathcal{L}_a(u, \underline{u}, \overline{u})$, thus concluding the proof. ■

3.1 Switching rule criterion

Since $x(t) \in \mathcal{L}_s(K_{(j)}, \overline{\nu}_{(j)})$ is guaranteed $\forall t \geq 0$ by Theorem 1, the following rule can be defined to select the active controller gain:

$$\begin{aligned} \sigma &= \mathrm{argmax}\, \mathrm{d}_{\mathrm{R}(h)} \\ \text{s.t. } h &\in \left\{ j \in \mathcal{I}_{[1,2^{n_u}]} : K_{(j)}x \in \mathcal{L}_a\left(u, \underline{u}, \overline{u}\right) \right\} \end{aligned} \tag{22}$$

It should be noted that this rule accounts for which of the controller gains provides an input signal contained in the asymmetric region of linearity (9), given the current value of the state. This is a novelty when compared to other existing approaches (see e.g., Yuan and Wu (2015) or Li and Lin (2018)), where the switching rule accounts for the control input signs instead. Furthermore, the rule (22) selects, among the non-saturating controller gains, the one that provides the largest guaranteed decay rate, therefore adjusting the closed-loop performance according to the criterion given in Definition 1. On the other hand, this work does not take into consideration the situation in which rule (22) is not defined due to saturating controller gains. The interested reader is referred to Hu (2001) and Tarbouriech et al. (2011) for a possible extension of the provided approach considering saturated control inputs.

4. EXTENSION TO THE LPV CASE

Let us extend the procedure described in Section 3 to the LPV case by modifying (6) as follows:

$$\dot{x} = A(\vartheta)x + B(\vartheta)\,\mathrm{sat}(K_{(j)}(\vartheta)x, \overline{\nu}_{(j)}), \tag{23}$$

with $\overline{\nu}_{(j)} \in \mathcal{V}$ and $j \in \mathcal{I}_{[1,2^{n_u}]}$. The parameter-dependent matrices $A(\vartheta) \in \mathbb{R}^{n_x \times n_x}$, $B(\vartheta) \in \mathbb{R}^{n_x \times n_u}$ and $K_{(j)}(\vartheta) \in \mathbb{R}^{n_u \times n_x}$ can be written as a convex combination of n_μ known vertices:

$$\left(A, B, K_{(j)}\right)(\vartheta) = \sum_{k=1}^{n_\mu} \mu_k\left(\vartheta\right)\left(A_{(k)}, B_{(k)}, K_{(j,k)}\right), \tag{24}$$

where $A_{(k)} \in \mathbb{R}^{n_x \times n_x}$, $B_{(k)} \in \mathbb{R}^{n_x \times n_u}$ and $K_{(j,k)} \in \mathbb{R}^{n_u \times n_x}$ stand for the given known vertex matrices and

$\vartheta \in \mathbb{P}_\vartheta \subseteq \mathbb{R}^{n_\vartheta}$ represents the scheduling parameter vector with $\mathbb{P}_\vartheta$ as a known, bounded and closed set. $\mu(\vartheta) \in \mathbb{R}^{n_\mu}$ corresponds to the polytopic weight vector belonging to the unit simplex:

$$\Delta_{n_\mu} \triangleq \left\{ \sum_{k=1}^{n_\mu} \mu_k(\vartheta) = 1, \mu_k(\vartheta) \geq 0, k \in \mathcal{I}_{[1,n_\mu]} \right\}. \quad (25)$$

Due to the controller gain being parameter-dependent, the symmetric region of linearity (10) becomes parameter-dependent as well $\forall j \in \mathcal{I}_{[1,2^{n_u}]}$:

$$\mathcal{L}_s(K_{(j)}(\vartheta), \overline{\nu}_{(j)}) \triangleq \left\{ x \in \mathbb{R}^{n_x} : |K_{[i](j)}(\vartheta)x| \leq \overline{\nu}_{i(j)}, i \in \mathcal{I}_{[1,n_u]} \right\}, \quad (26)$$

so that the set of inclusions (11) is modified as:

$$\mathcal{X}_R \subset \mathcal{E}(P) \subset \mathcal{L}_s(K_{(j)}(\vartheta), \overline{\nu}_{(j)}), \quad (27)$$

which leads to the extension of Theorem 1 to the LPV case.

Theorem 2. Given the regions (12), (13) and (26) with the known vertices $x_{(l)}$, a chosen Pólya's relaxation degree $d \in \mathbb{N}$, with $d \geq 2$, and the desired *switching decay rate* values $\mathrm{d}_{\mathrm{R}(j)} \in \mathbb{R}_+$, let there exist the decision matrices $P \in \mathbb{S}_+^{n_x}$ and $\Gamma_{(j,k)} \in \mathbb{R}^{n_u \times n_x}$ satisfying the LMI (15) and the following set of LMIs $\forall j \in \mathcal{I}_{[1,2^{n_u}]}$ and $\mathbf{k} \in \mathbb{I}^+_{(d,n_\mu)}$:

$$\sum_{\mathbf{p} \in \mathcal{P}(\mathbf{k})} \left(\mathrm{He}\left\{ A_{(p_1)}P + B_{(p_1)}\Gamma_{(j,p_2)} \right\} + 2\,\mathrm{d}_{\mathrm{R}(j)}\,P \right) \preceq 0 \quad (28)$$

$$\begin{bmatrix} \overline{\nu}^2_{i(j)} & \Gamma_{[i](j,k)} \\ \star & P \end{bmatrix} \succeq 0,\ \overline{\nu}_{(j)} \in \mathcal{V},\ \begin{array}{c} i \in \mathcal{I}_{[1,n_u]} \\ k \in \mathcal{I}_{[1,n_\mu]} \end{array}. \quad (29)$$

Then, the *guaranteed switching decay rate* performance defined in Definition 1 holds for all parameter-dependent terms appearing in (23), (26) and (27) with the controller gain computed as $K_{(j)}(\vartheta) = \sum_{k=1}^{n_\mu} \mu_k(\vartheta)\Gamma_{(j,k)}(\vartheta)P^{-1}$.

Proof. Similarly to the proof of Theorem 1, the following BMI can be obtained $\forall j \in \mathcal{I}_{[1,2^{n_u}]}$:

$$\mathrm{He}\left\{ A(\vartheta)P + B(\vartheta)K_{(j)}(\vartheta)P \right\} + 2\,\mathrm{d}_{\mathrm{R}(j)}\,P \preceq 0, \quad (30)$$

which can be transformed into the following LMI by means of the change of variable $\Gamma_{(j)}(\vartheta) = K_{(j)}(\vartheta)P$:

$$\mathrm{He}\left\{ A(\vartheta)P + B(\vartheta)\Gamma_{(j)}(\vartheta) \right\} + 2\,\mathrm{d}_{\mathrm{R}(j)}\,P \preceq 0. \quad (31)$$

By using the polytopic representation (24) of the parameter-dependent matrices appearing in (31), one gets:

$$\sum_{k_1=1}^{n_\mu} \sum_{k_2=1}^{n_\mu} \mu_{k_1}(\vartheta)\mu_{k_2}(\vartheta) \times \left(\mathrm{He}\left\{ A_{(k_1)}P + B_{(k_1)}\Gamma_{(j,k_2)} \right\} + 2\,\mathrm{d}_{\mathrm{R}(j)}\,P \right) \preceq 0. \quad (32)$$

Since the negative-definiteness of expression (32) involves multiple polytopic summations, the application of Pólya's relaxation theorem (Sala and Arino, 2007) is applied in order to get the LMI (28).

By multiplying the left-hand side of the LMI (29) by $\mu_k(\vartheta)$ and summing it up to $k \in \mathcal{I}_{[1,n_\mu]}$, one gets:

$$\begin{bmatrix} \overline{\nu}^2_{i(j)} & \Gamma_{[i](j)}(\vartheta) \\ \star & P \end{bmatrix} \succeq 0,\ \overline{\nu}_{(j)} \in \mathcal{V},\ \begin{array}{c} i \in \mathcal{I}_{[1,n_u]} \\ j \in \mathcal{I}_{[1,2^{n_u}]} \end{array}. \quad (33)$$

Then, let us pre- and post-multiply the expression (33) by $\mathrm{diag}\left\{1, P^{-1}\right\}$, thus obtaining:

$$\begin{bmatrix} \overline{\nu}^2_{i(j)} & K_{[i](j)}(\vartheta) \\ \star & P^{-1} \end{bmatrix} \succeq 0,\ \overline{\nu}_{(j)} \in \mathcal{V},\ \begin{array}{c} i \in \mathcal{I}_{[1,n_u]} \\ j \in \mathcal{I}_{[1,2^{n_u}]} \end{array}. \quad (34)$$

Finally, the following inequality is obtained by using Schur's complement and pre- and post-multiplication by $x^\top$ and x:

$$x^\top \frac{K_{[i](j)}(\vartheta)^\top K_{[i](j)}(\vartheta)}{\overline{\nu}^2_{i(j)}} x \leq x^\top P^{-1} x,$$

which leads to the parameter-dependent inclusions $\mathcal{E}(P) \subset \mathcal{L}_s(K_{(j)}(\vartheta), \overline{\nu}_{(j)})$. As a result, $x \in \mathcal{L}_s(K_{(j)}(\vartheta), \overline{\nu}_{(j)})$ is guaranteed for any $x \in \mathcal{E}(P)$, and so $u \in \mathcal{L}_a(u, \underline{u}, \overline{u})$, thus concluding the proof. ■

Note that the same rule (22) can be applied to the LPV case, although it will lead to the active controller being dependent on the varying parameter ϑ:

$$\begin{aligned} \sigma(\vartheta) &= \mathrm{argmax}\,\mathrm{d}_{\mathrm{R}(h)} \\ \text{s.t. } h &\in \left\{ j \in \mathcal{I}_{[1,2^{n_u}]} : K_{(j)}(\vartheta)x \in \mathcal{L}_a(u, \underline{u}, \overline{u}) \right\} \end{aligned} \quad (35)$$

5. ILLUSTRATIVE EXAMPLE

Let us consider an example taken from Tarbouriech et al. (2011), with modified saturation limits:

$$A = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3 \end{bmatrix},\ B = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix},\ \begin{array}{l} \underline{u} = [2,3]^\top \\ \overline{u} = [5,2]^\top \end{array}. \quad (36)$$

The characterization of the asymmetric saturation function (2) is obtained by a combination of 4 symmetric saturations, thus defining the following matrices:

$$D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, D_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, D_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\hat{D}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \hat{D}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \hat{D}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \hat{D}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (37)$$

Then, the symmetric saturation limits $\overline{\nu}_{(j)} \in \mathcal{V}$ of the switched closed-loop system (6) are obtained for $j \in \mathcal{I}[1,4]$ with $\mathcal{V}$ defined as:

$$\mathcal{V} = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}. \quad (38)$$

5.1 Design specifications

Consider that initial states of system (1) belongs to the polyhedral set $\mathcal{X}_R$ defined as:

$$\mathcal{X}_R = \mathrm{Co}\left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}. \quad (39)$$

Then, for each controller gain $K_{(j)}$ in (5), let us define the following desired *switching decay rate* values:

$$\mathrm{d}_{\mathrm{R}(1)} = 0,\ \mathrm{d}_{\mathrm{R}(2)} = 1.65,\ \mathrm{d}_{\mathrm{R}(3)} = 3.30,\ \mathrm{d}_{\mathrm{R}(4)} = 4.95, \quad (40)$$

which will lead to obtain the fastest guaranteed closed-loop convergence speed for the closed-loop system (6) associated with the controller gain $K_{(4)}$ and the saturation limits $\overline{\nu}_4 \in \mathcal{V}$, whereas, the slowest one is established for the system with $K_{(1)}$ and $\overline{\nu}_1 \in \mathcal{V}$. On the other hand, the decay rates $\mathrm{d}_{\mathrm{R}(2)}$ and $\mathrm{d}_{\mathrm{R}(3)}$ are chosen by considering 33% and 66% of $\mathrm{d}_{\mathrm{R}(4)}$, respectively.

Remark 1. It should be noted that there exists a trade-off between the provided values in (40) and the feasibility of the problem. For example, a larger value of $\mathrm{d}_{\mathrm{R}(1)}$ implies lower values in the remaining decay rates in order to reach a feasible solution. Furthermore, keep in mind that these values can be obtained by employing a linear search technique until Problem 1 becomes unfeasible for a given pre-determined design criterion.

After determining the design specifications, Problem 1 is solved by implementation of Theorem 1 in *MATLAB* environment via the *YALMIP* toolbox (Löfberg, 2004) and the use of the *SeDuMi* solver (Sturm, 1999).

5.2 Simulation results

This section illustrates the results of the designed switching state-feedback controller under three different initial conditions: $x_0^{(1)} \triangleq [3.5162, 0.6602]^\top$, $x_0^{(2)} \triangleq [2.9, -0.65]^\top$ and $x_0^{(3)} \triangleq [-1, -1]^\top$, denoted by a solid red line (—), a blue dashed line (- -) and a dotted green line (···), respectively.

Figs. 1-2 show the closed-loop state responses and the evolution of the control input over time, respectively. It is worth noting that regardless of the initial conditions, all states tend to the origin ensuring closed-loop stabilization. For the case of starting in $x_0^{(1)}$, the computed input signal has three discontinuities due to the activation of three different controller gains, as shown in Fig. 3, where the signal $\sigma(t)$ is plotted. Conversely, it can be seen that the input signals do not show any discontinuities in simulation $x_0^{(3)}$ as a result of using the same controller gain for the entire simulation.

Fig. 4 illustrates the phase plane delimited by the asymmetric regions of linearity obtained for each designed controller: $\sigma = 1$, $\sigma = 2$, $\sigma = 3$ and $\sigma = 4$. Furthermore, the regions $\mathcal{X}_R$ and $\mathcal{E}(P)$ are represented by the shaded black area and the solid violet line (—), respectively. Then, it can be seen that all region transitions produced by the evolution of the state correspond to the values of the signal $\sigma(t)$ shown in Fig. 3.

Remark 2. During the controller design stage, a unit Lyapunov level curve $\mathcal{E}(P)$ has been considered without loss of generality. It should be emphasized, however, that all level curves of the QLF (7) completely contained in the illustrated asymmetrical regions of linearity in Fig. 4, such as the one bordered by the solid yellow line (—), satisfy the stated closed-loop performance requirement.

Fig. 5 shows that the Lyapunov function $V(x)$ is strictly decreasing in all the simulations. Furthermore, note that the closed-loop convergence speed of the state trajectory corresponding to $x_0^{(1)}$ rises from the specified value $\mathrm{d}_{\mathrm{R}(2)}$ to the value of $\mathrm{d}_{\mathrm{R}(4)}$ according to current state vector, thus producing abrupt changes in $\dot{V}(x)$. Finally, Fig.6 shows that *guaranteed switching decay rate* (8), denoted by the colourful translucid manifold, is ensured for all the possible non-saturating state trajectories. This demonstrates the capacity of online adaptation of the closed-loop system.

6. CONCLUSIONS

The design of a control law for a linear system subject to asymmetric saturations has been studied in this work. The suggested approach is based on the transformation of an asymmetrically saturated linear system into an analogous

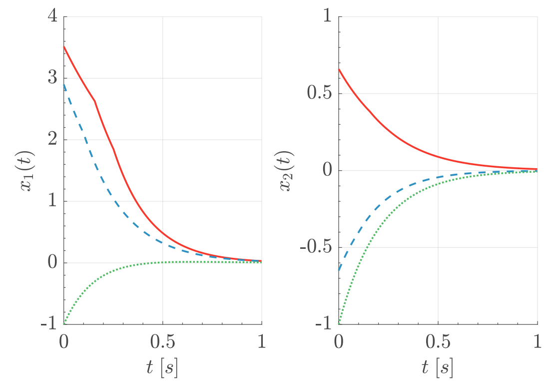

Fig. 1. Closed-loop state responses.

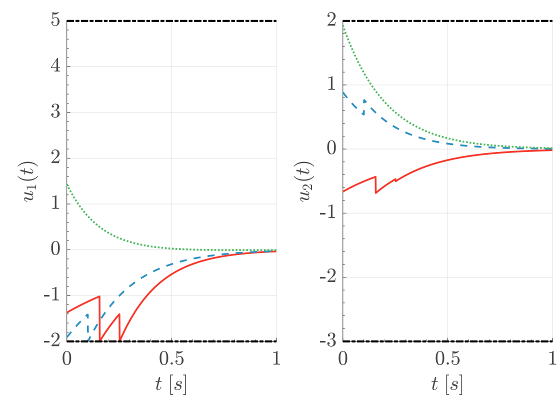

Fig. 2. Evolution of control inputs.

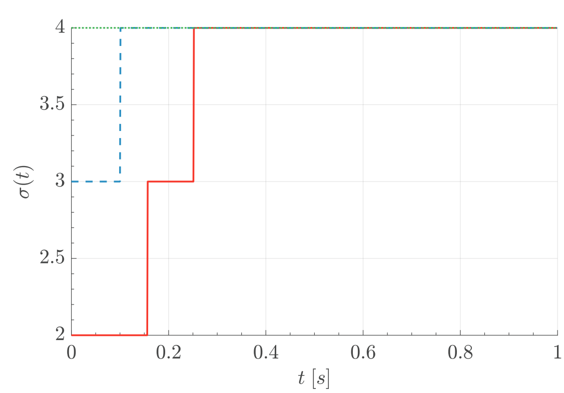

Fig. 3. Switching control rule signal.

switched system with symmetric saturations, yielding an LMI-based methodology for designing a switching state-feedback controller. The results show that the suggested switching rule criterion satisfies the *guaranteed switching decay rate* performance criterion. In this sense, the closed-loop system response modifies online its performance in terms of convergence speed based on the current state vector. Furthermore, it has been shown how to extend the proposed methodology to the parameter-varying case.

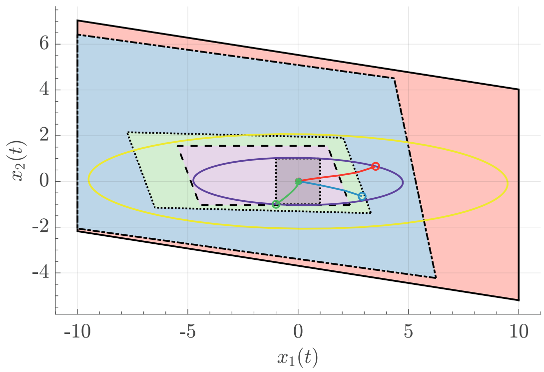

Fig. 4. Asymmetric regions of linearity.

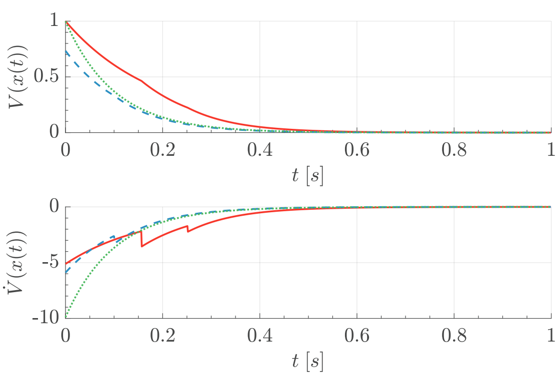

Fig. 5. Evolution of the Lyapunov function.

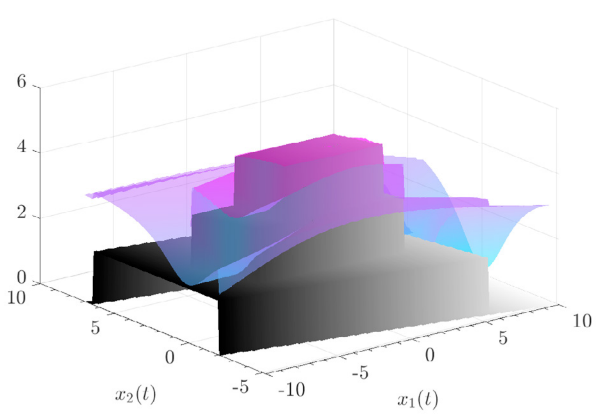

Fig. 6. Guaranteed switching decay rate. (■■ corresponds to $-\dot{V}(x)\,/\,(2V(x))$ and ■ denote $\mathrm{d}_{\mathrm{R}(\sigma)}$)

Future research will focus on developing alternative LMI-based methodologies to deal with the case where the saturation limits are time-varying by considering shifting specifications as in Ruiz et al. (2021). Furthermore, techniques that take into consideration the saturated control input will be explored.

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