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# Virtual actuator and sensor fault tolerant consensus for homogeneous linear multi-agent systems

Damiano Rotondo, Didier Theilliol, Jean-Christophe Ponsart

*Abstract*—**This paper presents a fault tolerant consensus protocol for homogeneous linear multi-agent systems using virtual actuators and virtual sensors. By means of the virtual actuators/sensors, the faulty system is reconfigured so that it can be brought into a block-triangular form through an appropriate change of variables. In this way, a Lyapunov-based approach can be used to obtain design conditions expressed as a feasibility problem involving linear matrix inequalities (LMIs). Both the linear time invariant (LTI) and the linear parameter varying (LPV) cases are discussed, the latter under the assumption of synchronized trajectories of the time-varying parameters. An academic example is used to illustrate the main features of the proposed fault tolerant consensus protocol. In particular, it is shown that by activating the virtual actuators/sensors, the agents achieve consensus in spite of severe faults, whereas instability of the synchronization errors occurs if no fault tolerant strategy is employed.**

*Index Terms*—**Fault tolerant consensus, multi-agent systems (MASs), linear matrix inequalities (LMIs), virtual actuators, virtual sensors.**

## I. Introduction

CONSENSUS in multi-agent systems (MASs) has been a topic researched over the past couple decades due to its importance in many applications, such as cooperative control of vehicular platoons, synchronization of coupled oscillators, formation control of autonomous vehicles, and rendezvous of space shuttles, among others [1], [2], [3]. As most systems, also MASs can be affected by faults that can degrade the performance or, in the worst case, cause a loss of stability that can have severe consequences on the agents' integrity. For this reason, fault-tolerant control (FTC) has been used to maintain the system performance close to a desirable value and preserve the stability properties even under faults. Among the proposed techniques, one finds adaptive control [4], [5], [6], fuzzy control [7], or sliding mode control [8].

The above mentioned techniques belong to a family of FTC approaches that modify or adapt the controller taking into account the fault occurrence. Another family of FTC approaches relies on the principle of hiding the fault from the controller's point of view by activating a dedicated block in the control loop that normally does not operate under non-faulty conditions. This *fault-hiding* strategy has two advantages. First, it allows implementing fault tolerance as a plug-and-play property; second, the nominal controller does not need to be disabled, so that all the valuable implicit knowledge acquired during the design cycle and embedded into the nominal control law is preserved.

When the above-mentioned dedicated block is used to tolerate sensor faults, it acts as a state observer that estimates the output of the faulty sensor, which is called *virtual sensor*. On the other hand, tolerance against actuator faults is achieved by replacing the faulty actuator with additional control effort assigned to the healthy actuators. In this case, the block is called *virtual actuator*, which is related via duality to the virtual sensor, in the same way as state-feedback controllers relate to state observers. Initially developed for LTI systems [9], virtual sensors and actuators have been applied successfully to piecewise affine [10], Hammerstein-Wiener [11], linear parameter varying (LPV) [12], [13], networked [14] and descriptor systems [15]. Practical validation of this FTC approach has been achieved using case studies such as turbofan engines [16] and pH neutralization plants [17], and a model-free adaptive version has been recently proposed in [18].

Some works have described how virtual actuators could be applied in the multi-agent framework to achieve a fault-tolerant consensus. For instance, [19] has shown how the tracking errors of the faulty agents could be kept bounded if the fault estimates were accurate. An adaptive version, suitable for use with MASs composed by heterogeneous agents, was proposed by [20]. On the other hand, [21] described a virtual actuator-based architecture for fault-tolerant leader-following consensus. However, the joint virtual actuator-virtual sensor design has not been considered so far in the existing literature.

Motivated by the above, in this paper we aim at developing a joint virtual actuator-virtual sensor formulation for fault tolerant consensus of homogeneous linear multi-agent systems. It is shown that the overall system comprising the estimate-feedback controller, the state observer, the virtual actuator and the virtual sensor can be brought into a block-triangular form by means of an appropriate change of variables. We provide a linear matrix inequalities (LMIs)-based formulation of the gain design. It is later shown how the provided results can be adapted to the linear parameter varying (LPV) case under the assumption of synchronized trajectories of the time-varying parameters, thus equipping the gain-scheduled protocol originally proposed in [22] with fault tolerant properties. In this way, a generalization to the multi-agent case of the results in [12] is obtained.

The remaining of the paper is organized as follows. Section II describes the multi-agent systems and the type of faults under consideration. Section III describes the components of the nominal consensus protocol (controller and observer), and

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those of the proposed fault tolerant approach (the virtual actuator and the virtual sensor). It is discussed that thanks to the introduction of these reconfiguration blocks, a block-triangular structure of the overall augmented system is recovered in spite of the fault occurrence. Section IV provides design conditions for obtaining the gains, based on linear matrix inequalities. Section V discusses the extension of the theoretical results to the LPV case. Section VI illustrates the main features of the proposed fault tolerant consensus protocol using a simulation example. Finally, Section VII summarizes the main conclusions.

*Notation:* Given a matrix $X \in \mathbb{R}^{n\times m}$, the symbol $X^\dagger$ denotes the Moore-Penrose inverse (a.k.a. pseudoinverse) of $X$ for which the following holds:

$$XX^\dagger X = X \tag{1}$$

The notation $\mathrm{diag}(a, b, \ldots, z)$ denotes the diagonal matrix which has $a, b, \ldots, z$ on the diagonal. Given a matrix $X = X^T$, the notation $X \prec 0$ must be interpreted in the sense of negative definiteness, which corresponds to all the eigenvalues of $X$ being negative. The symbol $\otimes$ is used to denote the Kronecker product between matrices.

## II. System Description

Let us consider a MAS with $N$ homogeneous agents which correspond to the nodes $\mathcal{V}$ of an undirected connected graph $\mathcal{G}(\mathcal{V}, \epsilon, \mathcal{A})$, with $\epsilon = \{(i, j) : i, j \in \mathcal{V}\} \subseteq \mathcal{V}\times\mathcal{V}$ denoting the set of edges, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ denoting the adjacency matrix, whose elements satisfy $a_{ii} = 0$ and $a_{ij} > 0$ if and only if $(i, j) \in \epsilon$. The matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ with $l_{ii} = \sum_{j\neq i} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$, is referred to as the *Laplacian matrix* of $\mathcal{G}$, which is positive semi-definite and with exactly one zero eigenvalue.

The agents can be affected by actuator and sensor faults, and their dynamics is described by:

$$\dot{x}_{fi}(t) = Ax_{fi}(t) + B_f\left(\phi_i(t)\right)\left(u_{fi}(t) + f_{ui}(t)\right) \tag{2}$$
$$y_{fi}(t) = C_f\left(\gamma_i(t)\right)x_{fi}(t) + f_{yi}(t) \tag{3}$$

where $i = 1, \ldots, N$, $x_{fi}(t) \in \mathbb{R}^{n_x}$ represents the state vector, $u_{fi}(t) \in \mathbb{R}^{n_u}$ denotes the control inputs and $y_{fi}(t) \in \mathbb{R}^{n_y}$ are the sensor outputs. The matrices $A \in \mathbb{R}^{n_x\times n_x}$, $B_f \in \mathbb{R}^{n_x\times n_u}$ and $C_f \in \mathbb{R}^{n_y\times n_x}$ are assumed to be constant and known for design purposes. The vector $f_{ui}(t) \in \mathbb{R}^{n_u}$ denotes the additive actuator faults affecting the $i$-th agent, whereas $\phi_i(t) \in \mathbb{R}^{n_u}$ denotes the multiplicative actuator faults, embedded in the input matrix $B_f$, as follows:

$$B_f\left(\phi_i(t)\right) = B\mathrm{diag}\left(\phi_{i,1}(t), \ldots, \phi_{i,n_u}(t)\right), \tag{4}$$

where $B$ denotes the faultless input matrix, and $0 \le \phi_{i,j}(t) \le 1$ represents the effectiveness of the $j$-th actuator ($j = 1, \ldots, n_u$) of the $i$-th agent, such that the extreme values $\phi_{i,j} = 0$ and $\phi_{i,j} = 1$ represent its complete loss and its healthy situation, respectively. Similarly, $f_{yi}(t) \in \mathbb{R}^{n_y}$ represents additive sensor faults, whereas $\gamma_i(t) \in \mathbb{R}^{n_y}$ are multiplicative sensor faults, such that the output matrix $C_f\left(\gamma_i(t)\right)$ is obtained as follows:

$$C_f\left(\gamma_i(t)\right) = \mathrm{diag}\left(\gamma_{i,1}(t), \ldots, \gamma_{i,n_y}(t)\right)C, \tag{5}$$

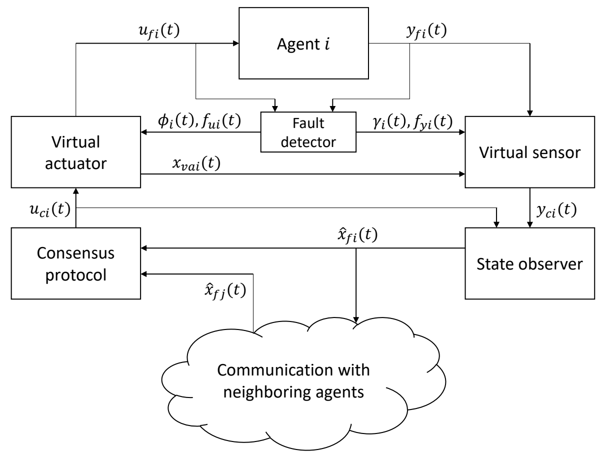


Fig. 1. Conceptual scheme of the proposed fault tolerant strategy.

where $C$ is the healthy output matrix, and $0 \le \gamma_{i,j}(t) \le 1$ denotes the effectiveness of the $j$-th sensor ($j = 1, \ldots, n_y$) of the $i$-th agent, so that the extreme values $\gamma_{i,j} = 0$ and $\gamma_{i,j} = 1$ denote its complete loss and healthy situation, respectively.

In the following, we will assume a perfect fault estimation to be available so that the fault functions $\phi_i(t)$, $f_{ui}(t)$, $\gamma_i(t)$ and $f_{yi}(t)$ can be used as if they were known. Although this assumption is unrealistic in practical settings, it serves the purpose of building the theoretical framework for the operation of the virtual sensors and actuators. It can be expected that fault estimation errors will deteriorate the performance of the proposed approach, which is an issue that deserves further investigation and that can be tackled using the ideas contained in [23].

## III. Fault Tolerant Consensus Using Virtual Actuators And Sensors

In this section, a joint virtual actuators and sensors approach is used for achieving fault tolerant consensus in the multi-agent system described in the previous section. The consensus protocol is based on observed states, as proposed by [24]. A conceptual scheme of the proposed fault tolerant consensus strategy, which shows the exchange of information between blocks and agents, is provided in Fig. 1. The main idea is to introduce these blocks in the control loop to mask the faults from the controller/observer point of view [25]. When an actuator is affected by a fault, the $i$-th virtual actuator will compute the vector $u_{fi}(t)$ from the output of the $i$-th agent's nominal controller $u_{ci}(t)$, taking into account the fault. Similarly, the $i$-th virtual sensor reconstructs the healthy output vector $y_{ci}(t)$ from the faulty output $y_{fi}(t)$, taking into account the sensor fault appearance.

Let us consider the following consensus protocol of the MAS based on using state estimated values:

$$u_{ci}(t) = K\sum_{j\in\mathcal{N}_i} a_{ij}\left(\hat{x}_{fi}(t) - \hat{x}_{fj}(t)\right), \quad i = 1, \ldots, N \tag{6}$$

where the matrix $K \in \mathbb{R}^{n_u\times n_x}$ is the controller gain to be designed, and $\mathcal{N}_i$ denotes the set of agents adjacent to $i$. The estimated states $\hat{x}_{fi}(t) \in \mathbb{R}^{n_x}$ are provided by local state observers of the form:

$$\dot{\hat{x}}_{fi}(t) = A\hat{x}_{fi}(t) + Bu_{ci}(t) + L\left[C\hat{x}_{fi}(t) - y_{ci}(t)\right] \quad (7)$$

where $L \in \mathbb{R}^{n_x \times n_y}$ is the state observer gain to be designed, and $y_{ci}(t)$ is the healthy output vector which, under sensor fault occurrence, is provided by the virtual sensor.

The signal $u_{ci}(t)$ is processed by the virtual actuator to obtain the actual value to be given to the (possibly faulty) actuators. The structure of the virtual actuator depends on how $\mathrm{rank}\left(B_f\left(\phi_i(t)\right)\right)$ compares to $\mathrm{rank}(B)$. If:

$$\mathrm{rank}\left(B\right) = \mathrm{rank}\left(B_f\left(\phi_i(t)\right)\right) \neq 0 \quad (8)$$

then the fault can be tolerated through an appropriate redistribution of the control inputs, as for example in case of partial faults or when the control efforts corresponding to the loss actuators can be achieved as a linear combination of the efforts of the remaining actuators. In this situation, each column of $B$ can be rewritten as a linear combination of the columns of $B_f\left(\phi_i(t)\right)$ which means that there exists a matrix $\Phi\left(\phi_i(t)\right)$ such that:

$$B = B_f\left(\phi_i(t)\right)\Phi\left(\phi_i(t)\right) \quad (9)$$

Consequently, by choosing the virtual actuator reconfiguration structure as:

$$u_{fi}(t) = N_{va}\left(\phi_i(t)\right)u_{ci}(t) - f_{ui}(t) \quad (10)$$

where the matrix function $N_{va}\left(\phi_i(t)\right)$ is given by:

$$N_{va}\left(\phi_i(t)\right) = B_f\left(\phi_i(t)\right)^{\dagger} B \quad (11)$$

Then, by replacing (9)-(11) into (2), and recalling (1), one eventually obtains:

$$\dot{x}_{fi}(t) = Ax_{fi}(t) + Bu_{ci}(t) \quad (12)$$

which clearly shows that the reconfigured system behaves as under non-faulty conditions.

On the other hand, in cases where:

$$\mathrm{rank}\left(B\right) > \mathrm{rank}\left(B_f\left(\phi_i(t)\right)\right) \neq 0 \quad (13)$$

the fault tolerance is achieved by adding a dynamical behavior to the virtual actuator, by introducing the virtual actuator state $x_{vai}(t)$ and the corresponding state equation. These cases are described by:

$$B^* = B_f\left(\phi_i(t)\right)N_{va}\left(\phi_i(t)\right) \quad (14)$$

where $B^*$ does not depend on $\phi_i(t)$ because the matrix function $N_{va}\left(\phi_i(t)\right)$ eliminates the effects of actuator partial faults, as originally shown in [26] and reported in Appendix I. Hence, the virtual actuator reconfiguration structure becomes:

$$u_{fi}(t) = N_{va}\left(\phi_i(t)\right)\left[u_{ci}(t) - M_{va}x_{vai}(t)\right] - f_{ui}(t) \quad (15)$$

where $M_{va}$ denotes the virtual actuator gain to be designed and the $i$-th agent virtual actuator state is updated as:

$$\dot{x}_{vai}(t) = \left[A + B^*M_{va}\right]x_{vai}(t) + \left[B - B^*\right]u_{ci}(t) \quad (16)$$

Similarly, the virtual sensor structure depends on how $\mathrm{rank}\left(C_f\left(\gamma_i(t)\right)\right)$ compares to $\mathrm{rank}\left(C\right)$. If:

$$\mathrm{rank}\left(C\right) = \mathrm{rank}\left(C_f\left(\gamma_i(t)\right)\right) \neq 0 \quad (17)$$

then the reconfiguration structure consists of a static block:

$$y_{ci}(t) = N_{vs}\left(\gamma_i(t)\right)\left[y_{fi}(t) + C_f\left(\gamma_i(t)\right)x_{vai}(t) - f_{yi}(t)\right] \quad (18)$$

where the matrix $N_{vs}\left(\gamma_i(t)\right)$ is given by:

$$N_{vs}\left(\gamma_i(t)\right) = CC_f\left(\gamma_i(t)\right)^{\dagger} \quad (19)$$

and by arguments similar to the virtual actuator case, equivalence of reconfigured and non-faulty system can be shown. Otherwise, if:

$$\mathrm{rank}\left(C\right) > \mathrm{rank}\left(C_f\left(\gamma_i(t)\right)\right) \neq 0 \quad (20)$$

does not hold, a dynamical behavior must be introduced with a virtual sensor state $x_{vsi}(t)$ and the corresponding state equation. These cases are described by matrices $C^*$ given by:

$$C^* = N_{vs}\left(\gamma_i(t)\right)C_f\left(\gamma_i(t)\right) \quad (21)$$

where the dependence on $\gamma_i(t)$ is removed thanks to the matrix $N_{vs}\left(\gamma_i(t)\right)$, which eliminates the effect of partial sensor faults (a proof of this fact can be obtained by applying the reasoning in Appendix I to the matrix $C^T$). Then, the reconfiguration structure is given by:

$$\begin{aligned} y_{ci}(t) = &N_{vs}\left(\gamma_i(t)\right)\left[y_{fi}(t) + C_f\left(\gamma_i(t)\right)x_{vai}(t) - f_{yi}(t)\right] \\ &+ \left[C - C^*\right]x_{vsi}(t) \end{aligned} \quad (22)$$

where $x_{vsi}(t)$ is the $i$-th agent virtual sensor state, updated as:

$$\begin{aligned} &\dot{x}_{vsi}(t) = \left[A + M_{vs}C^*\right]x_{vsi}(t) + Bu_{ci}(t) \quad (23) \\ &- M_{vs}N_{vs}\left(\gamma_i(t)\right)\left[y_{fi}(t) + C_f\left(\gamma_i(t)\right)x_{vai}(t) - f_{yi}(t)\right] \end{aligned}$$

where $M_{vs} \in \mathbb{R}^{n_x \times n_y}$ denotes the virtual sensor gain to be designed.

After connecting the agent dynamics (2)-(3), the consensus protocol (6), the local state observers (7), the virtual actuator reconfiguration structure (15)-(16) and the virtual sensor reconfiguration structure (22)-(23), one gets the following equations:

$$\begin{aligned} \dot{x}_{fi}(t) = &Ax_{fi}(t) - B^*M_{va}x_{vai}(t) && (24) \\ &+ B^*K\sum_{j\in\mathcal{N}_i} a_{ij}\left(\hat{x}_{fi}(t) - \hat{x}_{fj}(t)\right) \\ \dot{\hat{x}}_{fi}(t) = &A\hat{x}_{fi}(t) + BK\sum_{j\in\mathcal{N}_i} a_{ij}\left(\hat{x}_{fi}(t) - \hat{x}_{fj}(t)\right) && (25) \\ &+ LC\left(\hat{x}_{fi}(t) - x_{vsi}(t)\right) \\ &- LC^*\left(x_{fi}(t) + x_{vai}(t) - x_{vsi}(t)\right) \\ \dot{x}_{vai}(t) = &\left(A + B^*M_{va}\right)x_{vai}(t) && (26) \\ &+ \left(B - B^*\right)K\sum_{j\in\mathcal{N}_i} a_{ij}\left(\hat{x}_{fi}(t) - \hat{x}_{fj}(t)\right) \\ \dot{x}_{vsi}(t) = &\left(A + M_{vs}C^*\right)x_{vsi}(t) - M_{vs}C^*\left(x_{fi}(t) + x_{vai}(t)\right) \\ &+ BK\sum_{j\in\mathcal{N}_i} a_{ij}\left(\hat{x}_{fi}(t) - \hat{x}_{fj}(t)\right) && (27) \end{aligned}$$

Let us define new state variables $z_{1i}(t) = x_{vsi}(t) - x_{fi}(t) - x_{vai}(t)$, $z_{2i}(t) = \hat{x}_{fi}(t) - x_{vsi}(t)$, $z_{3i}(t) = x_{fi}(t) + x_{vai}(t)$ and $z_{4i}(t) = x_{vai}(t)$ which, as shown in the following, corresponds to a change of state coordinates that puts the augmented state matrix in a block-triangular form. Then, one obtains:

$$\dot{z}_{1i}(t) = (A + M_{vs}C^*)\, z_{1i}(t) \tag{28}$$

$$\dot{z}_{2i}(t) = (L - M_{vs})\, C^* z_{1i}(t) + (A + LC)\, z_{2i}(t) \tag{29}$$

$$\dot{z}_{3i}(t) = A z_{3i}(t) + BK \sum_{j\in\mathcal{N}_i} a_{ij} \sum_{k=1}^{3} (z_{ki}(t) - z_{kj}(t)) \tag{30}$$

$$\begin{aligned}\dot{z}_{4i}(t) = &(A + B^* M_{va})\, z_{4i}(t) \\ &+ (B - B^*)\, K \sum_{j\in\mathcal{N}_i} a_{ij} \sum_{k=1}^{3} (z_{ki}(t) - z_{kj}(t))\end{aligned} \tag{31}$$

By defining $z_1(t) = \left[z_{11}(t)^T, z_{12}(t)^T, \ldots, z_{1N}(t)^T\right]^T$, the following is obtained from (28):

$$\dot{z}_1(t) = [I_N \otimes (A + M_{vs}C^*)]\, z_1(t) \tag{32}$$

Similarly, the following is obtained from (29) by defining $z_2(t) = \left[z_{21}(t)^T, z_{22}(t)^T, \ldots, z_{2N}(t)^T\right]^T$:

$$\begin{aligned}\dot{z}_2(t) = &[I_N \otimes (L - M_{vs})\, C^*]\, z_1(t) \\ &+ [I_N \otimes (A + LC)]\, z_2(t)\end{aligned} \tag{33}$$

Let us now define synchronizing states as $\delta_{3i}(t) = z_{3i}(t) - (1/N)\sum_{j=1}^{N} z_{3j}(t) = z_{3i}(t) - \overline{z_3}(t)$. Then, one can verify that the vectors $\delta_3(t) = \left[\delta_{31}(t)^T, \delta_{32}(t)^T, \ldots, \delta_{3N}(t)^T\right]^T$ and $z_3(t) = \left[z_{31}(t)^T, z_{32}(t)^T, \ldots, z_{3N}(t)^T\right]^T$ are related by:

$$\delta_3(t) = \begin{bmatrix} \frac{N-1}{N} I_{n_x} & -\frac{1}{N} I_{n_x} & \cdots & -\frac{1}{N} I_{n_x} \\ -\frac{1}{N} I_{n_x} & \frac{N-1}{N} I_{n_x} & \cdots & -\frac{1}{N} I_{n_x} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{N} I_{n_x} & -\frac{1}{N} I_{n_x} & \cdots & \frac{N-1}{N} I_{n_x} \end{bmatrix} z_3(t) = \Upsilon z_3(t) \tag{34}$$

Then, the following can be written:

$$\begin{aligned}\dot{\delta}_3(t) = &\Upsilon\, [(\mathcal{L} \otimes BK)\, (z_1(t) + z_2(t)) \\ &+ (I_N \otimes A + \mathcal{L} \otimes BK)\, z_3(t)]\end{aligned} \tag{35}$$

Taking into account that $(\mathcal{L} \otimes BK) z_3(t) = (\mathcal{L} \otimes BK)\, \delta_3(t)$ since:

$$\begin{aligned}\sum_{j\in\mathcal{N}_i} a_{ij}\, (z_{3i}(t) - z_{3j}(t)) = \sum_{j\in\mathcal{N}_i} a_{ij} \left( \left( z_{3i}(t) - \frac{1}{N}\sum_{j=1}^{N} z_{3j}(t) \right) \right. \\ \left. - \left( z_{3j}(t) - \frac{1}{N}\sum_{j=1}^{N} z_{3j}(t) \right) \right) = \sum_{j\in\mathcal{N}_i} a_{ij}\, (\delta_{3i}(t) - \delta_{3j}(t))\end{aligned} \tag{36}$$

and that:

$$(I_N \otimes A)\, z_3(t) = (I_N \otimes A)\, (\delta_3(t) + \mathrm{col}\{\overline{z_3}(t)\}) \tag{37}$$

where $\mathrm{col}\{\overline{z_3}(t)\}$ is the column vector obtained by repeating $N$ times $z_3(t)$, then one obtains:

$$\begin{aligned}\dot{\delta}_3(t) = &\Upsilon\, [(\mathcal{L} \otimes BK)\, (z_1(t) + z_2(t)) \\ &+ (I_N \otimes A + \mathcal{L} \otimes BK)\, \delta_3(t)]\end{aligned} \tag{38}$$

where $\Upsilon(I_N \otimes A)\mathrm{col}\{\overline{z_3}(t)\} = 0$ has been exploited.

Finally, let $z_4(t) = \left[z_{41}(t)^T, z_{42}(t)^T, \ldots, z_{4N}(t)^T\right]^T$. Using (36), the following is obtained:

$$\begin{aligned}\dot{z}_4(t) = &[\mathcal{L} \otimes (B - B^*)\, K]\, (z_1(t) + z_2(t) + \delta_3(t)) \\ &+ [I_N \otimes (A + B^* M_{va})]\, z_4(t)\end{aligned} \tag{39}$$

Let $z(t) = \left[z_1(t)^T, z_2(t)^T, \delta_3(t)^T, z_4(t)^T\right]^T$, then by considering (32), (33), (38) and (39), we obtain the following autonomous system:

$$\dot{z}(t) = \begin{bmatrix} I_N \otimes (A + M_{vs}C^*) & 0 & 0 & 0 \\ I_N \otimes (L - M_{vs})\, C^* & I_N \otimes (A + LC) & 0 & 0 \\ \Upsilon\, (\mathcal{L} \otimes BK) & \Upsilon\, (\mathcal{L} \otimes BK) & \Upsilon\, [I_N \otimes A + \mathcal{L} \otimes BK] & 0 \\ \mathcal{L} \otimes (B - B^*)K & \mathcal{L} \otimes (B - B^*)K & \mathcal{L} \otimes (B - B^*)K & I_N \otimes (A + B^* M_{va}) \end{bmatrix} z(t) \tag{40}$$

Let us provide an interpretation of the value $z = 0$. Given the above definitions, $z(t) = 0$ corresponds to $x_{vai}(t) = 0$, $x_{vsi}(t) = x_{fi}(t) = \hat{x}_{fi}(t)$ and $x_{fi}(t) = \frac{1}{N}\sum_{j=1}^{N} x_{fj}(t) = \overline{x_f}$, which means that fault tolerant consensus is achieved. In the following section, we will provide sufficient LMI-based conditions for ensuring that $z(t) \to 0$ when $t \to \infty$.

## IV. LMI-based design conditions

Given the block-triangular structure of the augmented system obtained in (40), we can address the stability of the reconfigured control system by addressing the stability of different subsystems separately. Before doing so, let us note that, based on the symmetry property of undirected topology graphs $a_{ij} = a_{ji}$, which means that:

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} BK a_{ij}\, (z_{1i}(t) - z_{1j}(t) \\ + z_{2i}(t) - z_{2j}(t) + \delta_{3i} - \delta_{3j}(t)) = 0\end{aligned} \tag{41}$$

one finds out that the dynamics of $z(t)$ is driven by:

$$\dot{z}(t) = \Xi z(t) \tag{42}$$

with:

$$\Xi = \begin{bmatrix} I_N \otimes (A + M_{vs}C^*) & 0 & 0 & 0 \\ I_N \otimes (L - M_{vs})C^* & I_N \otimes (A + LC) & 0 & 0 \\ \mathcal{L} \otimes BK & \mathcal{L} \otimes BK & I_N \otimes A + \mathcal{L} \otimes BK & 0 \\ \mathcal{L} \otimes (B - B^*)K & \mathcal{L} \otimes (B - B^*)K & \mathcal{L} \otimes (B - B^*)K & I_N \otimes (A + B^* M_{va}) \end{bmatrix} \tag{43}$$

Then, based on (43), the following theorem (LMI-based design conditions) is obtained.

*Theorem 1:* Consider the closed-loop augmented system (40), obtained as the interconnection of the agent dynamics (2)-(3), the consensus protocol (6), the local state observers (7), the virtual actuator reconfiguration structure (15)-(16) and the virtual sensor reconfiguration structure (22)-(23), and let $\alpha_j$, $j = 2, \ldots, N$, be the non-zero eigenvalues of the

Laplacian matrix $\mathcal{L}$. If there exist symmetric matrices $P_1 \succ 0$, $P_2 \succ 0$, $P_3 \succ 0$, $P_4 \succ 0$, and matrices $G_{vs}$, $\Gamma$, $Y$ and $G_{va}$ of compatible dimensions such that:

$$\mathrm{He}\{P_1 A + G_{vs} C^*\} \prec 0 \tag{44}$$

$$\mathrm{He}\{P_2 A + \Gamma C\} \prec 0 \tag{45}$$

$$\mathrm{He}\left\{A P_3 + \alpha_j B Y\right\} \prec 0 \qquad \forall j = 2, \ldots, N \tag{46}$$

$$\mathrm{He}\left\{A P_4 + B^* G_{va}\right\} \prec 0 \tag{47}$$

holds for $j = 2, \ldots, N$, then fault tolerant consensus is achieved if the gains are chosen as $M_{vs} = P_1^{-1} G_{vs}$, $L = P_2^{-1}\Gamma$, $K = Y P_3^{-1}$ and $M_{va} = G_{va} P_4^{-1}$.

*Proof:* By applying a separation principle argument, we find out that the asymptotical stability of (42)-(43) can be assessed by proving that the following subsystems are asymptotically stable:

$$\dot{z}_1(t) = \left[I_N \otimes (A + M_{vs} C^*)\right] z_1(t) \tag{48}$$

$$\dot{z}_2(t) = \left[I_N \otimes (A + LC)\right] z_2(t) \tag{49}$$

$$\dot{\delta}_3(t) = \left[I_N \otimes A + \mathcal{L} \otimes BK\right] \delta_3(t) \tag{50}$$

$$\dot{z}_4(t) = \left[I_N \otimes (A + B^* M_{va})\right] z_4(t) \tag{51}$$

For the subsystem (48), let us choose the following Lyapunov function candidate:

$$V_1\left(z_1(t)\right) = z_1(t)^T \left(I_N \otimes P_1\right) z_1(t) \tag{52}$$

where $P_1 \succ 0$ is a matrix to be determined. Let us compute the derivative of $V_1\left(z_1(t)\right)$, thus obtaining:

$$\begin{aligned} \dot{V}_1\left(z_1(t)\right) &= \mathrm{He}\left\{z_1(t)^T \left(I_N \otimes P_1\right) \dot{z}_1(t)\right\} \\ &= \mathrm{He}\left\{z_1(t)^T \left(I_N \otimes P_1\right) \left(I_N \otimes (A + M_{vs} C^*)\right) z_1(t)\right\} \\ &= \mathrm{He}\left\{z_1(t)^T \left(I_N \otimes P_1 (A + M_{vs} C^*)\right) z_1(t)\right\} \end{aligned} \tag{53}$$

from which we obtain that $\dot{V}_1\left(z_1(t)\right) \prec 0$ for all $z_1(t) \neq 0$ as long as:

$$\mathrm{He}\left\{P_1 (A + M_{vs} C^*)\right\} \prec 0 \tag{54}$$

By performing the change of variables $G_{vs} = P_1 M_{vs}$, (44) is obtained.

For the subsystem (49), by choosing the Lyapunov function candidate:

$$V_2\left(z_2(t)\right) = z_2(t)^T \left(I_N \otimes P_2\right) z_2(t) \tag{55}$$

with $P_2 \succ 0$ to be determined, computing its derivative similarly to (53), and performing the change of variables $\Gamma = P_2 L$, (45) is obtained.

For the subsystem (50), by choosing the Lyapunov function candidate:

$$V_3\left(\delta_3(t)\right) = \delta_3(t)^T \left(I_N \otimes P_3^{-1}\right) \delta_3(t) \tag{56}$$

we obtain the time derivative:

$$\begin{aligned} \dot{V}_3\left(\delta_3(t)\right) &= \mathrm{He}\left\{\delta_3(t)^T \left(I_N \otimes P_3^{-1}\right) \dot{\delta}_3(t)\right\} \\ &= \mathrm{He}\left\{\delta_3(t)^T \left(I_N \otimes P_3^{-1}\right) \left(I_N \otimes A + \mathcal{L} \otimes BK\right) \delta_3(t)\right\} \\ &= \mathrm{He}\left\{\delta_3(t)^T \left(I_N \otimes P_3^{-1} A + \mathcal{L} \otimes P_3^{-1} BK\right) \delta_3(t)\right\} \end{aligned} \tag{57}$$

Let us now consider the spectral decomposition of $\mathcal{L} = \Pi \Lambda \Pi^T$, where the orthogonal matrix $\Pi \in \mathbb{R}^{N \times N}$ contains the eigenvectors of $\mathcal{L}$, while $\Lambda = \mathrm{diag}\left(\alpha_1, \ldots, \alpha_N\right) \in \mathbb{R}^{N \times N}$ where the eigenvalues are ordered so that $\alpha_1 < \alpha_2 \leq \ldots \leq \alpha_N$, with $\alpha_1 = 0$ due to the positive semidefiniteness of $\mathcal{L}$.

By applying the change of coordinates:

$$\zeta(t) = \left(\Pi^T \otimes I_{n_x}\right) \delta_3(t) \tag{58}$$

we obtain:

$$\dot{V}_3\left(\zeta(t)\right) = \mathrm{He}\left\{\zeta(t)^T \left(I_N \otimes P_3^{-1} A + \Lambda \otimes P_3^{-1} BK\right) \zeta(t)\right\} \tag{59}$$

which, due to the fact that $\zeta_1(t) = 0$, can be rewritten as:

$$\dot{V}_3\left(\zeta(t)\right) = \sum_{j=2}^{N} \zeta_j(t)^T \mathrm{He}\left\{P_3^{-1} A + \alpha_j P_3^{-1} BK\right\} \zeta_j(t) \tag{60}$$

for which we obtain that $\dot{V}_3\left(\delta_3(t)\right) \prec 0$ for all $\delta_3(t) \neq 0$ as long as:

$$\mathrm{He}\left\{P_3^{-1} A + \alpha_j P_3^{-1} BK\right\} \prec 0 \qquad \forall j = 2, \ldots, N \tag{61}$$

By pre- and post-multiplying (61) by $P_3$, and performing the change of variables $Y = K P_3$, (46) is obtained.

Finally, for the subsystem (51), let us choose:

$$V_4\left(z_4(t)\right) = z_4(t)^T \left(I_N \otimes P_4^{-1}\right) z_4(t) \tag{62}$$

that leads to the condition:

$$\mathrm{He}\left\{P_4^{-1}\left(A + B^* M_{va}\right)\right\} \prec 0 \tag{63}$$

which, through pre- and post-multiplication by $P_4$ and the change of variables $G_{va} = M_{va} P_4$ leads to (51), thus completing the proof. □

*Remark 1:* The conditions in Theorem 1 take the form of a typical linear matrix inequality (LMI)-based feasibility problem. Widely accessible toolboxes and solvers, such as YALMIP [27] and SeDuMi [28] can be used to find suitable matrices $P_1$, $P_2$, $P_3$, $P_4$, $G_{vs}$, $G_{va}$, $\Gamma$, $Y$ and use them to compute the gains $M_{vs}$, $L$, $K$, $M_{va}$ to be implemented.

## V. Extension to the LPV case

This section will discuss how the results presented in the previous sections can be extended to the LPV case following some elements contained in [22]. In the LPV case, each agent is associated to its vector of time-varying parameters $\theta_i(t)$, $i = 1, \ldots, N$, available in real-time and assumed to vary in a polytope $\Theta$. Consequently, the matrices $A$, $B$, $C$, $K$, $L$, $M_{va}$ and $M_{vs}$ appearing in the equations in Sections II-III are modified into the matrix functions $A\left(\theta_i(t)\right)$, $B\left(\theta_i(t)\right)$, $C\left(\theta_i(t)\right)$, $K\left(\theta_i(t)\right)$, $L\left(\theta_i(t)\right)$, $M_{va}\left(\theta_i(t)\right)$ and $M_{vs}\left(\theta_i(t)\right)$, which can be written as the convex combinations of appropriate *vertex matrices*, through parameter-varying coefficients, as follows[1]:

$$\begin{pmatrix} A\left(\theta_i(t)\right) \\ B\left(\theta_i(t)\right) \\ C\left(\theta_i(t)\right) \\ K\left(\theta_i(t)\right) \\ L\left(\theta_i(t)\right) \\ M_{va}\left(\theta_i(t)\right) \\ M_{vs}\left(\theta_i(t)\right) \end{pmatrix} = \sum_{h=1}^{S} \alpha_h\left(\theta_i(t)\right) \begin{pmatrix} A_h \\ B_h \\ C_h \\ K_h \\ L_h \\ M_{vah} \\ M_{vsh} \end{pmatrix} \tag{64}$$

where $S$ is the total number of vertices, and the coefficients $\alpha_h\left(\theta_i(t)\right)$ satisfy [**?**]:

$$\sum_{h=1}^{S} \alpha_h\left(\theta_i(t)\right) = 1, \quad \alpha_h\left(\theta_i(t)\right) \geq 0 \quad \forall \theta_i(t) \in \Theta \tag{65}$$

By defining the multi-agent vector of time-varying parameters $\theta(t) = \left[\theta_1(t)^T, \theta_2(t)^T, \ldots, \theta_N(t)^T\right]^T$ and the following shorthand notation:

$$\alpha_{ij}\left(\theta(t)\right) = \mathrm{diag}\left(\alpha_i\left(\theta_1(t)\right)\alpha_j\left(\theta_1(t)\right), \ldots, \alpha_i\left(\theta_N(t)\right)\alpha_j\left(\theta_N(t)\right)\right) \tag{66}$$

the steps described in Section III can be repeated, as described in the Appendix, to show that in the LPV case (40) becomes (dependence of $\theta$ on $t$ is not explicitly shown):

$$\dot{z} = \sum_{h=1}^{S}\sum_{l=1}^{S}\left\{\left[\begin{matrix} \alpha_{hl}\left(\theta\right) \otimes \left(A_h + M_{vsh}C_l^*\right) \\ \alpha_{hl}\left(\theta\right) \otimes \left(L_h - M_{vsh}\right)C_l^* \\ \Upsilon\alpha_{hl}\left(\theta\right)\mathcal{L} \otimes B_l K_h \\ \alpha_{hl}\left(\theta\right)\mathcal{L} \otimes (B_l - B_l^*)K_h \end{matrix}\right.\right. \quad \cdots \tag{67}$$

$$\cdots \quad \begin{matrix} 0 \\ \alpha_{hl}\left(\theta\right) \otimes \left(A_h + L_h C_l\right) \\ \Upsilon\alpha_{hl}\left(\theta\right)\mathcal{L} \otimes B_l K_h \\ \alpha_{hl}\left(\theta\right)\mathcal{L} \otimes (B_l - B_l^*)K_h \end{matrix} \quad \begin{matrix} 0 \\ 0 \\ \Upsilon\alpha_{hl}\left(\theta\right)\left[I_N \otimes A_h + \mathcal{L} \otimes B_l K_h\right] \\ \alpha_{hl}\left(\theta\right)\mathcal{L} \otimes (B_l - B_l^*)K_h \end{matrix} \quad \cdots$$

$$\cdots \quad \left.\left.\begin{matrix} 0 \\ 0 \\ 0 \\ \alpha_{hl}\left(\theta\right) \otimes \left(A_h + B_l^* M_{vah}\right) \end{matrix}\right] z + \begin{bmatrix} 0 \\ 0 \\ \Upsilon f_{hl}\left(\theta, \overline{z_3}\right) \\ 0 \end{bmatrix}\right\}$$

where:

$$f_{hl}\left(\theta(t), \overline{z_3}(t)\right) = \begin{bmatrix} \alpha_h\left(\theta_1(t)\right)\alpha_l\left(\theta_1(t)\right)A_h\overline{z_3} \\ \vdots \\ \alpha_h\left(\theta_N(t)\right)\alpha_l\left(\theta_N(t)\right)A_h\overline{z_3} \end{bmatrix} \tag{68}$$

By looking at (67), it is clear that in cases where $\Upsilon f_{hl}\left(\theta(t), \overline{z_3}(t)\right) \neq 0$, the dynamics for $z(t)$ is not described by an autonomous system, so that in general consensus would not be achieved. A sufficient condition for $\Upsilon f_{hl}\left(\theta(t), \overline{z_3}(t)\right) = 0$ is that $\theta_i(t) = \theta_j(t)\,\forall i, j = 1, \ldots, N$. In fact, in this case:

$$\Upsilon f_{hl}(\theta, \overline{z_3}) = \begin{bmatrix} \frac{N-1}{N}I_{n_x} & \cdots & -\frac{1}{N}I_{n_x} \\ -\frac{1}{N}I_{n_x} & \cdots & -\frac{1}{N}I_{n_x} \\ \vdots & \ddots & \vdots \\ -\frac{1}{N}I_{n_x} & \cdots & \frac{N-1}{N}I_{n_x} \end{bmatrix} \begin{bmatrix} \alpha_h\left(\theta\right)\alpha_l\left(\theta\right)A_h\overline{z_3} \\ \vdots \\ \alpha_h\left(\theta\right)\alpha_l\left(\theta\right)A_h\overline{z_3} \end{bmatrix}$$

$$\begin{bmatrix} \left(1 - \sum_{i=1}^{N}\frac{1}{N}\right)\alpha_h(\theta)\alpha_l(\theta)\overline{z_3} \\ \vdots \\ \left(1 - \sum_{i=1}^{N}\frac{1}{N}\right)\alpha_h(\theta)\alpha_l(\theta)\overline{z_3} \end{bmatrix} = 0 \tag{69}$$

For this reason, in the following we will provide sufficient conditions for fault tolerant consensus under the assumption that $\theta_i(t) = \theta_j(t)\,\forall i, j = 1, \ldots, N$, which is referred to as *synchronization* (whereas the case of different varying parameter trajectories is referred to as *non-synchronization*).

*Remark 2:* It is worth noting that when a single agent is considered ($N = 1$), by replacing $\delta_3(t)$ with $z_3(t)$ in the definition of $z(t)$, the behavior of the augmented system in [12] is recovered, so that the virtual actuator and sensor approach developed in this paper is indeed an extension to the multi-agent case of the results in [12].

The following theorem is the LPV version of Theorem 1.

*Theorem 2:* Consider the LPV system (67) under the synchronization and polytopic assumptions, i.e., $\theta_1(t) = \theta_2(t) = \ldots = \theta_N(t)$ and (64)-(65) hold. For any $s \in \mathbb{N}$, with $s \geq 2$, and for given eigenvalues $\alpha_j$, $j = 2, \ldots, N$, of the Laplacian matrix $\mathcal{L}$, if there exist symmetric matrices $P_1 \succ 0$, $P_2 \succ 0$, $P_3 \succ 0$, $P_4 \succ 0$ and matrices $G_{vs1}, \ldots, G_{vsS}$, $\Gamma_1, \ldots, \Gamma_S$, $Y_1, \ldots, Y_S$ and $G_{va1}, \ldots, G_{vaS}$ of compatible dimensions such that:

$$\sum_{\vec{r} \in \mathcal{P}(\vec{p})} \mathrm{He}\{P_1 A_{r_1} + G_{vsr_1}C_{r_2}^*\} \prec 0 \tag{70}$$

$$\sum_{\vec{r} \in \mathcal{P}(\vec{p})} \mathrm{He}\{P_2 A_{r_1} + \Gamma_{r_1}C_{r_2}\} \prec 0 \tag{71}$$

$$\sum_{\vec{r} \in \mathcal{P}(\vec{p})} \mathrm{He}\left\{A_{r_1}P_3 + \alpha_j B_{r_2}Y_{r_1}\right\} \prec 0 \quad \forall j = 2, \ldots, N \tag{72}$$

$$\sum_{\vec{r} \in \mathcal{P}(\vec{p})} \mathrm{He}\left\{A_{r_1}P_4 + B_{r_2}^* G_{var_1}\right\} \prec 0 \tag{73}$$

holds for $j = 2, \ldots, N$ and $\forall \vec{p} \in \mathbb{P}_s^+$, where:

$$\mathbb{P}_s = \{\ \vec{p} = [p_1, p_2, \ldots, p_s] \in \mathbb{N}^s | 1 \leq p_k \leq s\ \forall k = 1, \ldots, s\} \tag{74}$$

$$\mathbb{P}_s^+ = \{\vec{p} \in \mathbb{P}_s | p_k \leq p_{k+1}, k = 1, \ldots, s-1\} \tag{75}$$

and $\mathcal{P}(\vec{p}) \subset \mathbb{P}_s$ denotes the set of permutations, with possible repeated elements, of the multi-index $\vec{p}$, then fault tolerant consensus is achieved if the gains are chosen as $M_{vsi} = P_1^{-1}G_{vsi}$, $L_i = P_2^{-1}\Gamma_i$, $K_i = Y_i P_3^{-1}$ and $M_{vai} = G_{vai}P_4^{-1}$ for $i = 1, \ldots, S$.

*Proof.* The first part of the proof follows the reasoning of the proof of Theorem 1 to show that fault tolerant consensus is achieved if $\forall \theta \in \Theta$:

$$\mathrm{He}\{P_1 A(\theta) + G_{vs}(\theta)C^*(\theta)\} \prec 0 \tag{76}$$

$$\mathrm{He}\{P_2 A(\theta) + \Gamma(\theta)C(\theta)\} \prec 0 \tag{77}$$

$$\mathrm{He}\left\{A(\theta)P_3 + \alpha_j B(\theta)Y(\theta)\right\} \prec 0 \qquad \forall j = 2, \ldots, N \tag{78}$$

$$\mathrm{He}\left\{A(\theta)P_4 + B(\theta)^* G_{va}(\theta)\right\} \prec 0 \tag{79}$$

After accounting for the polytopic assumption (64), the conditions to be assessed become:

$$\sum_{h=1}^{S}\sum_{l=1}^{S} \alpha_h(\theta)\alpha_l(\theta)\mathrm{He}\{P_1 A_h + G_{vsh}C_l^*\} \prec 0 \tag{80}$$

$$\sum_{h=1}^{S}\sum_{l=1}^{S} \alpha_h(\theta)\alpha_l(\theta)\mathrm{He}\{P_2 A_h + \Gamma_h C_l\} \prec 0 \tag{81}$$

[1]Note that according to (4)-(5), if $B\left(\theta_i(t)\right)$ and $C\left(\theta_i(t)\right)$ are written as convex combinations, the matrices $B_f\left(\cdot\right)$ and $C_f\left(\cdot\right)$ are too.

$$\sum_{h=1}^{S}\sum_{l=1}^{S}\alpha_h(\theta)\alpha_l(\theta)\mathrm{He}\left\{A_hP_3+\alpha_jB_lY_h\right\}\prec 0 \tag{82}$$

$$\sum_{h=1}^{S}\sum_{l=1}^{S}\alpha_h(\theta)\alpha_l(\theta)\mathrm{He}\left\{A_hP_4+B_l^*G_{vah}\right\}\prec 0 \tag{83}$$

which correspond to the problem of verifying the negativity of double polytopic sums. By applying Polya's theorem on definite quadratic forms, as proposed e.g. by [30], (72)-(73) are obtained, thus completing the proof. □

## VI. Example

The example considered in this paper is taken from [31] and consists of a network of five aircrafts with a communication topology described by:

$$\mathcal{L}=\begin{bmatrix}2 & 0 & -1 & 0 & -1\\ 0 & 2 & -1 & -1 & 0\\ -1 & -1 & 2 & 0 & 0\\ 0 & -1 & 0 & 1 & 0\\ -1 & 0 & 0 & 0 & 1\end{bmatrix}$$

Each aircraft is described by its state and input vector:

$$x_i(t)=\begin{bmatrix}p_{Si}(t)\\ p_{Ri}(t)\\ p_{Yi}(t)\end{bmatrix}\qquad u_i(t)=\begin{bmatrix}q_{DTi}(t)\\ q_{ATi}(t)\\ q_{RUi}(t)\end{bmatrix}$$

where $p_{Si}(t)$ is the side-slip angle, $p_{Ri}(t)$ is the roll angle, $p_{Yi}(t)$ is the yaw rate, $q_{DTi}(t)$ is the differential tail deflection, $q_{AIi}(t)$ is the aileron deflection, and $q_{RUi}(t)$ is the rudder deflection, respectively.

The state-space matrices are given as follows:

$$A=\begin{bmatrix}-0.059 & 0.496 & -0.868\\ -5.513 & -0.939 & 0.665\\ 0.068 & 0.026 & -0.104\end{bmatrix}$$

$$B=\begin{bmatrix}0.006 & 0.006 & 0.004\\ 1.879 & 1.328 & 0.029\\ -0.109 & -0.096 & -0.084\end{bmatrix}$$

$$C=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}$$

The fault considered in this example is a complete loss of the rudder and of the second sensor, which lead to:

$$B_f=\begin{bmatrix}0.006 & 0.006 & 0\\ 1.879 & 1.328 & 0\\ -0.109 & -0.096 & 0\end{bmatrix}$$

$$C_f=\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

Note that these are severe faults, which correspond to (13) and (20) not hold. Hence, the implementation of virtual actuator and virtual sensor requires the computation of matrices $B^*$ and $C^*$ using (14) and (21), which gives:

$$B^*=\begin{bmatrix}0.006 & 0.006 & 0.008\\ 1.879 & 1.328 & 0.029\\ -0.109 & -0.096 & -0.084\end{bmatrix}$$

$$C^*=\begin{bmatrix}1 & 1 & 1\\ 0.667 & 0.667 & 0.667\end{bmatrix}$$

By using the YALMIP toolbox [27] with SeDuMi solver [28], the LMIs in Theorem 1 are solved, thus obtaining the following matrices:

$$P_1=\begin{bmatrix}1.964 & 0.388 & 0.370\\ 0.388 & 0.434 & 0.056\\ 0.370 & 0.056 & 1.099\end{bmatrix}$$

$$P_2=\begin{bmatrix}1.012 & 0.065 & -0.078\\ 0.065 & 0.845 & 0.085\\ -0.078 & 0.085 & 0.999\end{bmatrix}$$

$$P_3=\begin{bmatrix}0.293 & -0.320 & 0.035\\ -0.320 & 2.501 & 0.161\\ 0.035 & 0.161 & 0.367\end{bmatrix}$$

$$P_4=\begin{bmatrix}1.087 & -0.121 & 0.221\\ -0.121 & 1.013 & -0.034\\ 0.221 & -0.034 & 1.082\end{bmatrix}$$

$$M_{vs}=\begin{bmatrix}1.280 & 0\\ -1.455 & 0\\ -0.405 & 0\end{bmatrix}\qquad L=\begin{bmatrix}-0.333 & 2.045\\ 2.997 & -2.821\\ -0.836 & 0.398\end{bmatrix}$$

$$M_{va}=10^4\begin{bmatrix}-0.08 & -0.80 & -0.01\\ 0.12 & 1.13 & 0.01\\ -0.03 & -0.26 & 0.00\end{bmatrix}$$

$$K=\begin{bmatrix}632.0 & 237.0 & -176.7\\ -890.4 & -333.8 & 231.8\\ 133.9 & 24.8 & -0.1\end{bmatrix}$$

The simulation scenario considered in the following lasts 30 seconds, and presents faults in the agent #1 at time 10 seconds. We show and compare four different cases:

- no fault tolerant strategy is implemented (Fig. 2);
- the proposed fault tolerant strategy based on virtual actuators and sensors is activated as soon as the fault occurs, which corresponds to a rather ideal setting (Fig. 3);
- the proposed fault tolerant strategy is activated at time 10.5 seconds, i.e., 0.5 seconds after the fault occurrence. This corresponds to the more realistic situation where a fault diagnosis algorithm requires some time to collect and process some faulty data before diagnosing the presence and location of the faults (Fig. 4);
- faults are related to the estimates used by the FTC algorithm (denoted by $\hat{}$) as follows:

$$\begin{aligned}\hat{f}_{ui,j}(t)&=f_{ui,j}(t)+\varepsilon_{fui,j}(t)\\ \hat{f}_{yi,j}(t)&=f_{yi,j}(t)+\varepsilon_{fyi,j}(t)\\ \hat{\phi}_{i,j}(t)&=\phi_{i,j}(t)+\varepsilon_{\phi i,j}(t)\\ \hat{\gamma}_{i,j}(t)&=\gamma_{i,j}(t)+\varepsilon_{\gamma i,j}(t)\end{aligned}$$

  where the terms $\epsilon_{\dots}(t)$ denote uniformly distributed random variables in the intervals $\varepsilon_{fui,j}\in[-0.1,0.1]$, $\varepsilon_{fyi,j}\in[-0.1,0.1]$, $\varepsilon_{\phi i,j}(t)\in[-0.05,0.05]$, $\varepsilon_{\gamma i,j}(t)\in[-0.05,0.05]$. Moreover, in spite of the abrupt change from 1 to 0 of $\phi_{1,3}(t)$ and $\gamma_{1,2}(t)$ at time $t_f$, the transition of the expected value of the corresponding estimates

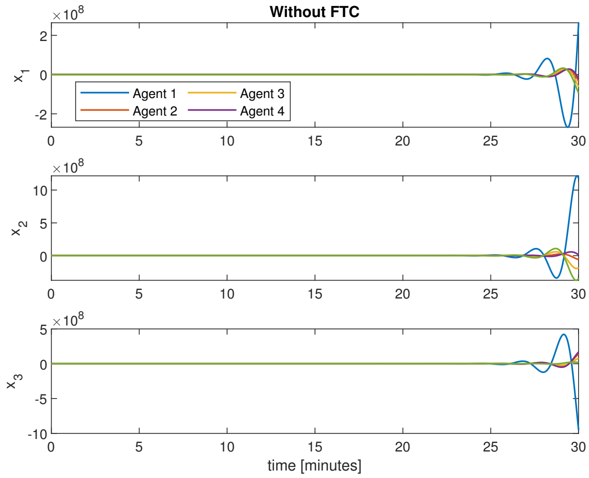


Fig. 2. Simulation results without fault tolerant strategy.

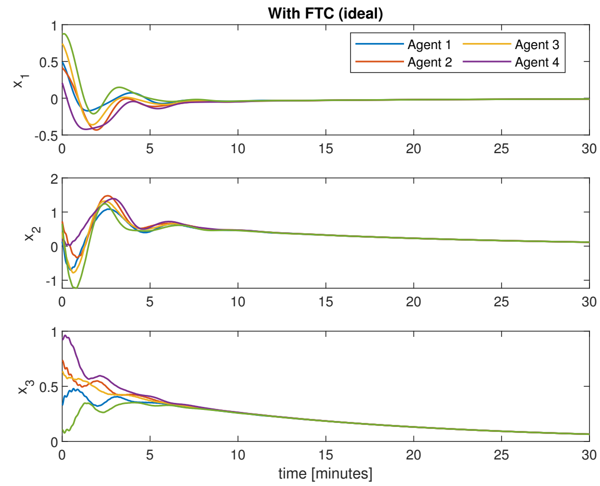


Fig. 3. Simulation results with fault tolerant strategy (ideal diagnosis).

$\hat{\phi}_{1,3}(t)$ and $\hat{\gamma}_{1,2}(t)$ from 1 to 0 happens linearly over a time interval of 0.5 seconds. This corresponds to the even more realistic situation where model uncertainties and measurement noise contribute to fault estimation errors (Fig. 5).

From the simulation, it can be seen that the effect of the fault is so severe that the dynamics of the synchronization error becomes unstable if no actions are taken to restore a good behavior of the consensus protocol despite the fault occurrence (see Fig. 2). Thanks to the proposed fault tolerant strategy, the effect of the fault is hidden from the controller and observer point of view, and the stability of the synchronization error is recovered so that the agents reach again consensus in spite of the fault. Under an ideal diagnosis assumption, the agents become completely insensitive to the fact that one of them is working with less actuation and sensoring capabilities (see Fig. 3). A delay in the activation of the FTC strategy causes a temporary instability of the synchronization error under fault occurrence, which leads to a loss of consensus. However, the activation of the virtual actuators and sensors recovers the stability of the consensus protocol, and the agents eventually reach consensus in spite of the faults (see Fig. 4). Finally, imperfections in the fault estimation degrade the performance of the multi-agent system under both nominal and faulty operation, although the proposed FTC strategy shows some inherent robustness which allows for recovery of the consensus in spite of the estimation errors (see Fig. 5).

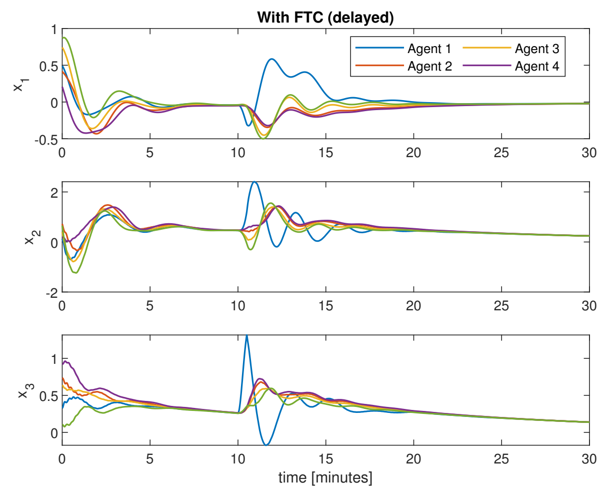


Fig. 4. Simulation results with fault tolerant strategy (delayed diagnosis).

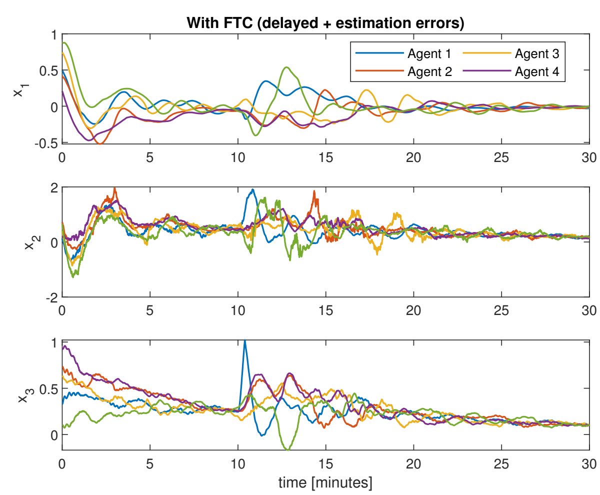


Fig. 5. Simulation results with fault tolerant strategy (imperfect estimation).

## VII. Conclusions

This paper has presented a fault-tolerant consensus protocol using virtual actuators and sensors. A theoretical proof of stability based on the separation principle has led to LMI-based design conditions for choosing the gains of the different components of the control system: the controller, the observer, and the two virtual components. The extension of the proposed fault tolerant consensus strategy to the LPV case has been discussed, and some connection with previous results concerning single agent systems has been established. The simulation results have confirmed the effectiveness of the proposed fault tolerant protocol, although they have shown that the quality of the fault diagnosis and fault estimation would impact the ability of the virtual actuators and sensors to provide fault tolerance without degrading the consensus performance. It clear that a theoretical investigation of the issues arising from an imperfect fault diagnosis is an important topic for future research.

## Appendix I

Let the nominal input matrix $B$ be given by:

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n_u} \\ b_{21} & b_{22} & \cdots & b_{2n_u} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n_x1} & b_{n_x2} & \cdots & b_{n_xn_u} \end{bmatrix} \tag{84}$$

Without loss of generality, we can assume that the first $n_f$ actuators are completely lost, i.e., $\phi_{i,j}(t) = 0$ for $j = 1, \ldots, n_f$, and $\phi_{i,j}(t) \neq 0$ for $j = n_f + 1, \ldots, n_u$. Hence, the matrix $B_f(\phi_i(t))$ calculated using (4) can be rewritten as:

$$B_f(\phi_i(t)) = \Psi\Omega(\phi_i(t)) \tag{85}$$

with:

$$\Psi = \begin{bmatrix} b_{1(n_f+1)} & b_{1(n_f+2)} & \cdots & b_{1n_u} \\ b_{2(n_f+1)} & b_{2(n_f+2)} & \cdots & b_{2n_u} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n_x(n_f+1)} & b_{n_x(n_f+2)} & \cdots & b_{n_xn_u} \end{bmatrix} \tag{86}$$

$$\Omega(\phi_i(t)) = \begin{bmatrix} 0 & \cdots & 0 & \phi_{i,n_f+1}(t) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \phi_{i,n_f+2}(t) & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \phi_{i,n_u}(t) \end{bmatrix} \tag{87}$$

The matrix $B_f(\phi_i(t))^\dagger$ can be calculated as follows:

$$B_f^\dagger = \Omega^T\left(\Omega\Omega^T\right)^{-1}\left(\Psi^T\Psi\right)^{-1}\Psi^T \tag{88}$$

where dependence of $B_f$ and $\Omega$ on $\phi_i(t)$ has been omitted. It is quite straightforward to check that:

$$\Omega^T\left(\Omega\Omega^T\right)^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \frac{1}{\phi_{i,n_f+1}(t)} & 0 & \cdots & 0 \\ 0 & \frac{1}{\phi_{i,n_f+2}(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\phi_{i,n_u}(t)} \end{bmatrix} \tag{89}$$

so that:

$$\begin{aligned} B^* &= B_f(\phi_i(t))\,B_f(\phi_i(t))^\dagger B \qquad (90) \\ &= \Psi\Omega\Omega^T\left(\Omega\Omega^T\right)^{-1}\left(\Psi^T\Psi\right)^{-1}\Psi^T B \\ &= \Psi\left(\Psi^T\Psi\right)^{-1}\Psi^T B \end{aligned}$$

which does not depend on $\phi_i(t)$.

## Appendix II

After connecting the LPV versions of equations (2)-(3), (6)-(7), (15)-(16) and (22)-(23), and defining the new state variables $z_{1i} = x_{vsi} - x_{fi} - x_{vai}$, $z_{2i} = \hat{x}_{fi} - x_{vsi}$, $z_{3i} = x_{fi} + x_{vai}$ and $z_{4i} = x_{vai}$, one obtains:

$$\dot{z}_{1i} = \left(A(\theta_i) + M_{vs}(\theta_i)C^*(\theta_i)\right)z_{1i} \tag{91}$$

$$\begin{aligned} \dot{z}_{2i} =& \left(L(\theta_i) - M_{vs}(\theta_i)\right)C^*(\theta_i)z_{1i} \qquad (92) \\ &+ \left(A(\theta_i) + L(\theta_i)C(\theta_i)\right)z_{2i} \end{aligned}$$

$$\dot{z}_{3i} = A(\theta_i)z_{3i} + B(\theta_i)K(\theta_i)\sum_{j\in\mathcal{N}_i} a_{ij}\sum_{k=1}^{3}(z_{ki} - z_{kj}) \tag{93}$$

$$\begin{aligned} \dot{z}_{4i} =& \left(A(\theta_i) + B^*(\theta_i)M_{va}(\theta_i)\right)z_{4i} \qquad (94) \\ &+ \left(B(\theta_i) - B^*(\theta_i)\right)K(\theta_i)\sum_{j\in\mathcal{N}_i} a_{ij}\sum_{k=1}^{3}(z_{ki} - z_{kj}) \end{aligned}$$

Accounting for the shorthand notation (66), the following is obtained from (91), taking into account the polytopic assumption (64):

$$\dot{z}_1 = \sum_{h=1}^{S}\sum_{l=1}^{S}\left[\alpha_{hl}(\theta) \otimes \left(A_h + M_{vsh}C_l^*\right)\right]z_1 \tag{95}$$

Similarly, the following is obtained from (92):

$$\begin{aligned} \dot{z}_2 = \sum_{h=1}^{S}\sum_{l=1}^{S}&\left\{\left[\alpha_{hl}(\theta) \otimes (L_h - M_{vsh})C_l^*\right]z_1 \qquad (96)\right. \\ &\left. + \left[\alpha_{hl}(\theta) \otimes (A_h + L_hC_l)\right]z_2\right\} \end{aligned}$$

By considering the relationship (34), the following can be obtained from (93):

$$\begin{aligned} \dot{\delta}_3 = \sum_{h=1}^{S}\sum_{l=1}^{S}\Upsilon&\left\{\left[\alpha_{hl}(\theta)\mathcal{L} \otimes B_lK_h\right](z_1 + z_2) \qquad (97)\right. \\ &\left. + \left[\alpha_{hl}(\theta) \otimes A_h + \alpha_{hl}(\theta)\mathcal{L} \otimes B_lK_h\right]z_3\right\} \end{aligned}$$

Taking into account that $\mathcal{L} \otimes B_lK_hz_3 = \mathcal{L} \otimes B_lK_h\delta_3$ due to (36), and that:

$$\alpha_{hl}(\theta)\left(I_N \otimes A_h\right)z_3 = \left(\alpha_{hl}(\theta) \otimes A_h\right)\delta_3 + f_{hl}\left(\theta, \overline{z_3}\right) \tag{98}$$

with $f_{hl}(\theta, \overline{z_3})$ defined as in (68), then one obtains:

$$\begin{aligned} \dot{\delta}_3 = \sum_{h=1}^{S}\sum_{l=1}^{S}\Upsilon&\left\{\left[\alpha_{hl}(\theta)\mathcal{L} \otimes B_lK_h\right](z_1 + z_2) \qquad (99)\right. \\ &\left. + \left[\alpha_{hl}(\theta) \otimes A_h + \alpha_{hl}(\theta)\mathcal{L} \otimes B_lK_h\right]\delta_3 + f_{hl}(\theta, \overline{z_3})\right\} \end{aligned}$$

Finally, (94) leads to:

$$\begin{aligned} \dot{z}_4 = \sum_{h=1}^{S}\sum_{l=1}^{S}&\left\{\left[\alpha_{hl}(\theta)\mathcal{L} \otimes \left(B_l - B_l^*\right)K_h\right](z_1 + z_2 + \delta_3)\right. \\ &\left. + \alpha_{hl}(\theta) \otimes (A_h + B_l^*M_{vah})z_4\right\} \qquad (100) \end{aligned}$$

By writing (95)-(96) and (99)-(100) in a compact form, (67) is obtained.

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