Students’ reasoned dialogs during problem solving in a Norwegian thinking classroom

INGUNN VALBEKMO AND RAYMOND BJULAND

This study explores students’ collective mathematical problem-solving processes. Grade-seven Norwegian students were observed during regular problem-solving sessions to discover how reasoned dialogs might develop in a thinking-classroom context. An analytical framework associated with *sociocultural discourse analysis* was used to identify utterances (dialog moves) that were essential in revealing a dynamic and continuous scaffolding process with symmetrical interaction between the students, and where the vertical whiteboards supported students during the reasoned dialogs. The context of a thinking classroom created an environment suitable for highly interactive learning where students constructed and refined their ideas in collaboration with each other. The findings also point to the crucial role of the teacher as a facilitator of classroom dialogs to get students to dig deeper into the ideas of others.

A recent review of literature points out the crucial role of collaboration in problem solving, emphasizing the importance of considering problem solving as highly situated and socially constructed, and where context matters (Liljedahl & Cai, 2021). The review highlights one specific choice-rich problem-solving environment, called the thinking-classroom context. For instance, Pruner and Liljedahl (2021) examined what happens when students in collaborative groups of three working on vertical whiteboards have access to resources outside the group. One important finding from this study showed that members of a group looked at the visible work of other groups, allowing the students to be engaged with that work, illustrating that the resources students gained from groupmates were important for the problem-solving process. Liljedahl and Cai (2021)

Ingunn Valbekmo, *Norwegian University of Science and Technology*
Raymond Bjuland, *University of Stavanger*

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call for more research to better understand what forms problem solving takes in different contextualized situations.

Referring to literature reviews of many perspectives on classroom research in recent decades, Webb et al. (2021) have pointed out the importance of giving students opportunities to explain their thinking, where engaging with others' mathematical ideas, and justifying their solutions, are productive for student learning. These authors call for research that more specifically examines in depth how participating in these ways supports learning. Meeting the call to investigate problem solving in various contextualized situations (Liljedahl & Cai, 2021) and to delve into how students benefit from explaining and engaging with groupmates' mathematical ideas (Webb et al., 2021), the present study aims to dig deeper into classroom sessions where students are working on problems while using vertical whiteboards. More specifically, the aim is to show how this collective problem-solving process takes place by undertaking a fine-grained analysis of the classroom dialog. In line with Warwick et al. (2016), and Fauskanger and Bjuland (2021), we apply an analytical framework (see method section) associated with sociocultural discourse analysis to explore teacher-student and student-student interactions for the potential they have "to engage in reasoned dialogue" (Littleton & Mercer, 2013, p. 112). The study addresses the following research question.

> *How do specific utterances of reasoned classroom dialogs show the students' collective mathematical problem-solving process in the context of a thinking classroom?*

To answer this research question, a case study (Stake, 1995) was conducted to develop an in-depth understanding of reasoned dialogs (see the section on reasoned dialogs) among students in a grade-seven Norwegian class (12–13 years old). The classroom was observed over five approximately 90-minute-long problem-solving sessions over a period of four months. Since the context is a thinking classroom, one aim has also been to focus on how the vertical whiteboards support students in their reasoned dialogs during their problem solving. In this study we use the term vertical whiteboard about A2-sized, electro-static surfaces hanging on a wall.

Theoretical background

The field of mathematics education has considered problem solving to be an essential part of teaching and learning in mathematics for more than 50 years. The aim is that students learn to solve problems and learn mathematics by solving problems (Liljedahl et al., 2016; Liljedahl & Cai, 2021). Our literature review is structured around relevant research within the specific choice-rich problem-solving environment in a thinking-classroom context with collectively reasoned classroom dialogs.

Whiteboards as a tool for changing a classroom into a thinking classroom

It appears that whiteboards had their renaissance after Liljedahl's work over the course of 15 years with whiteboards in mathematics classrooms (Liljedahl, 2021). To support teaching and learning through problem solving where students collaborate with others, Liljedahl (2016) presented a framework called "building thinking classrooms". In such a classroom students work with mathematical problem solving in an environment that elicits thinking, both individually and collectively. Teachers in a thinking classroom foster and expect thinking, and students work in visibly random groups, changing from lesson to lesson. They work on highly engaging thinking tasks (Liljedahl, 2021) using vertical non-permanent surfaces, such as whiteboards, that make their work visible to other students in the classroom.

In one of his studies, Liljedahl (2016) found that when students work on vertical whiteboards they get down to work quicker, have better persistence, are more eager, and collaborate better than when working at their desks. Megowan-Romanowicz (2016) has similar findings where the focus is on high-school physics classrooms, showing that whiteboards make it possible to change the focus from just looking for correct answers to a more open-ended activity of sense-making. Whiteboards were then used as a tool for mediating this activity while students were collaborating on revising and coordinating different representations of their solutions.

In a thinking classroom, students' help-seeking was more subtle than just asking the teacher. When students reached an impasse they sought help from other groups and their whiteboards. Students will use other whiteboards as resources in their work when it is necessary (Pruner & Liljedahl, 2021). Knowledge and understanding are passed on, co-constructed, or shared, and knowledge is dispersed amongst the students during their work (Liljedahl, 2021). Liljedahl discovered that there was greater mobility of knowledge in the classroom. When knowledge mobility increases in a classroom, the teacher is no longer the only source of knowledge. Students start to view themselves and their classmates as competent (Liljedahl, 2021). The findings in a recent Norwegian study in a thinking-classroom context also point out that students in a grade-seven classroom either used their classmates' whiteboards on their own initiative or were encouraged to do so by their teacher (Valbekmo & Svorkmo, 2021).

Reasoned classroom dialogs

The present study builds on a sociocultural perspective that is based on the work of Vygotsky (1978) who described language as a cultural tool in which knowledge is shared and developed amongst participants of communities (Mercer et al., 2019). Webb et al. (2021) highlight many perspectives that have all focused on the importance of explaining one's thinking and engaging with others' ideas in classroom dialogs. In this study, we will particularly highlight one of these perspectives: Exploratory talk (Mercer et al., 2019). According to Mercer et al. (2019), in exploratory talk students engage constructively in dialogs based on the wish to understand and engage in each other's ideas. Students ask each other questions, answer them, and members in the dialog try to reach an agreement before they continue. This perspective can be "described as a cultural tool for reasoning collectively" (Littleton & Mercer, 2013, p. 22). A dialog is considered to be reasoned when participants both engage critically and constructively with each other's ideas and treat the ideas as worthy of consideration (Littleton & Mercer, 2013).

In classroom dialogs the teacher wants to engage students, so they express their understandings, critique other students' opinions, and develop their thinking, reasoning, and knowledge through dialogs (Cui & Teo, 2021). The classroom dialogs can be student directed; students discuss their work, their strategies, or their solutions without any interference from the teacher. The dialog can also be guided by the teacher in a whole-class discussion or in small groups (Michaels & O'Connor, 2015). Bearing this in mind, we consider the notion of the *Zone of proximal development* (ZPD) and the notion of scaffolding to be important in the thinking classroom.

The notion of the ZPD was initially applied to situations where a teacher or adult explicitly supported a learner to complete a task the learner could not accomplish while working alone (Fernández et al., 2001). The difference between independent performance and aided performance is characterized as the child's ZPD. Wood et al. (1976) introduced scaffolding to describe teaching that helps a child to achieve more than he or she could do alone. The notion of scaffolding was used as a metaphor for the way a more experienced other can support a child during the process of solving a relatively challenging task. It was seen as temporary intellectual support that moved the learner towards a higher level of understanding and included a handover of independence and a transfer of responsibility (Bakker et al., 2015). Students can achieve more demanding tasks with the help of adults or more capable peers (Vygotsky, 1978).

Fernández et al. (2001) argue that students can scaffold each other more dynamically and continuously than a teacher can manage. When students scaffold each other in pairs, the intellectual support cannot be seen as temporary. In these situations, there are no expert others but a more symmetrical interaction between participants. To use the notion of *scaffolding* in situations where students work together in pairs a reconceptualization of it may be needed. Students are not as aware of the intentions of the scaffolding process as the teacher. The intentions for students collaborating in pairs are to finish the task and get the work done. They are to ask and answer questions to understand and solve the problem, and this can lead to a deeper understanding than what they might have attained when working alone (Wegerif & Mercer, 2000).

According to Webb et al. (2021), a shift in the understanding of mathematical ideas, marked changes in problem-solving strategies, or the generation of new mathematical ideas or problem-solving strategies can be seen as the result of mathematical advances. These authors documented a relationship between students' participation in explaining their work to others and/or engaging with others' ideas and their mathematical advances. In the present study, the students' shifts in problem-solving processes identified from their reasoned dialogs might indicate mathematical advances.

Method

Participants and setting

The class teacher, Kristin (all names are pseudonyms), and her 7th grade class, consisting of 25 students, were selected through purposeful sampling (Creswell & Poth, 2018) based on their prior experience of collaborating on mathematics problem solving. Kristin had used problem-solving tasks weekly since she started to teach mathematics to the class on the fifth grade. Kristin hoped that introducing vertical whiteboards and randomized grouping in the class would support her in teaching students how to learn from each other and collaborate on finding solutions for demanding mathematics problems. Even though Kristin felt that her students were good at solving problems *in* pairs, she pointed out that there was little collaboration *between* the pairs. The students had tried out vertical whiteboards during problem solving once before the first observed session. The only information given to the class was that they were going to use the whiteboards on the walls instead of using paper and pencil to solve the problems. In between the five problem-solving sessions, Kristin used vertical whiteboards in her regular lessons when she found this to be beneficial. The structure of the sessions will be explained in detail below (see analytical approach).

Data collection

The data material consists of audio and video recordings from five problem-solving sessions. These sessions were snapshots of regular practice. Kristin formed new random groups and selected two student pairs for observation in each session. She chose students who were able to collaborate, could verbalize their thinking, and were not too shy in front of the camera.

To ensure good sound quality, each of the students in the observed pairs and Kristin were equipped with a microphone connected to an audio recorder. Each observed pair was filmed by a researcher with a handheld camera. When the observed groups split up to discuss their work with the two members of another group, the researcher with the camera filmed the two students (one from each group) standing by the originally observed group's whiteboard. The other two students were not filmed, but they were equipped with audio recorders when they discussed their work on the other group's whiteboard. It took about 10–15 seconds to rearrange the sound-recording equipment in this change. The change was done before the students started to talk, to avoid interruptions in the dialog.

In the last three sessions, Kristin challenged her students to come together, visiting a whiteboard next to them to discuss solutions and solution strategies. Particularly in sessions three and four we could observe how two student pairs that had worked with the problem on two different whiteboards came together and discussed their problem-solving strategies. Following Webb et al. (2021), who pointed out the importance of giving students the opportunity to explain their thinking and engage with others' ideas, the present study aims to dig into situations from sessions 3 and 4, respectively. Below we present the two problems given for these two sessions.

The problems

The ice-cream problem was used in the third observed session. The students were told that guests were served ice cream at a birthday party. They could choose between four different flavors and each guest could choose two scoops. How many different combinations could they choose between? One of the students asked if they were allowed to choose two scoops of the same flavor, and Kristin told them it was up to each pair to decide. It was also up to each pair to decide if flavor one and then flavor two were the same or a different combination from flavor two and then flavor one.

The table-chair problem was used in the fourth observed lesson. A picture of a rectangular table was presented (see figure 1). The students were told that four chairs could be placed around the table, one on each side.

Figure 1. *Rectangular table used to illustrate the table-chair problem*

To make a longer table to accommodate more than four people, tables could be placed side-by-side. The class was asked how many chairs there could be around two tables placed together. They agreed this had to be six; no chairs could be placed at the two table sides that had been placed together. Then the students were asked to work on the whiteboards in pairs to find out how many chairs there would be if they placed five and ten tables together in rows. After a while, Kristin also asked her students about 17-table and then 100-table problems. The final part of the problem was to find a general expression for how many chairs there could be around any number of tables (*n*-tables).

Analytical approach

First, we carefully read the transcripts from the five problem-solving sessions. Then we focused on sessions 3, 4, and 5 as the students were encouraged to discuss their work across groups during these sessions. Based on the overarching structure, these three sessions were divided into the following seven steps: 1) Kristin presenting the problem (about 3–5 minutes); 2) Students working in pairs on the problems on their whiteboards while Kristin walked around monitoring their work; 3) Two student groups situated next to each other were encouraged to debate different solution strategies; 4) The students kept on working in their original pairs, telling each other what they had learned from discussing solution strategies with the student pair next to them; 5) Kristin leading a whole-class discussion, standing at one whiteboard, challenging different student pairs to explain their thinking; 6) The students returning to their original whiteboard, continuing their work on the problem; and 7) Kristin leading a whole-class discussion, summing up from the session.

To make a fine-grained analysis, we dig particularly into two student pairs, one from each session. In session 4, we followed Arne and Vetle and their work on the whiteboard. They were standing next to Mari and Synne. When Kristin encouraged her students to debate their solution strategies by studying the whiteboard of a neighboring student pair, Mari visited Arne, and Vetle visited Synne. By observing a particular student pair (e.g. Arne/Vetle) in depth our aim has been to show how their collective problem-solving process developed.

Warwick et al. (2016, p. 567) identified utterances with the potential to establish a reasoned dialog, emphasizing the following five dialog moves.

[DM1] – Requesting information, opinion or clarification

[DM2] – Making positive and supportive contributions

[DM3] – Expressing shared ideas and agreements

[DM4] – Providing evidence or reasoning

[DM5] – Challenging ideas or re-focusing talk

In our study, an utterance is considered to be a participant's verbalization as long as the interlocutor has the floor. This in-depth analysis of the utterances during the interactions from selected specific situations was coded with the five dialog moves developed by Warwick et al. (2016) to identify reasoned dialogs during problem solving on vertical whiteboards.

The utterances in table 1 illustrate examples of the dialog moves. The data material was coded individually for reasoned dialogs by the two authors. Then we worked together to discuss and agree on the coding.

Table 1. *Examples of dialog moves*

Utterance	Dialog moves [DM]
Mari: Where did you get 16?	[DM1]: Requesting clarification, inviting Arne to elaborate on his and Vetle's solution process.
Mari: Yes, I think so, but I'm a bit confused. What you just said doesn't match your calculation.	[DM2]: Supportive confirmation. [DM5]: Challenging Arne to clarify the mismatch between verbal explanation and the written calculation.
Arne: When we were doing 17-table problem, we first added the chairs for the 5-table problem and then for the 10-table problem.	[DM3]: Recapitulating and building on the prior discussion with his partner Vetle.
Arne: 16 is from the 5-table problem and the 2-table problem, 12 plus 6 is 18. But when I put two tables together the seats at one end of each table disappear, that's why we take minus two.	[DM4]: Building and elaborating on the solution process, making further reasoning.

Findings

The main goal of our work has been to better understand how students critically and productively discuss their work and engage in the ideas of others, in addition to examining how these reasoned dialogs might lead to a change in mathematical thinking. This could then lead to some sort of mathematical advance during their problem solving. We will here present our findings and discuss situations in sessions three and four, respectively.

The reasoned dialog with the ice-cream problem with Frida and Lea

Lea and Frida have been working on the problem for a while. They have used different colors to represent different flavors of ice cream and placed the different flavors side by side (see figure 2). They are unsure if they have a strategy that will help them solve the problem completely.

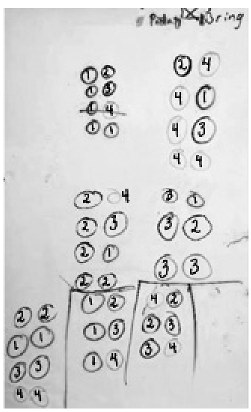

Figure 2. *Lea and Frida's whiteboard when they discuss with the other group*

Note.The numbers are marked by the authors to represent the different colors in the original picture: 1 is red, 2 is black, 3 is blue and 4 is green.

About 25 minutes into the session Kristin instructs the student pairs to split up. She encourages her students to describe to others how they have solved the problem, their use of representations, and which conditions they have chosen. The discussion that takes place when Synne visits Frida at their whiteboard is shown in table 2.

The identified moves indicate a dialog where the girls challenge each other's opinions (1) and request clarification (4) according to the work on the whiteboard. Frida seems a bit unsure about what they have done and invites Synne to say what she thinks Frida and Lea have done (1). Synne responds to Frida's request by providing reasoning about the representation and written work on the whiteboard (2). Both girls support each

Table 2. *Dialog between Frida and Synne about Frida and Lea's solution strategy*

Utterances	Dialog moves [DM]
1. Frida: Yes, we have done it like, eh hehe. Or what do you think we have done?	[DM5]: Challenging Synne to try to understand the group's approach.
2. Synne: I think the different colors represent different flavors, and you have found all the different combinations with the different flavors.	[DM4]: Providing reasoning, explaining what she thinks is the solution strategy.
3. Frida: Yes, we first took all the different flavors with raspberry, and then all the combinations with oreo, and then there will be one less for each flavor.	[DM2] and [DM3]: Expressing agreement and building further on Synne's observation.
4. Synne: Yes, I think we solved the task quite similarly, except we didn't do that last part. When you say that there will be one less for each flavor, I think I see what you mean, but I'm not quite sure. What does it mean?	[DM2]: Supportive contribution to Frida's utterance. [DM1]: Asking for clarification.
5. Frida: Eh, if, when we have made all the combinations with red and then move over to blue, we've already made the combination with red and blue. We don't not have to take that once again, so there's one less combination.	[DM4]: Responding to the request by providing reasoning for why there is one less combination for each flavor.
6. Synne: Yes!	[DM2]: Supportive confirmation.

other positively during their conversation (3, 4, 6), which indicates that they treat each other's ideas as worthy of consideration. Synne engages critically but constructively (Littleton & Mercer, 2013) with Frida and Lea's ideas when she expresses how she understands their work (2). When Frida builds on Synne's observation (3) she offers relevant information. Synne also responds critically to Frida's explanation and requests further information (5). The whiteboard becomes an important cultural tool in the reasoned dialog as the girls are standing side by side, looking at the work from the same angle, and using the drawings as the background for their dialog.

Michaels and O'Connor (2015) argue that it is challenging to get students to dig deeper into their own or others' reasoning to explore ideas. Table 3 below shows the dialog when Lea returns to Frida and they discuss both the solution Lea saw and debated at Synne and Ada's whiteboard and their own solution.

Lea suggests that Ada and Synne have a better overview of their result and it appears as if she wants to borrow the strategy from them (1). However, Frida explains that they have almost solved the problem themselves (2). She keeps to their own solution without borrowing the solution from the other group. She has discovered an argument for why there is one less combination for each flavor (2). Lea seems to be convinced that they have a good overview, and then also keeping to their own solution process (3).

Table 3. *Dialog between Frida and Lea after discussions with Synne and Ada*

Utterances	Dialog moves [DM]
1. Lea: Because it seemed smarter, I think they had a better overview of their result. They wrote it vertically, with the two flavors on top of each other. Then they had all the combinations with strawberry and then all with pistachio. We don't have the same overview; we're a bit messy. (Lea looks over at Ada and Synne's whiteboard during her observations).	[DM3]: Elaborating on the strategy she saw on the other whiteboard.
2. Frida: But I think we have pretty good control. There's one less combination for each flavor because we have made one combination with the new flavor with the previous flavors.	[DM4]: Expressing that she stands by their own strategy and provides reasoning about their own work.
3. Lea: Yeah, so we have a pretty good overview. The other group used some other conditions for their solution, but that was up to each group. We can just continue our work. That's good!	[DM2]: Supportive contribution. [DM4]: Providing reasoning when she compares the work of the two groups.

The reasoned dialog with the table-chair problem with Arne and Vetle

The students have been working with the first part of the problem, how many chairs are there for the 5- and 10-table problems. About 27 minutes into the session, they start working on the next part of the problem, how many chairs are there for 17 tables? Kristin asks the class if they can write a calculation to help them find an answer. Arne and Vetle's solutions for the 5-table and the 10-table problems are shown on their whiteboard, (see figure 3). They have drawn tables with chairs (marked with dots) and counted the dots. To find the solution for the 17-table problem they add the chairs from the 10-table, 5-table, and 2-table problems (22 + 12 + 6) and then subtract four chairs because no chairs can be placed at the two table sides that have been placed together, finding a solution of 36 chairs (see figure 4).

About 35 minutes into the session Kristin instructs the student pairs to discuss their work across groups. The following dialog takes place (see table 4) when Mari visits Arne's whiteboard (step 3 of the problem-solving session).

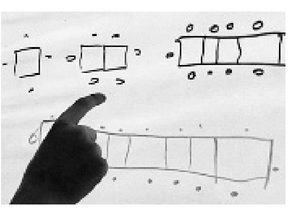

Figure 3. *Solution for 5- and 10-table problem*

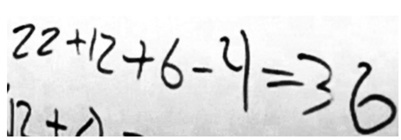

Figure 4. *Calculation for the 17-table problem*

Table 4. *Dialog between Arne and Mari about the 17-table solution*

Utterances	Dialog moves [DM]
1. Arne: When we (Arne and Vetle) were doing the 17-table problem, we first added the chairs for the 5-table and then for the 10-table problems.	[DM3]: Recapitulating and building on the prior discussion with his partner Vetle.
2. Mari: Why?	[DM5]: Challenging Arne to give reasons for this solution strategy.
3. Arne: Because 10 and 5 and 2 make 17. When we put them together, we got 6 and 12 chairs, minus the two in the middle. That's the 7-table solution, and then 16 plus 22 chairs for the 10-table solution, minus the two in the middle here. Then we got 36 chairs for the 17-table problem.	[DM4]: Responding to the challenge by providing reasoning, explaining the solution strategy.
4. Mari: Where did you get 16?	[DM1]: Requesting clarification, inviting Arne to elaborate on his and Vetle's solution process.
5. Arne: 16 is from the 5-table and the 2-table solutions, 12 plus 6 is 18. But when I put two tables together the seats at one end of each table disappear, that's why we take minus two.	[DM4]: Building and elaborating on the solution process, making further reasoning.
6. Mari: When you set the 7-table and the 10-table solutions together you have to take away two?	[DM1]: Requesting further clarifications about the solution process.
7. Arne: Yes, that's 38 minus 2. Do you get it?	[DM2]: Supportive move, confirming the 36-chairs solution. [DM1]: Requesting confirmation.
8. Mari: Yes, I think so, but I'm a bit confused. What you just said doesn't match your calculation.	[DM2]: Supportive confirmation [DM5]: Challenging Arne to clarify the mismatch between the verbal explanation and the written representation.
9. Arne: No, that's right. When we were doing the 17-table problem, we first added the chairs for the 10-table solution and then for the 5-table and the 2-table solutions, and then we had to take away four chairs between the tables. (Arne writes a calculation on the whiteboard that matches what he tells Mari, figure 5)	[DM2]: Supportive confirmation [DM4]: Building on prior utterances, clarifying their solution process.

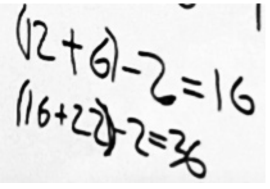

Figure 5. *Calculation, clarifying the two boys' solution process*

The dialog moves identified in the discussion between Mari and Arne illustrate how Mari challenges (2, 8) and requests clarifications about the solution process (4, 6), inviting Arne to build on and give further reasons for the solution strategy for the 17-table problem (3, 5, 9). The supportive moves (7–9) also indicate an atmosphere in which the two students are willing to explain their thinking and engage with others' ideas. Säljö (1995, p. 91) points out the importance of thinking "with and through artifacts", and we observe how the vertical whiteboard supports the reasoned dialog, which is particularly clear when Arne adds a calculation on the whiteboard while explaining to Mari, helping her to understand their thinking (9). Mari does not tell Arne about her and Synne's solution, but when Vetle returns to Arne, he explains how the two girls have solved the 17-table problem: "they had found many ways to find a solution, for instance 17 + 17 + 2".

The dialog in table 5 shows how the teacher guides a reasoned dialog while focusing on Ada and Kaja's solution, 17 · 2 + 2.

Table 5. *Whole-class dialog about solutions for the 17-table problem*

Utterances	Dialog Moves [DM]
1. Kristin: Can you (Ada and Kaja) explain how you found the solution (17 · 2 + 2)? The rest of you should try to understand their explanation.	[DM5]: Challenging Ada and Kaja to present their solution, explaining to the rest of the class.
2. Kaja: First, we calculated 17 · 2, because there were 17 chairs on one side (of the table) and 17 chairs on the other side.	[DM4]: Responding to the challenge by providing reasoning, explaining their solution strategy.
3. Kristin: Maybe we need a drawing. Can you make a quick drawing?	[DM1]: Requesting clarifications about the solution process, suggesting a drawing.
4. Kaja: I can just do it like this (she draws an oblong rectangle and puts dots along the long sides, figure 6). Then there are 17 here (points to the dots on the upper side of the rectangle) and 17 here (points to the dots on the lower side of the rectangle. That's 34 chairs together, then we miss two here (puts a dot on each short side of the rectangle), so we add 2, and then there are 36 (chairs).	[DM4]: Building on prior utterances, clarifying their reasoning leading to the solution.

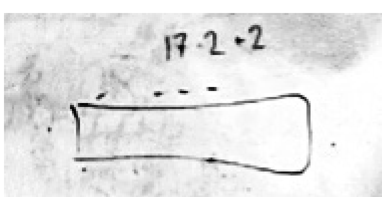

Figure 6. *Whiteboard after Kaja's explanation*

Kristin plays a crucial role in this reasoned dialog. Her dialog moves challenge (1) and request clarification about Ada and Kaja's solution strategy (3), inviting them to explain their thinking to the rest of the class. Kaja responds to the challenge by providing reasoning, explaining, and clarifying their solution process (2, 4). We observe how Kristin invites Kaja (3) to make a drawing on the whiteboard (4). It seems as if Kristin uses the drawing on the whiteboard as an important tool for the collectively reasoned dialog that is established in the class.

Kristin asks the class if they can solve the 100-table problem. The students return to their original whiteboard and attempt to respond to this challenge. The dialog between Arne and Vetle at their whiteboard is shown in table 6.

Table 6. *Arne and Vetle are solving the 100-table problem*

Utterances	Dialog Moves [DM]
1. Arne: 17 · 2 + 2. We do the same, just 100 · 2 + 2.	[DM3]: Recapitulating and building on the prior whole-class discussion.
2. Vetle: Yes ... (Arne writes 100 · 2 – 2 = 202) (Vetle looks at the whiteboard for about 10 secs.) ... eh, no.	[DM2]: Supportive move.
3. Arne: No?	[DM1]: Requesting clarification, responding to Vetle's "no".
4. Vetle: No, 100 times 2 is 200, minus 2, that's not 202.	[DM3]: Building on (2), expressing shared ideas and agreements, only to correct the calculation.
5. Arne: Yes, it wasn't minus 2. It was plus 2. (Arne revises the written representation, showing 100 · 2 + 2 = 202) ... Now we'll find a "recipe" that works no matter how many tables we put together.	[DM2]: Supportive move. [DM5]: Challenging them to find an expression, a solution for *n*-tables.
6. Vetle: I know, we can write like this. (Vetle writes: ? · 2 + 2 = ?)	[DM3]: Responding to the challenge, suggesting a general expression.
7. Arne: That's our "math recipe"!	[DM2]: Supportive move.
8. Vetle: Yes, our "recipe" for this problem.	[DM3]: Expressing shared agreement.

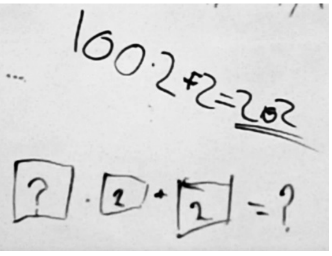

Figure 7. *Arne and Vetle's whiteboard*

We have particularly focused on Arne and Vetle's whiteboard, and one of the whole-class discussions based on three situations from this problem-solving session. The two boys found a solution for the 17-table problem. We argue that while their first solution was correct, it is incomplete for finding a solution for the 100-table or for any number of tables. The findings section has revealed some mathematical advances (Webb et al., 2021) during the focused situations. Mari and Synne's solution (17 + 17 + 2) and Kristin's guided whole-class dialog, emphasizing Ada and Kaja's solution, $17 \cdot 2 + 2$, are important influences on the shift in Arne and Vetle's understanding of the mathematical-solution process. They immediately see the connection (1) between the solution for a 17-table problem ($17 \cdot 2 + 2$) and a 100-table problem ($100 \cdot 2 + 2$) when they return to their original whiteboard (step 6). The reasoned dialogs from the different situations illustrated by the dialog moves, in combination with using whiteboards and drawings, reveal how Arne and Vetle are supported in expressing a new mathematical idea/solution. The dialog (see table 6) illustrates that the two boys are satisfied with an expression written as "$? \cdot 2 + 2 = ?$" (6–8). They have made the written representation, $100 \cdot 2 + 2 = 202$, and this representation is showing that the unknown "?" represents two different unknowns in the expression. The first "?" represents the number of chairs on one side of the table. The second "?" represents the total amount of chairs around the table. The two boys have made substantial progress in their solution process.

Concluding discussion

To answer the calls to investigate problem solving in various contextualized situations (Liljedahl & Cai, 2021) and to investigate how students benefit from explaining and engaging with groupmates' mathematical ideas (Webb et al., 2021), this study has addressed the following research question: How do specific utterances of reasoned classroom dialogs show the students' collective mathematical problem-solving process in the context of a thinking classroom? In line with Warwick et al. (2016), and Fauskanger and Bjuland (2021), five dialog moves have been used to identify reasoned dialogs in two problem-solving sessions with Norwegian grade-seven students in the context of a thinking classroom (Liljedahl, 2021). A possible criticism is that there could be a need to ask for more specific criteria for determining whether interactions are reasoned dialogs or not. Following Warwick et al. (2016), all five dialog moves have the potential to make a reasoned dialog. Our selection of situations illustrates chains of utterances, showing the students' collaborative efforts to participate in building on each other's initiatives.

Our analysis suggests that when students participate in reasoned dialogs, they both challenge and support each other's ideas. They clarify their thinking by listening to others and putting thoughts into words. When they debate solutions across groups, they have to be specific about their work, both when it comes to their use of representations and their strategies. In line with Fernández et al. (2001), our study illustrates the dynamic and continuous scaffolding process with symmetrical interaction between the students. Norwegian teachers might argue that the students would need a well-rehearsed procedure or a great deal of help from the teacher to solve these types of tasks. On the other hand, the table-chair problem, for example, is a problem used quite frequently in the algebra literature that students successfully engage with. The findings reveal that the context of a thinking classroom created an environment suitable for highly interactive learning where students constructed and refined their ideas in collaboration with each other.

Clancey (2008) argues that knowledge has to be actively constructed, it is not passively received when working in collaborative contexts. In the third problem-solving session we see Frida struggling to understand how and why Lea wants to change their strategy and their representations of the ice-cream problem. They do not passively copy the work of Ada and Synne, Frida has to understand how a change can improve their work. It might be tempting to ask that when students gain access to their classmates' work, will they stop thinking and just copy from them? Only once in the data material did we see a student pair passively receive knowledge from another whiteboard. This occurred in session 2 where a boy copied a solution for the number-tower problem (see Valbekmo & Svorkmo, 2021). In this situation, the student himself defined this as stealing, saying: "I stole it from the other whiteboard." In all the other situations where students used ideas from their classmates and their whiteboards, they analyzed and evaluated the idea before they used it.

We have illustrated students' mathematical advances (Webb et al., 2021) by showing the shift in Arne and Vetle's understanding of the mathematical-solution process when working on the table-chair problem. They made a correct solution for the 17-table problem. However, the reasoned dialogs with other students and the scaffolded teacher-led whole-class dialog helped Arne and Vetle to solve the problem in a more sophisticated way, finding a solution for the 100-table problem and making a good attempt at solving any table-number problem.

One essential, but challenging part of classroom dialogs is to get students to dig deeper into the mathematical ideas of others (Michaels & O'Connor, 2015). Liljedahl (2021) points out that the role of the teacher is to mobilize the knowledge available in the classroom. Even though

Kristin had an active role in initiating the groups in their collaboration and discussion, we see from the reasoned dialogs between students (see table 4 as an example) that they can debate their work without further teacher involvement. These students are capable of and engaged in understanding and creating meaning on their own. Kristin gives her students new opportunities to learn by facilitating discussions between groups where the students can use each other's work as a scaffold in their construction of new knowledge.

In sum, this study provides insight into students' collective mathematical problem-solving processes. During reasoned dialogs students challenge each other's ideas and contribute their knowledge to solve problems together. The teacher plays a crucial role in facilitating mathematical discussions, both in small groups and whole-class discussions. This study contributes to knowledge on how vertical whiteboards can support students during their reasoned dialogs as a tool for directing attention and as a tool with the possibility to easily refine solutions and solution strategies during dialogs.

As also found by Fauskanger and Bjuland (2021), one limitation of the analytical approach used in this study is that we do not foreground individual students' learning trajectories. One idea for future research will therefore be to focus on individual students' opportunities to participate in a reasoned dialog in a thinking-classroom context over time. By choosing case pupils to represent or typify learner groups (Dudley, 2013), such case students can improve our understanding of proximal development needs in the context of a thinking classroom.

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Ingunn Valbekmo

Ingunn Valbekmo is assistant professor at the Norwegian Centre for Mathematics Education at the Norwegian University of Science and Technology. Her research interests concern how students can learn mathematics through problem solving and how teachers can facilitate learning through problem solving.

ingunn.valbekmo@matematikksenteret.no

Raymond Bjuland

Raymond Bjuland is professor of mathematics education at the University of Stavanger, Norway. His research interests are related to mathematical knowledge for teaching, collaborative mathematical problem solving and classroom research with a special focus on teacher-student dialogues. Bjuland is involved in the research and PD project *Mastering ambitious mathematics teaching* (MAM) in which teachers are provided with opportunities for learning ambitious mathematics teaching practices by participating in learning cycles.

raymond.bjuland@uis.no