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Completeness of certain exponential systems and zeros of lacunary polynomials

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A B S T R A C T

Let Γ be a subset of $\{0, 1, 2, ...\}$. We show that if Γ has 'gaps' then the completeness and frame properties of the system $\{t^k e^{2\pi int} : n \in \mathbb{Z}, k \in \Gamma\}$ differ from those of the classical exponential systems. This phenomenon is closely connected with the existence of certain uniqueness sets for lacunary polynomials.

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1. Introduction

Let Λ be a *separated* set of real numbers. Denote by

$$E(\Lambda) := \{e^{2\pi i\lambda t}, \lambda \in \Lambda\}$$

the corresponding exponential system.

Approximation and representation properties of exponential systems in different function spaces is a classical subject of investigation. In particular, the completeness and frame problems of $E(\Lambda)$ for the space $L^2(a, b)$ can be stated as follows: Determine if

(a) (*Completeness property of* $E(\Lambda)$) every function F in $L^2(a, b)$ can be approximated arbitrarily well in L^2-norm by finite linear combinations of exponential functions from $E(\Lambda)$;

(b) (*Frame property of* $E(\Lambda)$) there exist two positive constants A and B such that for every $F \in L^2(a, b)$ we have

$$A\|F\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle F, e^{2\pi i\lambda t}\rangle|^2 \leq B\|F\|_2^2,$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product in $L^2(a, b)$.

Note that the notion of a frame is very important and can be defined in a similar manner for an arbitrary system of elements $E = \{e_\lambda\}$ in a Hilbert space H. If E is a frame in H, then every element f from H admits a (possibly, non-unique) representation

$$f = \sum_{e_\lambda\in E} c_\lambda e_\lambda,$$

for some l^2-sequence of complex numbers c_λ (see e.g. [3]).

It is easy to check that the completeness property of $E(\Lambda)$ is translation-invariant: If $E(\Lambda)$ is complete in $L^2(a, b)$, then it is complete in $L^2(a + c, b + c)$, for every $c \in \mathbb{R}$. As a 'measure of completeness', we can introduce the so-called *completeness radius* of $E(\Lambda)$:

$$CR(\Lambda) = \sup\{a \geq 0 : E(\Lambda) \text{ is complete in } L^2(-a, a)\}.$$

Similarly, the frame property of $E(\Lambda)$ is also translation-invariant, and we can introduce the *frame radius* as

$$FR(\Lambda) = \sup\{a \geq 0 : E(\Lambda) \text{ is a frame in } L^2(-a, a)\}.$$

Both radii above can be expressed in terms of certain densities:

(A) The celebrated Beurling–Malliavin theorem [1] states that $CR(\Lambda) = D^*(\Lambda)$. Here D^* is the so-called upper (or external) Beurling–Malliavin density.

(B) It follows from the classical 'Beurling Sampling Theorem' [2] (see also a detailed discussion in [7]) that $FR(\Lambda) = D^-(\Lambda)$, where Λ is a separated (also called uniformly discrete) set and $D^-(\Lambda)$ is the lower uniform density of Λ.

We refer the reader to [8] or [11] for a complete description of exponential frames for the space $L^2(a, b)$. Note that it is not given in terms of a density of Λ.

Observe that the proofs of (A) and (B) use techniques from the complex analysis.

The density D^* can be defined and the Beurling–Malliavin formula for the completeness radius remains valid for the *multisets* $(\Lambda, \Gamma(\lambda))$, where $\Lambda \subset \mathbb{R}$ and $\Gamma(\lambda) = \{0, ..., n(\lambda) - 1\}$, i.e. for the systems

$$E(\Lambda, \Gamma(\lambda)) := \{t^k e^{2\pi i\lambda t} : \lambda \in \Lambda, k = 0, ..., n(\lambda) - 1\}. \tag{1}$$

Here and in what follows, for convenience of notation, we assume that $t^0 = 1$ for every $t \in \mathbb{R}$. In (1), by $n(\lambda)$ we denote the multiplicity (number of occurrences) of the element $\lambda \in \Lambda$. The same is true for the frame radius, see [4]. In particular, if $\Lambda = \mathbb{Z}$ and $\Gamma(\lambda) = \Gamma_N := \{0, ..., N - 1\}$, $\lambda \in \Lambda$, then we have

$$CR(\mathbb{Z}, \Gamma_N) = FR(\mathbb{Z}, \Gamma_N) = N/2 = \#\Gamma_N/2, \tag{2}$$

where $\#\Gamma$ is the number of elements of Γ, and $CR(\mathbb{Z}, \Gamma_N)$ and $FR(\mathbb{Z}, \Gamma_N)$ are the completeness and frame radius of $E(\mathbb{Z}, \Gamma_N)$, respectively. Moreover, the system $E(\mathbb{Z}, \Gamma_N)$ is a Riesz basis in $L^2(-N/2, N/2)$, see [12].

We can consider the completeness property of systems from (1) in other function spaces, such as $L^p(a, b)$ and $C([a, b])$. For each of these spaces, the completeness property is translation-invariant. Clearly, the completeness in $C([-a, a])$ implies the completeness in $L^p(-a, a)$ for every $1 \le p < \infty$. Observe that if $E(\Lambda, \Gamma(\Lambda))$ is not complete in $C([-a, a])$, its *deficiency* in $C([-a, a])$ is at most 1, i.e. by adding to the system an exponential function $e^{2\pi iat}, a \notin \Lambda$, the new lager system becomes complete in $C([-a, a])$ (see e.g. discussion in [10]). It easily follows that every system in (1) has the same completeness radius for every space considered above.

2. Statement of problem and results

We will now introduce somewhat more general systems. Assume that $\Lambda \subset \mathbb{R}$ is a discrete set and that to every $\lambda \in \Lambda$ there corresponds a finite or infinite set $\Gamma(\lambda) \subset \mathbb{N}_0 := \{0, 1, 2, 3, ...\}$. Set

$$E(\Lambda, \Gamma(\lambda)) = \{t^\gamma e^{2\pi i\lambda t} : \lambda \in \Lambda, \gamma \in \Gamma(\lambda)\}.$$

Inspired by the recent work of Hedenmalm [5], we ask: What are the completeness and frame properties of $E(\Lambda, \Gamma(\lambda))$? In this paper, we restrict ourselves to the case $\Lambda = \mathbb{Z}$

and $\Gamma(n) = \Gamma \subset \mathbb{N}_0, n \in \mathbb{Z}$, is a fixed set. That is, we will consider the completeness and frame properties of the system

$$E(\mathbb{Z}, \Gamma) := \{t^\gamma e^{2\pi i n t} : n \in \mathbb{Z}, \gamma \in \Gamma\}, \quad \Gamma \subset \mathbb{N}_0.$$

Let us now introduce the formal analogues of the completeness and frame radii:

$$CR(\mathbb{Z}, \Gamma) := \sup\{a \geq 0 : E(\mathbb{Z}, \Gamma) \text{ is complete in } L^2(-a, a)\},$$
$$FR(\mathbb{Z}, \Gamma) := \sup\{a \geq 0 : E(\mathbb{Z}, \Gamma) \text{ is a frame in } L^2(-a, a)\}.$$

We also define the completeness radius $CR_C(\mathbb{Z}, \Gamma)$ in the spaces of continuous functions:

$$CR_C(\mathbb{Z}, \Gamma) := \sup\{a \geq 0 : E(\mathbb{Z}, \Gamma) \text{ is complete in } C([-a, a])\}.$$

In what follows, to exclude trivial remarks, we will always assume that $0 \in \Gamma$.

Set

$$\Gamma_{even} = \Gamma \cap 2\mathbb{Z} \quad \text{and} \quad \Gamma_{odd} = \Gamma \cap (2\mathbb{Z} + 1),$$

and introduce the following number

$$r(\Gamma) := \begin{cases} \#\Gamma_{odd} + \frac{1}{2}, \text{ if } \#\Gamma_{odd} < \#\Gamma_{even}, \\ \#\Gamma_{even}, \text{ if } \#\Gamma_{odd} \geq \#\Gamma_{even}. \end{cases}$$

Observe that $r(\Gamma) < \#\Gamma/2$ unless $\#\Gamma_{even} = \#\Gamma_{odd}$ or $\#\Gamma_{even} = \#\Gamma_{odd} + 1$.

It turns out that the completeness and frame properties of $E(\mathbb{Z}, \Gamma)$ may differ from the ones for the systems considered above. In particular, we have

Theorem 1. *Given any finite or infinite set $\Gamma \subset \mathbb{N}_0$ satisfying $0 \in \Gamma$. Then*

(i) $CR(\mathbb{Z}, \Gamma) = \#\Gamma/2$;

(ii) $CR_C(\mathbb{Z}, \Gamma) = FR(\mathbb{Z}, \Gamma) = r(\Gamma)$.

Below we prove more precise results.

Theorem 1 shows that property (2) is no longer true for the systems $E(\mathbb{Z}, \Gamma)$.

The proof of part (i) uses mainly basic linear algebra. We will see that the completeness property of $E(\mathbb{Z}, \Gamma)$ in $L^2(a, b)$ is translation-invariant, and so $CR(\mathbb{Z}, \Gamma)$ still can be viewed as a 'measure of completeness' of $E(\mathbb{Z}, \Gamma)$.

On the other hand, neither the frame property in $L^2(a, b)$ nor the completeness property in $C([a, b])$ is translation-invariant in the sense that both of them depend on the length of the interval (a, b) and also on its position. This phenomenon is intimately connected with the solvability of certain systems of linear equations and also with the existence of certain uniqueness sets for lacunary polynomials, see Theorem 2 below.

Given any finite set $M \subset \mathbb{N}_0$, let $P(M)$ denote the set of complex polynomials with exponents in M:

$$P(M) := \{P(x) = \sum_{m_j \in M} c_j x^{m_j} : c_j \in \mathbb{C}\}.$$

If $M \subset \mathbb{N}_0$ consists of n elements (shortly, $\#M = n$), then clearly no set $X \subset \mathbb{R}$ satisfying $\#X \leq n-1$ is a uniqueness set for $P(M)$, i.e. there is a non-trivial polynomial $P \in P(M)$ which vanishes on X. This is no longer true if $\#X = n$. Moreover, there exist real uniqueness sets X, $\#X = n$, that are uniqueness sets for every space $P(M), \#M = n$. Indeed, by Descartes' rule of signs, each $P \in P(M)$ may have at most $n-1$ distinct positive zeros, and so every set of n positive points is a uniqueness set for $P(M)$ (usually, Descartes' rule of signs is formulated only for the real polynomials P, but applying it to $\operatorname{Re} P$ and $\operatorname{Im} P$, we get the result for complex polynomials P as well). Here we present a less trivial example of such a set. Given N distinct real numbers $t_1, \dots, t_N$, set

$$S(t_1, \dots, t_N) := \{(-1)^k t_k\}_{k=1}^N. \tag{3}$$

Theorem 2. *Assume that $0 < t_1 < t_2 < \cdots < t_N$. Then both sets $\pm S(t_1, \dots, t_N)$ are uniqueness sets for every space $P(M), M \subset \mathbb{N}_0, \#M = N$.*

The rest of the paper is organized as follows: In Section 3 several auxiliary results are proved. Theorem 2 is proved in Section 4. We consider the completeness property of $E(\mathbb{Z}, \Gamma)$ in $L^2(a, b)$ and in $C([a, b])$ in Sections 5 and 6, respectively. Finally, in Section 7 we consider the frame property of $E(\mathbb{Z}, \Gamma)$ and also present some remarks.

3. Auxiliary lemmas

Given $N \in \mathbb{N}, \mathbf{x} = \{x_0, \dots, x_{N-1}\} \subset \mathbb{R}$, and $\Gamma = \{\gamma_0, \gamma_1, \dots, \gamma_{N-1}\} \subset \mathbb{N}$ we denote by $V(\mathbf{x}, \Gamma)$ a *generalized $N \times N$ Vandermonde matrix*,

$$V(\mathbf{x}; \Gamma) := \begin{pmatrix} x_0^{\gamma_0} & x_1^{\gamma_0} & x_2^{\gamma_0} & \dots & x_{N-1}^{\gamma_0} \\ x_0^{\gamma_1} & x_1^{\gamma_1} & x_2^{\gamma_1} & \dots & x_{N-1}^{\gamma_1} \\ \dots & \dots & \dots & \dots & \dots \\ x_0^{\gamma_{N-1}} & x_1^{\gamma_{N-1}} & x_2^{\gamma_{N-1}} & \dots & x_{N-1}^{\gamma_{N-1}} \end{pmatrix}. \tag{4}$$

We will usually assume that $0 \in \Gamma$. Note that if $\Gamma = \{0, 1, \dots, N-1\}$, then the matrix $V(\mathbf{x}; \Gamma)$ is a standard Vandermonde matrix, and it is easy to compute its determinant and establish whether it is invertible or not. However, if Γ has gaps, the situation is more complicated. In the case when $x_i > 0$ for all $i = 0, \dots, n-1$, we can use the following result from the theory of *totally positive matrices*, see e.g. [6] and [9].

Proposition 1. (see [9], section 4.2) *If $0 < x_0 < x_1 < \cdots < x_N$ and $\gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_N$, then $V(\mathbf{x}; \Gamma)$ is a totally positive matrix. In particular, it is invertible.*

This statement is no longer true if $\mathbf{x}$ contains both positive and negative coordinates.

We will be interested in a particular case where $\mathbf{x} = (s, s+1, ..., s+N-1)$ for some $s \in \mathbb{R}$. Consider the problem: Describe the set of points $s \in \mathbb{R}$ such that the matrix $V((s, \dots, s+N-1); \Gamma)$ is invertible *for every* $\Gamma \subset \mathbb{N}_0, \#\Gamma = N$.

Lemma 1. *$V((x_0, x_1, \dots, x_{N-1}); \Gamma)$ is not invertible if and only if there exists a polynomial $P \in P(\Gamma)$ which vanishes on the set $\{x_0, x_1, ..., x_{N-1}\}$.*

Proof. Write $\Gamma = \{\gamma_0, \gamma_1, \dots, \gamma_{N-1}\}$. The matrix $V((x_0, x_1, \dots, x_{N-1}); \Gamma)$ is not invertible if and only if its transpose is not. The latter means that there exists a non-zero vector $\mathbf{a} = (a_0, \dots, a_{N-1})$ satisfying $V((x_0, x_1, \dots, x_{N-1}); \Gamma)^T \mathbf{a}^T = 0$. This means that the polynomial $\sum_{j=0}^{N-1} a_j x^{\gamma_j}$ vanishes at the points $x_0, \dots, x_{N-1}$. $\square$

Lemma 2. *Given $N \geq 2$, the matrix $V((s, \dots, s+N-1); \Gamma)$ is invertible for every $\Gamma \subset \mathbb{N}_0, \#\Gamma = N$, $0 \in \Gamma$, if*

(i) *$s \geq 0$;*

(ii) *$s \in (-N/2, -N/2+1) \setminus (1/2)\mathbb{Z}$.*

For part (i), if $s > 0$ then the statement directly follows from Proposition 1 (and we do not even need to assume that $0 \in \Gamma$). Now, we consider the case $s = 0$. Since $0 \in \Gamma$, we see that the first column of our matrix is just $(1, 0, \dots, 0)^T$, so after the column expansion, it suffices to note that the matrix $V((s+1, s+2, \dots, s+N-1); \Gamma \backslash \{0\})$ is invertible by Proposition 1.

Part (ii) follows from Lemma 1, Theorem 2, and the observation that for every $s \in (-N/2, -N/2+1)$ such that s is not equal to $k/2$ for some $k \in \mathbb{Z}$, the set $\{s, \dots, s+N-1\}$ can be written as $\pm S$, where S is defined in (3).

Clearly, by Lemma 2, the determinant of $V((s, \dots, s+N-1); \Gamma)$ is a non-trivial polynomial of s. Hence, for every fixed Γ, this matrix is invertible for every s outside of a finite number of points.

In what follows, by measure we mean a finite, complex Borel measure on $\mathbb{R}$.

Given a measure μ, we denote by $\hat{\mu}$ its Fourier–Stieltjes transform

$$\hat{\mu}(x) = \int_{\mathbb{R}} e^{-2\pi i x t} \, d\mu(t).$$

We also denote by δ_x the δ-measure concentrated at the point x.

Lemma 3. *Let μ be a measure supported on the interval $[\alpha, \alpha+1]$. The following are equivalent:*

(i) *$\hat{\mu}$ vanishes on $\mathbb{Z}$;*

(ii) *$\mu = A(\delta_\alpha - \delta_{\alpha+1})$, for some $A \in \mathbb{C}$.*

Proof. We present a proof of (i) $\Rightarrow$ (ii). The converse implication is trivial.

Since $\operatorname{supp}\mu \subset [\alpha, \alpha+1]$, it is easy to see that the function

$$f(z) := e^{2\pi i(\alpha+1/2)z}\hat{\mu}(z)$$

is entire and satisfies

$$|f(x+iy)| \le Ce^{\pi|y|}, \quad x, y \in \mathbb{R}, \tag{5}$$

with some constant C. Since f vanishes on $\mathbb{Z}$, the function $g(z) := f(z)/(\sin \pi z)$ is also entire. It is easy to see that there is a positive constant B such that

$$|\sin(\pi(x+iy))| \ge Be^{\pi|y|}, \quad \text{for all } x, y \in \mathbb{R}, \ \inf_{n\in\mathbb{Z}} |x+iy-n| \ge 1/4.$$

This, (5) and the maximum modulus principle imply that $g(z)$ is bounded in $\mathbb{C}$. Hence, g is a constant function, from which the lemma follows by the uniqueness of the Fourier–Stieltjes transform. $\square$

Let us now consider measures μ that are "orthogonal" to $E(\mathbb{Z}, \Gamma)$:

$$\int_{\mathbb{R}} t^\gamma e^{-2\pi int}\, d\mu(t) = 0, \quad \text{for all } \gamma \in \Gamma, n \in \mathbb{Z}. \tag{6}$$

Lemma 4. *Assume that $\Gamma \subset \mathbb{N}_0, \#\Gamma = N, 0 \in \Gamma$, and that a measure μ is concentrated on $[\alpha, \alpha+N]$. If μ satisfies (6), then there is a finite set $S \subset (\alpha, \alpha+1)$ and measures μ_s, $s \in S$, and ν such that*

(i) *$\mu = \sum_{s\in S} \mu_s + \nu$;*

(ii) *ν and $\mu_s, s \in S$, satisfy (6);*

(iii) *The following representations hold:*

$$d\nu = \sum_{j=1}^{N+1} a_j \delta_{\alpha+j-1}, \quad d\mu_s = \sum_{j=1}^{N} c_{s,j}\delta_{s+j-1}, \quad s \in S, \, c_{s,j} \in \mathbb{C}, \, a_j \in \mathbb{C}. \tag{7}$$

Note that μ_s satisfies (6) if and only if

$$\sum_{j=1}^{N} (s+j-1)^\gamma c_{s,j} = 0, \quad \text{for every } \gamma \in \Gamma, \ s \in S. \tag{8}$$

A similar observation is true for the measure ν.

Proof of Lemma 4. Clearly, μ admits a unique representation

$$d\mu(x) = \sum_{j=1}^{N} d\mu_j(x-j+1), \tag{9}$$

where each μ_j is a measure supported on $[\alpha, \alpha+1)$ for $j = 1, \ldots, N-1$, and $\operatorname{supp} \mu_N \subset [\alpha, \alpha+1]$. Then (6) is equivalent to

$$\int\limits_{[\alpha,\alpha+1]} e^{-2\pi i n t} \sum_{j=1}^{N} (t+j-1)^{\gamma}\, d\mu_j(t) = 0, \quad \text{for every } \gamma \in \Gamma, n \in \mathbb{Z}.$$

It follows from Lemma 3 that μ_j satisfy the system of N equations

$$\sum_{j=1}^{N} (t+j-1)^{\gamma} d\mu_j(t) = C_{\gamma}(\delta_{\alpha} - \delta_{\alpha+1}), \quad \text{for every } \gamma \in \Gamma. \tag{10}$$

The corresponding matrix on the left-hand-side is $V((t, \ldots, t+N-1), \Gamma)$. As we mentioned above, the subset $S \subset (\alpha, \alpha+1)$ of the zeros of its determinant is finite. Therefore, (10) implies that each measure $\mu_j, 1 \le j < N$, may only be concentrated at $\{\alpha\}$ and on S, while the support of μ_N belongs to $\{\alpha, \alpha+1\} \cup S$. We may therefore write:

$$d\mu_j = \sum_{s \in S} c_{s,j} \delta_s + a_j \delta_{\alpha}, \quad 1 \le j \le N-1;$$
$$d\mu_N = \sum_{s \in S} c_{s,N} \delta_s + a_N \delta_{\alpha} + a_{N+1} \delta_{\alpha+1}.$$

This and (9) prove part (i) of the lemma, where ν and μ_j are defined in (7).

Finally, part (ii) easily follows from (10). □

4. Uniqueness sets for lacunary polynomials

In this section, we will prove Theorem 2. Clearly, if $S(t_1, \ldots, t_N)$ is a uniqueness set for $P(M)$, then so is $-S(t_1, \ldots, t_N)$, since $P(-x) \in P(M)$ whenever $P(x) \in P(M)$. Therefore, it suffices to prove that $S(t_1, \ldots, t_N)$ is a uniqueness set for every space $P(M)$, $\#M = N$.

Assume that a polynomial $P \in P(M)$ vanishes on $S(t_1, \ldots, t_N)$. Without loss of generality, we can assume that P has real coefficients, since otherwise, we can just prove the theorem for $\operatorname{Re} P$ and $\operatorname{Im} P$ separately, because they both also must vanish on $S(t_1, \ldots, t_N)$. If P is even or odd, we have $P(t_k) = 0$, $1 \le k \le N$, and by Descartes' rule of signs, we deduce that $P \equiv 0$. Thus, we can assume that $P \not\equiv 0$ is neither even nor odd and derive a contradiction from there.

Consider the polynomials

$$P_e(x) = \sum_{m_j \in M, 2|m_j} c_j x^{m_j} = \frac{1}{2}(P(x) + P(-x))$$

and

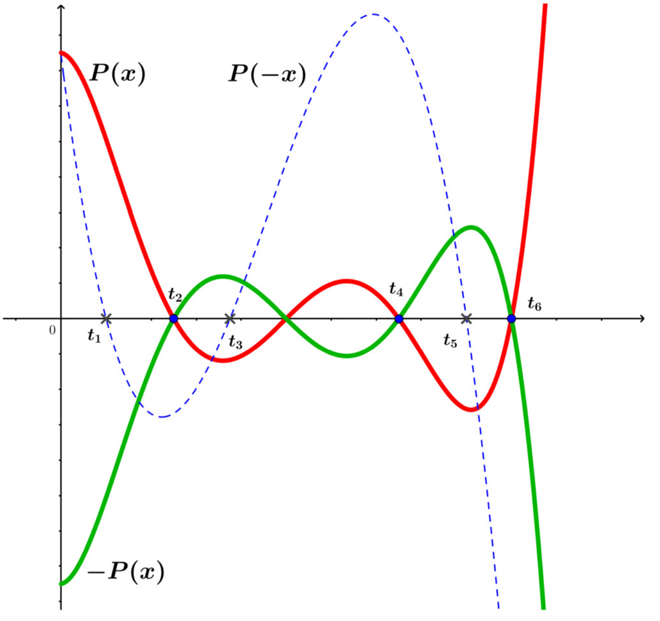

Fig. 1. $P(-x)$ has many intersections with either $P(x)$ or $-P(x)$.

$$P_o(x) = \sum_{m_j \in M, 2 \nmid m_j} c_j x^{m_j} = \frac{1}{2}(P(x) - P(-x)).$$

If one of them is identically zero, then P is even or odd and we are done. Let M have K even elements and $N - K$ odd elements. Then P_e has at most $K - 1$ positive roots and P_o has at most $N - K - 1$ positive roots by Descartes' rule of signs. We are going to show that P_e and P_o together have at least $N - 1$ positive roots thus getting the contradiction we need.

Let us consider the graphs of $P(x)$, $-P(x)$ and $P(-x)$, see Fig. 1. Since we assumed that P is neither even nor odd, these are three different polynomials. For simplicity, we first cover the case when $P(x)$ and $P(-x)$ do not have common positive zeroes. We indicate t_k with odd indices by crosses.

By assumption each cross except the first one and the last one is separated from the other crosses by the zeroes of $P(x)$. That is, it is contained in a connected component bounded by the pieces of the curves $y = P(x)$ and $y = -P(x)$. Thus, to get from the cross number m to the cross number $m+1$ we have to exit the component containing the first and enter the next one, giving us at least two intersections of the curve $y = P(-x)$ with curves $y = P(x)$ and $y = -P(x)$. Additionally, if N is even, then we also have to exit the last connected component as well, since there must be at least one more zero of $P(x)$ after the last cross. In total we will always have at least $N - 1$ intersections, that is P_e and P_o together have at least $N - 1$ positive roots as we wanted.

Now, we indicate the necessary changes in the case when $P(x)$ and $P(-x)$ have common positive roots. If we have two crosses that are not zeroes of $P(x)$ but between them there is a zero of $P(x)$, then the curve $y = P(-x)$ can go directly from the connected component of the first cross to the connected component of the second cross through this zero. But if $P(x_0) = P(-x_0) = 0$ then x_0 is a zero for both P_e and P_o, thus we anyway get two zeroes.

It remains to consider the case when we have a cross which is also a zero of $P(x)$. Assume that crosses from the number m to $m+l$ are zeroes of $P(x)$ and crosses number $m-1$ and $m+l+1$ are not (or there are no crosses with these indices). Then each of these $l+1$ zeroes are both zeroes for P_e and P_o, thus giving us two intersections. Finally, since the $m+l$'th cross is separated from $m+l+1$'st by at least one more zero of $P(x)$ we have to enter the connected component corresponding to this zero and the same between m'th and $m-1$'st zero, thus giving us the same number of intersections as in the case when $P(x)$ and $P(-x)$ did not have common zeroes.

5. Completeness of $E(\mathbb{Z}, \Gamma)$ in $L^2(a,b)$

Part (i) of Theorem 1 follows from the following theorem.

Theorem 3. *Given any finite set $\Gamma \subset \mathbb{N}_0$, the system $E(\mathbb{Z}, \Gamma)$ is complete in $L^2(a,b)$ if and only if $b - a \le \#\Gamma$.*

Proof. (i) Assume $b - a \le N := \#\Gamma$. It is then a simple consequence of Lemma 4 that $E(\mathbb{Z}, \Gamma)$ is complete in $L^2(a,b)$. Indeed, if the system is not complete then there exists non-trivial $f \in L^2(a,b)$ which is orthogonal to our system. Therefore, the measure $f\,dx$ is also orthogonal to the system, but it can not be a sum of delta measures unless f is identically zero.

(ii) Assume that $b - a > N$. We have to prove that $E(\mathbb{Z}, \Gamma)$ is not complete in $L^2(a,b)$, i.e. that there is a non-trivial function $F \in L^2(a,b)$ such that

$$\int_a^b t^\gamma e^{-2\pi int} F(t)\,dt = 0, \qquad \text{for every } \gamma \in \Gamma, n \in \mathbb{Z}. \tag{11}$$

The existence of such a function follows essentially from elementary linear algebra.

We have $b = a + N + \delta$, for some $\delta > 0$, and we can assume that $\delta < 1$. Write F in the form

$$F(t) = \sum_{j=0}^{N} F_j(t - j), \quad t \in (a, a + N + \delta),$$

where $F_j(t) := F(t + j)\mathbf{1}_{(a,a+1)}(t)$ vanish outside $(a, a + 1)$ for $j = 0, ..., N - 1$, and f_N vanishes outside $(a, a + \delta)$. Here $\mathbf{1}_{(a,a+1)}$ is the characteristic function of $(a, a + 1)$. Clearly, to prove (11) it suffices to find $N + 1$ non-trivial functions F_j as above satisfying for a.e. $t \in (a, a + 1)$ the system of N equations

$$\sum_{j=0}^{N}(t+j)^{\gamma}F_j(t)=0, \quad \text{for all } \gamma\in\Gamma,\ t\in(a,a+1).$$

Rewrite this system in the matrix form

$$V(t)\cdot(F_0(t),...,F_{N-1}(t))^T=-((t+N)^{\gamma_1},...,(t+N)^{\gamma_N})^T\cdot F_N(t), \quad \Gamma=\{\gamma_1,...,\gamma_N\},$$

where $V(t):=V((t,t+1,\dots,t+(N-1);\Gamma)$ is a generalized Vandermonde matrix defined above, whose determinant has only finite number of real zeroes. Therefore, there is an interval $I\subset(a,a+\delta)$ where $V(t)$ is invertible and satisfies

$$\sup_{t\in I}\ \sup_{\mathbf{x}\in\mathbb{R}^N,\,\|x\|=1}\|V^{-1}(t)\cdot\mathbf{x}\|<\infty.$$

Now, we can simply choose $F_N(t):=\mathbf{1}_I(t)$ and set

$$(F_0(t),...,F_{N-1}(t))^T:=-V^{-1}(t)\cdot((t+N)^{\gamma_1},...,(t+N)^{\gamma_N})^T\cdot\mathbf{1}_I(t). \quad \square$$

Remark 1. One can check that the above result on completeness of $E(\mathbb{Z},\Gamma)$ in $L^2(a,b)$ remain true for the space $L^p(a,b)$, $1\le p<\infty$.

6. Completeness of $E(\mathbb{Z},\Gamma)$ in $C([-a,a])$

Theorem 4. *$E(\mathbb{Z},\Gamma)$ is complete in $C([-a,a])$ if and only if $a<r(\Gamma)$.*

Clearly, this theorem implies $CR_C(\mathbb{Z},\Gamma)=r(\Gamma)$.

Proof. 1. Suppose that $a\ge r(\Gamma)$. We have to check that the system is not complete in $C([-a,a])$. Clearly, it suffices to produce a bounded measure μ on $[-r(\Gamma),r(\Gamma)]$ which satisfies (6).

Set $\mathbb{O}:=\#\Gamma_{odd},\mathbb{E}:=\#\Gamma_{even}$ and

$$f(x)=\begin{cases}\sin(\pi x)+\sum\limits_{k=1}^{\mathbb{O}}\alpha_k\sin\left((2k+1)\pi x\right),\ \text{if } \mathbb{O}<\mathbb{E},\\ 1+\sum\limits_{k=1}^{\mathbb{E}}\alpha_k\cos\left(2\pi kx\right),\ \text{if } \mathbb{O}\ge\mathbb{E},\end{cases} \tag{12}$$

where $\{\alpha_k\}\subset\mathbb{R}$.

Lemma 5. *There exist numbers α_k in (12) such that f satisfies*

$$f^{(\gamma)}(n)=0, \quad \gamma\in\Gamma,\, n\in\mathbb{N}. \tag{13}$$

It is easy to check that f in (12) is the Fourier-Stieltjes transform of a measure supported by $[-r(\Gamma), r(\Gamma)]$. We can therefore easily see that Lemma 5 proves the necessity in part 1 of Theorem 4.

Proof of Lemma 5. Consider the case $\mathbb{E} \leq \mathbb{O}$.

We wish to find α_k so that the function

$$f(x) = 1 + \alpha_1 \cos(2\pi x) + \cdots + \alpha_{\mathbb{E}} \cos(2\pi \mathbb{E} x)$$

satisfies (13).

It is clear that every *odd* derivative of f vanishes on $\mathbb{Z}$. Therefore, it suffices to find the coefficients so that $f^{(\gamma)}$ vanishes on $\mathbb{Z}$ for every $\gamma \in \Gamma_{even}$ (in particular, for $\gamma = 0$). This is equivalent to saying that the coefficients must satisfy the following system of $\mathbb{E}$ linear equations:

$$\gamma = 0: \quad \alpha_1 + \cdots + \alpha_{\mathbb{E}} = -1$$

and

$$\gamma \in \Gamma_{even}, \gamma \neq 0: \quad (2\pi)^\gamma \alpha_1 + (4\pi)^\gamma \alpha_2 \cdots + (2\pi\mathbb{E})^\gamma \alpha_{\mathbb{E}} = 0.$$

This system has a *unique non-trivial solution* by Proposition 1.

The case $\mathbb{E} > \mathbb{O}$ is similar, and we leave the proof to the reader. □

We return now to the proof of Theorem 4.

2. Assume $a < r(\Gamma)$. We have to show that $E(\mathbb{Z}, \Gamma)$ is complete in $C([-a, a])$, i.e. that the only measure μ on $[-a, a]$ which satisfies (6) is trivial.

We will consider the case $\mathbb{E} \leq \mathbb{O}$, i.e. $r(\Gamma) = \mathbb{E}$. Clearly, we can assume that $\mathbb{E} = \mathbb{O}$ and so $\mathbb{E} = N/2$, where $N := \#\Gamma$ is an even number. Also, to avoid trivial remarks, we assume that $N \geq 4$.

Assume that μ is concentrated on $[-a, a]$ and satisfies (6). By (7) and Lemma 4, since $\mu(\{\pm N/2\}) = 0$, we have

$$d\mu = \sum_{s\in S} d\mu_s + d\nu = \sum_{s\in S} \sum_{j=1}^{N} c_{s,j} \delta_{s+j-1} + \sum_{j=2}^{N} a_j \delta_{-N/2+j-1},$$

where S is a finite subset of $(-N/2, -N/2 + 1)$ and the coefficients $c_{s,j}$ satisfy for every $s \in S$ the system of equations (8). By part (ii) of Lemma 2, this system has only trivial solutions $c_{s,j} = 0, j = 1, ..., N, s \in S \setminus (1/2)\mathbb{Z}$, and so

$$\mu = \nu_1 + \nu, \quad d\nu_1 := \sum_{j=1}^{N} c_j \delta_{-N/2+j-1/2},$$

where ν and ν_1 both are orthogonal to $E(\mathbb{Z},\Gamma)$.

Let us check that $\nu = 0$. It is more convenient to write ν in the form

$$\nu = \sum_{k=-N/2+1}^{N/2-1} b_k \delta_k, \quad b_k := a_{N/2+k+1}.$$

Then clearly, (6) is equivalent to the system of $N-1$ equations:

$$\sum_{k=-N/2+1}^{N/2-1} k^\gamma b_k = 0, \qquad \text{for every } \gamma \in \Gamma.$$

This is equivalent to the following systems:

$$\sum_{k=0}^{N/2-1} k^\gamma (b_{-k} + b_k) = 0,\ \gamma \in \Gamma_{even}, \quad \sum_{k=1}^{N/2-1} k^\gamma (b_{-k} - b_k) = 0,\ \gamma \in \Gamma_{odd}.$$

We can now use Proposition 1 to deduce that $b_{-k} + b_k = b_{-k} - b_k = 0$, for every k, thus $b_k = b_{-k} = 0$ for every k, that is $\nu = 0$. Similarly, we can check that $\nu_1 = 0$, and so $\mu = 0$.

The proof of the case $\mathbb{O} < \mathbb{E}$ is similar, and so we leave it to the reader. $\square$

Remark 2. One can prove that for $a \in [r(\Gamma), \#\Gamma/2]$, the deficiency of $E(\mathbb{Z},\Gamma)$ in $C([-a,a])$ is always finite.

7. Frame property of $E(\mathbb{Z},\Gamma)$

The frame property of $E(\mathbb{Z},\Gamma)$ in $L^2(a,b)$ is closely connected with the completeness property of $E(\mathbb{Z},\Gamma)$ in $C([a,b])$:

Theorem 5. *Assume $a < b$ and $\epsilon > 0$.*

(i) *If $E(\mathbb{Z},\Gamma)$ is complete in $C([a,b])$, then $E(\mathbb{Z},\Gamma)$ is a frame in $L^2(a,b)$.*

(ii) *If $E(\mathbb{Z},\Gamma)$ is not complete in $C([a,b])$, then $E(\mathbb{Z},\Gamma)$ is not a frame in $L^2(a-\epsilon, b+\epsilon)$.*

Observe that to finish the proof of Theorem 1, it remains to show that $FR(\mathbb{Z},\Gamma) = r(\Gamma)$. This follows from Theorem 4 and Theorem 5.

Proof of Theorem 5. (i) Assume that the system $E(\mathbb{Z},\Gamma)$ is complete in $C([a,b])$. We have to show that it is a frame in $L^2(a,b)$.

Recall that $E(\mathbb{Z},\Gamma)$ is a frame in $L^2(a,b)$ if there are positive constants A, B such that

$$A\|F\|_2^2 \le \sum_{n\in\mathbb{Z}} \sum_{\gamma\in\Gamma} |\langle F, t^\gamma e^{2\pi i n t}\rangle|^2 \le B\|F\|_2^2, \quad \text{for every } F \in L^2(a,b). \tag{14}$$

Using the Fourier transform, this is equivalent to the condition

$$A\|f\|_2^2 \le \sum_{n\in\mathbb{Z}}\sum_{\gamma\in\Gamma}|f^{(\gamma)}(n)|^2 \le B\|f\|_2^2, \tag{15}$$

where f is the inverse Fourier transform of F.

It is standard to check that the right-hand-side inequality in (14) (and in (15)) holds for every interval (a,b), see e.g. [7], Lecture 2. So, we only prove the left-hand-side inequality.

By Theorem 1, $E(\mathbb{Z},\Gamma)$ is not complete, and so is not a frame in $L^2(a,b)$ when $b-a > N := \#\Gamma$. Therefore, in what follows we may assume that $a+k-1 < b \le a+k$, for some $k\in\mathbb{N}, k\le N$.

Write

$$F(t) = \sum_{j=0}^{k-1} F_j(t-j), \quad F_j(t) := F(t+j)\cdot \mathbf{1}_{(a,a+1)}(t). \tag{16}$$

Then we have

$$\langle F, t^\gamma e^{2\pi int}\rangle = \int_a^{a+1} e^{2\pi int}\left(\sum_{j=0}^{k-1}(t+j)^\gamma F_j(t)\right)dt.$$

Hence,

$$\sum_{n\in\mathbb{Z}}|\langle F, t^\gamma e^{2\pi int}\rangle|^2 = \|\sum_{j=0}^{k-1}(t+j)^\gamma F_j(t)\|_2^2.$$

We see that the left-hand-side inequality in (14) is equivalent to

$$\|V_k(t)\cdot(F_0(t),\dots,F_{k-1}(t))^T\|_2^2 \ge A\|F\|_2^2, \tag{17}$$

where

$$V_k(t) := V(t,\dots,t+k-1;\Gamma)^T$$

denotes the $k\times N$ matrix which consists of the first k columns of $V(t,\dots,t+N-1;\Gamma)$, and we set

$$\|(G_1,\dots,G_k)^T\|_2^2 := \|G_1\|_2^2+\dots+\|G_k\|_2^2.$$

Let us first consider the case $b=a+k$. Since $E(\mathbb{Z},\Gamma)$ is complete in $C([a,b])$, there is no measure μ on $[a,b]$ orthogonal to this system. Then, since any measure of the form

$$d\mu = \sum_{j=0}^{k-1} x_j \delta_{t+j}, \quad (x_1, \dots, x_k) \in \mathbb{R}^k \setminus \{\mathbf{0}\}, \ t \in [a, a+1],$$

is not orthogonal to $E(\mathbb{Z}, \Gamma)$, we see that $V_k(t) \cdot \mathbf{x}^T \neq \mathbf{0}$, for every $\mathbf{x} \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ and $t \in [a, b]$. Therefore, there is a constant A such that

$$\|V_k(t) \cdot \mathbf{x}^T\|^2 \geq A\|\mathbf{x}\|^2, \quad t \in [a, a+1],$$

which implies (17).

Now, let us assume that $b = a + k - 1 + \delta$, where $0 < \delta < 1$. Then the function F_{k-1} in (16) satisfies $F_{k-1}(t) = 0, \delta < t < 1$. Similarly to above, for every vectors $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^{k-1}$ we have

$$\|V_k(t) \cdot \mathbf{x}\| \geq A_1\|\mathbf{x}\|, \ t \in [a, a+\delta], \ \|V_{k-1}(t) \cdot \mathbf{y}\| \geq A_2\|\mathbf{y}\|, \ t \in [a+\delta, a+1],$$

from which (17) follows.

(ii) Assume that the system $E(\mathbb{Z}, \Gamma)$ is not complete in $C([a, b])$. We have to show that it is not a frame in $L^2(a - \epsilon, b + \epsilon)$, for every $\epsilon > 0$. We can assume that $0 < \epsilon < 1/2$.

Let g be the inverse Fourier transform of a measure μ on $[a, b]$ that is orthogonal to the system. Then $g^{(\gamma)}$ vanishes on $\mathbb{Z}$, for every $\gamma \in \Gamma$.

Choose any r, $0 < r < \epsilon$, and consider the function

$$f(x) := g(x)\varphi(x), \quad \varphi(x) := \frac{\sin(\pi r x)}{\pi r x}.$$

Then, clearly, f is the (inverse) Fourier transform of an absolutely continuous measure on $(a - r, b + r) \subset (a - \epsilon, b + \epsilon)$, and

$$\|f\|_2 > C > 0, \quad \text{where } C \text{ does not depend on } \epsilon. \tag{18}$$

We will need

Lemma 6. *There is a constant C such that*

$$\sum_{n \in \mathbb{Z}} |\varphi^{(j)}(n)|_2^2 \leq C^j r^j, \quad j \in \mathbb{N}. \tag{19}$$

The proof of the lemma follows from two observations:

(i) φ is the Fourier transform of $\mathbf{1}_{(-r/2, r/2)}(t)/r$, and so $\varphi^{(j)}$ is the Fourier transform of

$$(-2\pi i t)^j \mathbf{1}_{(-r/2, r/2)}(t)/r.$$

It easily follows that $\|\varphi^{(j)}\|_2^2 \leq C r^j, j \in \mathbb{N}$.

(ii) The sum in (19) is equal to the norm $\|\varphi^{(j)}\|_2^2$.

Using (19), since $g^{(\gamma)}, \gamma \in \Gamma$, vanishes on $\mathbb{Z}$ and the functions $g^{(j)}, j \in \mathbb{N}$, are bounded on $\mathbb{R}$, we can easily see that

$$\sum_{n\in\mathbb{Z}}\sum_{\gamma\in\Gamma}|f^{(\gamma)}(n)|^2 = \sum_{n\in\mathbb{Z}}\sum_{\gamma\in\Gamma}|(g\varphi)^{(\gamma)}(n)|^2 \le Cr,$$

for some C. This and (18) imply that the left-hand-side inequality in (15) is not true if we choose r small enough. □

Remark 3. By a similar argument to the proof of Theorem 1, using Lemma 2 case (i), we can show that if $0 \in \Gamma$ then $E(\mathbb{Z}, \Gamma)$ is complete in $C([a, b])$ for $0 \le a < b$ if and only if $b - a < N$ (and if $a > 0$ then we do not need to assume that $0 \in \Gamma$).

Remark 4. Let us come back to the exponential systems $E(\mathbb{Z}, \Gamma(n))$ defined at the beginning of Section 2. Here we present a simple example that illustrates that such systems can have strikingly different completeness properties in L^2-spaces and C-spaces.

Let $f(x) = \sin(\pi x/2)$. Then $f^{(2k)}(2n) = f^{(2k+1)}(2n+1) = 0$, for every $k \in \mathbb{N}_0, n \in \mathbb{Z}$. Then, since f is the inverse Fourier transform of $(\delta_{1/4} - \delta_{-1/4})/2i$, the system

$$\{t^{2k}e^{4\pi int} : k \in \mathbb{N}_0, n \in \mathbb{Z}\} \bigcup \{t^{2k+1}e^{2\pi i(2k+1)t} : k \in \mathbb{N}_0, n \in \mathbb{Z}\}$$

is not complete in $C([-1/4, 1/4])$. On the other hand, one can check that it is complete in $L^2(I)$ on every finite interval $I \subset \mathbb{R}$.

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